1. Prove that if $G$ is a graph of order $n \geq 3$ such that $d(u)+d(v) \geq n$ for any $(u,v) \notin E(G)$, then $G$ is Hamiltonian.

**Solution:** It is pretty much the same proof as the Dirac’s theorem. For the connectedness, pick $u$ and $v$ from two different components, then $d(u) + d(v) \leq n - 2$. Contradiction.

The rest of the proof is almost identical. The only thing to note is that $x_0$ and $x_{\ell}$ cannot be adjacent, as otherwise we either have a Hamiltonian cycle or can construct a longer path.

2. Show that Dirac’s theorem is the best possible in terms of minimal degrees. In other words, for any odd $n \geq 3$, construct a graph $G$ such that $\delta(G) = \frac{n-1}{2}$ and $G$ is not Hamiltonian.

**Solution:** Note that we want to avoid a cycle of length $n$, which is odd. It is natural to construct a bipartite graph. Then, to spread degrees evenly, consider $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$.

3. Let $k < n$ be two positive integers. Let $G$ be a connected graph of order $n$ such that $\delta(G) \geq \frac{k}{2}$. Prove that $G$ must have a cycle of length $\lceil \frac{k}{2} \rceil + 1$, and this is tight.

**Solution:** Once again, consider the maximal length path $P_\ell = \{x_0, x_1, \ldots, x_\ell\}$. Consider $\Gamma(x_0)$. By maximality, $\Gamma(x_0) \subset P_\ell \setminus \{x_0\}$. Since $|\Gamma(x_0)| \geq k/2$, there exists a $t \geq \lceil k/2 \rceil$ such that $x_t \in \Gamma(x_0)$ (otherwise there is not enough “room” for $\Gamma(x_0)$). This creates a cycle of length $\lceil k/2 \rceil + 1$.

This is best possible. Consider a bunch of copies of $K_{\lceil k/2 \rceil + 1}$. This graph satisfies the degree constraint and has no cycles of length $\lceil \frac{k}{2} \rceil + 2$. However it is not connected. To make it connected without creating cycles, connect these copies by one edge in a tree-like structure.

In technical terms, we “contract” each $K_{\lceil k/2 \rceil + 1}$ into a vertex. Then construct a spanning tree.