1. Prove that there exists a two-edge-colouring of $K_n$ with at most
\[ \binom{n}{a} 2^{1 - \left( \frac{a}{2} \right)} \]
monochromatic $K_a$.

**Solution:** Consider a uniformly at random 2-edge-colouring of $K_n$. Let $X$ be the (random) total number of monochromatic cliques of size $a$ in $K_n$, and $X_S$ be the indicator variable of the event that the set $S$ is monochromatic. Then $X = \sum_{S, |S|=a} X_S$. Note that
\[ \mathbb{E} X_S = \Pr(X_S = 1) = \frac{2}{2^{\left( \frac{a}{2} \right)}}. \]

By the linearity of expectations,
\[ \mathbb{E} X = \sum_{S, |S|=a} \mathbb{E} X_S = \binom{n}{a} 2^{1 - \left( \frac{a}{2} \right)}. \]

Hence, there must exist a colouring such that $X \geq \mathbb{E} X = \binom{n}{a} 2^{1 - \left( \frac{a}{2} \right)}$.

2. Using the alteration method, prove that the Ramsey number $R(4, k)$ satisfies
\[ R(4, k) \geq \Omega((k / \log k)^2) \]

**Solution:** Recall that in the class we have showed that For any integer $n$ and $p \in (0, 1)$,
\[ R(4, k) > n - \binom{n}{4} p^\left( \frac{4}{2} \right) - \binom{n}{k} (1 - p)^\left( \frac{k}{2} \right). \]

A rough estimate is to set $\binom{n}{4} p^\left( \frac{4}{2} \right) < n/4$ and $\binom{n}{k} (1 - p)^\left( \frac{k}{2} \right) < n/4$ and then $R(4, k) > n/2$. Note that
\[ \binom{n}{4} p^\left( \frac{4}{2} \right) < \frac{n^4}{24} p^6. \]
Hence we should set $p = n^{-1/2}$. For the other constraint we use once again $1 - p < e^{-p}$ and $\binom{n}{k} \leq \left(\frac{ne}{k}\right)^k$, which yields,

$$\binom{n}{k}(1 - p)^{\binom{k}{2}} \leq \left(\frac{ne}{k}\right)^{k} e^{-p^{k}} = \left(\frac{ne}{k}\right)^{k} e^{-k(k-1)/2\sqrt{n}} \leq \left(\frac{ne}{k}\right)^{k} e^{-k^2/4\sqrt{n}},$$

where we use a loose bound $\frac{k(k-1)}{2} \geq \frac{k^2}{4}$ to make the calculation easier. Our goal is to set the right hand side less than $n/4$. Equivalently, taking the logarithm, we want

$$k \log n + k - k \log k - \frac{k^2}{4\sqrt{n}} \leq \log n - \log 4.$$

As hinted by the question, we should take $n = c \cdot \left(\frac{k}{\log k}\right)^2$. One can verify that this choice does the job if $c = 1/16$, and thus $R(4, k) \geq n/2 = \frac{1}{32} \left(\frac{k}{\log k}\right)^2$.

Another way of deriving $n = \Omega \left(\left(\frac{k}{\log k}\right)^2\right)$ is the following. First note that $n$ should be larger than $k$, and thus $k \log k$, $k$, and $\log n$ are of lower order of magnitude comparing to $k \log n$ and $\frac{k^2}{4\sqrt{n}}$. Thus all we need to do is to ensure that

$$\frac{k^2}{4\sqrt{n}} \geq k \log n$$

$$\iff 4\sqrt{n} \log n \leq k,$$

which indicates that we should set $n = \Omega \left(\left(\frac{k}{\log k}\right)^2\right)$.

### 3. An subset $S$ of vertices in a hypergraph $H = (V, E)$ is independent if there is no $e \in E$ such that $e \subseteq S$. In other words, $S$ does not completely contain any (hyper-)edge.

Prove that every 3-uniform hypergraph with $n$ vertices and $m \geq n/3$ edges contains an independent set of size at least

$$\frac{2n^{3/2}}{3\sqrt{3m}}.$$

**Solution:** This is similar to the graph case in class. Let $S$ be a random set by choosing each vertex with probability $p$ independently. Let $X = |S|$ and $Y$ be the number of occupied hyperedges. Let $Y_e$ be the indicator variable that all vertices in a hyperedge $e$ are chosen. Thus, $Y = \sum_{e \in E} Y_e$. For a fixed $e \in E$,

$$\mathbb{E} Y_e = \Pr(Y_e = 1) = p^3.$$

Due to linearity of expectations,

$$\mathbb{E} X = np,$$
whereas
\[ E Y = \sum_{e \in E} E Y_e = mp^3. \]

Similar to the graph case, the alteration is that we remove one vertex of each hyperedge, leaving an independent set \( I \). Then \( |I| = X - Y \). Again, by linearity of expectations,
\[ E |I| = E X - E Y = np - mp^3. \]

We will set \( np \) and \( mp^3 \) to have the same order of magnitude. Thus, \( p \) should be set to \( c \cdot \sqrt{n/m} \) for some constant \( c \). Plugging it back in, we have
\[ E |I| = (c - c^3) \cdot \frac{n^{3/2}}{\sqrt{m}}. \]
Optimizing \( c - c^3 \), we get that \( c = 1/\sqrt{3} \) and
\[ E |I| = \frac{2}{3\sqrt{3}} \cdot \frac{n^{3/2}}{\sqrt{m}}. \]

4. Let \( G = (V, E) \) be a graph. Associate each \( v \in V \) a list \( S(v) \) of colours of size at least \( 10d \) for some \( d \geq 1 \). Moreover, suppose that for each \( v \in C \) and \( c \in S(v) \), there are at most \( d \) neighbours \( u \) of \( v \) such that \( c \in S(u) \).

Prove that there is a proper colouring of \( G \) assigning to each vertex \( v \) a colour from its list \( S(v) \).

**Solution:** Consider a random colouring by assigning each \( v \) uniformly and independently a colour \( c \in S(v) \). For each edge \( (u, v) \in E \), let \( A_{uv}^c \) be the “bad” event that \( u \) and \( v \) have the same colour \( c \).

What events are correlated with \( A_{uv}^c \)? There are three possibilities (1) \( A_{uv}^{c'} \) where \( v' \neq v \) but \( c' \) is arbitrary, (2) \( A_{v'u}^{c'} \) where \( u' \neq u \) but \( c' \) is arbitrary, or (3) \( A_{uv}^{c'} \) where \( c' \neq c \). The number of choices for the three cases are:

(1) at most \( 10d \) choices for \( c' \) and \( d \) choices for \( v' \) by assumption, implying at most \( 10d^2 \) choices in total;

(2) same as case (1), at most \( 10d^2 \) choices;

(3) at most \( (10d - 1) \) choices for \( c' \).

Thus, the total number of dependent events of any \( A_{uv}^c \) is at most \( \Delta = 20d^2 + 10d \leq 30d^2 \).

The probability of \( A_{uv}^c \) is \( p = \Pr(A_{uv}^c) = \left( \frac{1}{10d} \right)^2 = \frac{1}{100d^2} \). We want to apply the symmetric version of the Lovász Local Lemma. It only needs to verify that \( ep\Delta = e \cdot 30d^2 \cdot \frac{1}{100d^2} < 1 \). Hence, there is a colouring such that none of \( A_{uv}^c \) holds, namely a proper colouring.
5. Let $G = (V, E)$ be a cycle of length $11n$, and $V = V_1 \cup V_2 \cup \cdots \cup V_n$ be an arbitrary partition of its $11n$ vertices; that is, $V_i \cap V_j = \emptyset$ for any $1 \leq i \neq j \leq n$. Moreover, $|V_i| = 11$ for every $i \in [n]$.

Prove that there exists an independent set of $G$ that contains precisely one vertex from each $V_i$.

**Solution:** Uniformly at random choose one vertex from each $V_i$. For each $(u, v) \in E$ and $u, v$ do not belong to the same $V_i$, let $A_{uv}$ be the event that both $u$ and $v$ are chosen. Clearly $\Pr(A_{uv}) = \frac{1}{11^2} = \frac{1}{121}$.

We want to apply the symmetric version of the Lovász Local Lemma. How many events are correlated with $A_{uv}$? Since $G$ is a cycle, there are at most two events having the form $A_{uv}$ and $A_{vv'}$. Another possibility is $A_{xy}$ where $x$ or $y$ is in the same partition as $u$ or $v$. There are at most 10 choices for the vertex and there are two events associated with the vertex, implying at most 20 possibilities for each $u$ and $v$, and 40 possibilities in total. Hence, the maximum degree $\Delta$ in the dependency graph is at most $2 + 40 = 42$. We may verify that

$$ep(\Delta + 1) = \frac{43e}{121} < 1.$$  

The condition of symmetric LLL is met, and there must exist an independent set that contains precisely one vertex from each $V_i$. 