#BIS-Hardness for 2-Spin Systems on Bipartite Bounded Degree Graphs in the Tree Nonuniqueness Region

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Counting Independent Sets

Independent Sets
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Independent Sets

![Graph with independent sets]
Counting independent sets:

$$\#\text{IS} = \sum_{\sigma: V \rightarrow \{0,1\}} w(\sigma)$$

where $w(\sigma) = 1$ if $\sigma$ induces an independent set and $w(\sigma) = 0$ otherwise.
Counting Independent Sets

Independent Sets

Hardcore gas model:

\[ Z_G(\lambda) = \sum_{\sigma:V \rightarrow \{0,1\}} w(\sigma) \]

where \( w(\sigma) = \lambda^{\lvert \sigma \rvert} \) if \( \sigma \) induces an independent set and \( w(\sigma) = 0 \) otherwise.
The Ising Model

Edge interaction $\begin{bmatrix} \beta & 1 \\ 1 & \beta \end{bmatrix}$:
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\[ Z_G(\beta) = \sum_{\sigma: V \rightarrow \{0,1\}} w(\sigma) \]

where \( w(\sigma) = \beta^{m(\sigma)} \), \( m(\sigma) \) is the number of monochromatic edges under \( \sigma \).
2-Spin Models

Parametrization: edge function \([\begin{array}{c} \beta \\ 1 \\ \gamma \end{array}\] and vertex weight \([\frac{1}{\lambda}].\)
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- Ising model without external fields: \( [\beta \ 1] \) and \( [1] \).
2-Spin Models

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- Hardcore gas: $[\frac{1}{1} \frac{1}{0}]$ and $[\frac{1}{1}]$.
- Ising model without external fields: $\begin{bmatrix} \beta & 1 \\ 1 & \beta \end{bmatrix}$ and $[\frac{1}{1}]$.

Partition function:

$$Z_G(\beta, \gamma, \lambda) = \sum_{\sigma: V \rightarrow \{0, 1\}} w(\sigma)$$

where $w(\sigma) = \beta^{m_0(\sigma)} \gamma^{m_1(\sigma)} \lambda^{n_1(\sigma)}$,

$m_i(\sigma)$ is the number of $(i, i)$ edges under $\sigma$,

$n_1(\sigma)$ is the number of 1 vertices under $\sigma$. 
Gibbs Measure on Infinite Trees

Let $\mathbb{T}_\Delta$ be the infinite $\Delta$-regular tree.

- A Gibbs measure on $\mathbb{T}_\Delta$ is a measure such that for any finite subtree $T \subset \mathbb{T}_\Delta$, the induced distribution on $T$ conditioned on the outer boundary is the same as that given by $\Pr(\sigma) = \frac{w(\sigma)}{Z}$.
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- (Semi-)translation-invariant: invariant under all (parity-preserving) automorphisms of $T_\Delta$. 

Phase transition: the uniqueness of (semi-)translation invariant Gibbs measures may change as parameters change.

For anti-ferro ($\langle 1 \rangle$) systems, translation invariant Gibbs measure is always unique, whereas semi-translation invariant ones may not be.
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- **(Semi-)translation-invariant**: invariant under all (parity-preserving) automorphisms of $T_\Delta$.

- **Phase transition**: the uniqueness of (semi-)translation invariant Gibbs measures may change as parameters change.

- For anti-ferro ($\beta \gamma < 1$) systems, translation invariant Gibbs measure is always unique, whereas semi-translation invariant ones may not be.
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- On the algorithmic side, there exists an FPTAS for the partition function of the parameter set $(\beta, \gamma, \lambda, \Delta)$ satisfying the uniqueness condition
  
  [Weitz 06], [Sinclair, Srivastava, Thurley 12], [Li, Lu, Yin 12, 13].
Computational Transition of Anti-Ferro Systems

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- On the hardness side, it is NP-hard to approximate the partition function beyond the uniqueness threshold [Sly 10], [Galanis, Štefankovič, Vigoda 12], [Sly, Sun 12].
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- \#BIS: Counting Bipartite Independent Set.
  
  Conjectured to have intermediate complexity in approximation.

- Neither algorithm nor hardness reduction is known for \#BIS.
Main Results

**Theorem**

For all tuples of parameters $(\beta, \gamma, \lambda, \Delta)$ with $\Delta \geq 3$ and $\beta \gamma < 1$, if $T_\Delta$ is in the non-uniqueness region, then approximating $Z_G(\beta, \gamma, \lambda)$ on bipartite graphs with maximum degree $\Delta$ is #BIS-equivalent, except for the case $(\beta = \gamma, \lambda = 1)$, which has an FPRAS.

**Corollary**

Approximately counting independent sets in bipartite graphs with maximum degree 6 is as hard as without the degree constraint.
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Approximately counting independent sets in bipartite graphs with maximum degree 6 is as hard as without the degree constraint.
Nearly Independent Phase Correlated Spins

Sly’s gadget [Sly 10]:

A bipartite graph $G$ with a subset of terminal vertices $T = T^+ \cup T^-$ satisfying the following properties:

Two phases: a typical configuration will choose more vertices from left than right or vice versa. They should happen with non-trivial probabilities. Conditional on the phase, terminals in $T$ are drawn nearly independently. Moreover, in the $+$ phase, vertices in $T$ are drawn with probability $p$. In the $-$ phase, it is reversed. We call this nearly independent phase correlated spins.
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Nearly Independent Phase Correlated Spins - Definition

Given \( t \) and \( \epsilon \), let \( G \) be drawn from \( \mathcal{G}(t, n(t, \epsilon), \Delta) \). The following should hold with probability at least \( 3/4 \):

1. The phases are roughly balanced, i.e.,

\[
\Pr_{G; \beta, \gamma, \lambda}(Y(\sigma) = +) \geq \frac{1}{f(t, \epsilon)} \quad \text{and} \quad \Pr_{G; \beta, \gamma, \lambda}(Y(\sigma) = -) \geq \frac{1}{f(t, \epsilon)} .
\]

2. For a configuration \( \sigma : V \rightarrow \{0, 1\} \) and any \( \tau : T \rightarrow \{0, 1\} \),

\[
\left| \frac{\Pr_{G; \beta, \gamma, \lambda}(\sigma|_T = \tau \mid Y(\sigma) = +)}{Q^+(\tau)} - 1 \right| \leq \epsilon \quad \text{and} \quad \left| \frac{\Pr_{G; \beta, \gamma, \lambda}(\sigma|_T = \tau \mid Y(\sigma) = -)}{Q^-(\tau)} - 1 \right| \leq \epsilon,
\]

where \( Q^+ \) is the joint distribution where each vertex in \( T^\pm \) is drawn independently with probability \( p^\pm \), and swapping \( p^+ \) and \( p^- \) gives \( Q^- \).
Symmetry Breaking

**Definition**

A tuple of parameters \((\beta, \gamma, \lambda, \Delta)\) supports symmetry-breaking if there is a bipartite graph \(H\) whose vertices have degree at most \(\Delta\) with a distinguished degree-1 vertex \(v_H\) such that \(\Pr_{H; \beta, \gamma, \lambda}(\sigma_{v_H} = 1) \not\in \{0, \lambda/(1 + \lambda), 1\}\).
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Intuitively, this means the system is not “perfectly symmetric”.

"Subgraph" chain: an FPRAS for the partition function of ferromagnetic Ising models [Jerrum, Sinclair 93].

On bipartite graphs, anti-ferro Ising models without external fields \((\gamma = < 1, \lambda = 1)\) can be reduced to ferromagnetic systems, by flipping one side’s assignments. We showed that all other cases support symmetry breaking.
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A Sufficient Condition of #BIS-hardness

General Graphs
If a parameter set \((\beta, \gamma, \lambda, \Delta)\) supports nearly independent phase correlated spins, then Sly showed a reduction from Max-Cut to approximating \(Z_G(\beta, \gamma, \lambda)\) [Sly 10].
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**Bipartite Graphs - Our Result**

If a parameter set \((\beta, \gamma, \lambda, \Delta)\) supports both nearly independent phase correlated spins and symmetry breaking, then approximating \(Z_G(\beta, \gamma, \lambda)\) is \#BIS-equivalent.
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## Bipartite Graphs - Our Result

If a parameter set \((\beta, \gamma, \lambda, \Delta)\) supports both nearly independent phase correlated spins and symmetry breaking, then approximating \(Z_G(\beta, \gamma, \lambda)\) is **#BIS-equivalent**.

- Non-uniqueness \(\Rightarrow\) nearly independent phase correlated spins.
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Bipartite Graphs - Our Result

If a parameter set \((\beta, \gamma, \lambda, \Delta)\) supports both nearly independent phase correlated spins and symmetry breaking, then approximating \(Z_G(\beta, \gamma, \lambda)\) is \#BIS-equivalent.

- Non-uniqueness \(\Rightarrow\) nearly independent phase correlated spins.
- All parameters except \((\beta = \gamma, \lambda = 1)\) \(\Rightarrow\) symmetry breaking.
The first step is from $\#\text{BIS}$ to an Ising model with the edge interaction $\beta$ and non-uniform external field $\lambda$ for any $0 < \beta < 1$ and $\lambda \neq 1$. 
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The first step is from $\#\text{BIS}$ to an Ising model with the edge interaction $\beta$ and non-uniform external field $\lambda$ for any $0 < \beta < 1$ and $\lambda \neq 1$.

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- Replace every edge $(u, v)$ by the gadget on the right, where the two blue nodes are pinned to 1.
- The effective weights are $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ as $\lambda$ goes to 0.
Reductions - the First Step

- The first step is from \#BIS to an Ising model with the edge interaction $\beta$ and non-uniform external field $\lambda$ for any $0 < \beta < 1$ and $\lambda \neq 1$.
  - With the external field, we can effectively pin variables to 0 or 1.
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- With the external field, we can effectively pin variables to 0 or 1.
- Replace every edge $(u, v)$ by the gadget on the right, where the two blue nodes are pinned to 1.
- The effective weights are $\begin{bmatrix} \beta & \beta \\ \beta & \beta \end{bmatrix}$. 

$\frac{3}{2} \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ as $\lambda$ goes to 0.
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- With the external field, we can effectively pin variables to 0 or 1.

- Replace every edge $(u, v)$ by the gadget on the right, where the two blue nodes are pinned to 1.

- The effective weights are $\begin{bmatrix} \beta & \beta \\ \beta & \beta^3 \end{bmatrix}$. 

\begin{center}
\begin{tikzpicture}
  \node (u) at (0,0) [circle,fill=blue] {$u$};
  \node (v) at (1,0) [circle,fill=blue] {$v$};
  \node (1) at (2,1) [circle,fill=blue] {$1$};
  \node (1') at (2,-1) [circle,fill=blue] {$1$};
  \draw (u) -- node [above] {$\beta$} (v);
  \draw (v) -- node [above] {$\beta$} (1);
  \draw (u) -- node [above] {$\beta$} (1');
  \end{tikzpicture}
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The first step is from #BIS to an Ising model with the edge interaction $\beta$ and non-uniform external field $\lambda$ for any $0 < \beta < 1$ and $\lambda \neq 1$.

- With the external field, we can effectively pin variables to 0 or 1.
- Replace every edge $(u, v)$ by the gadget on the right, where the two blue nodes are pinned to 1.
- The effective weights are $\begin{bmatrix} \beta & \beta^3 \\ \beta & \beta^3 \end{bmatrix} = \beta \begin{bmatrix} 1 & 1 \\ 1 & \beta^2 \end{bmatrix}$. 
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- $\begin{bmatrix} 1 & 1 \\ 1 & \beta^2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ as $\beta$ goes to 0.
Reductions - the Second Step

The second step is to use nearly independent phase correlated spins and symmetry breaking to simulate this Ising model.
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- Replace each vertex by a n-i p-c spins gadget.

Replace each edge by connecting terminals: $+$ to $+$ and $-$ to $-$. 
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- Effective edge interaction is of the Ising type.
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- Replace each vertex by a n-i p-c spins gadget.
  Replace each edge by connecting terminals: + to + and − to −.
- Effective edge interaction is of the Ising type.

However, like Sly's gadget, we only require the two phases of the gadget to be polynomially balanced. This induces an unpleasant polynomially large external field on each vertex.
Balance the Gadget

- Sly's MAX-CUT reduction works when the two phases occur with probabilities that are bounded below by an inverse polynomial.
Balance the Gadget

- Sly's Max-Cut reduction works when the two phases occur with probabilities that are bounded below by an inverse polynomial.
- We want to control the external field on each vertex. Hence we need the two phases to occur with roughly equal probability, i.e. about $1/2$. 
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  - Construct a new gadget by gluing two gadgets together, and connect many terminals between them. With high probability the two gadgets will have different phases.
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  - Construct a new gadget by gluing two gadgets together, and connect many terminals between them. With high probability the two gadgets will have different phases.
  - Define the phase of the whole gadget to be the phase of the first. The two phases are balanced as
    \[
    \Pr(+-) = \Pr(-+).
    \]
Summary

Theorem

For all tuples of parameters \((\beta, \gamma, \lambda, \Delta)\) with \(\beta \gamma < 1\) and \(\Delta \geq 3\), the following holds:

1. If the parameters satisfy strict-uniqueness then there is a FPTAS for the partition function for all graphs \([Li, Lu, Yin 13]\).

2. If the parameters satisfy non-uniqueness then:
   1. it is \#SAT-hard to approximate the partition function on graphs \([Sly, Sun 12]\).
   2. it is \#BIS-hard to approximate the partition function on bipartite graphs, except when \(\beta = \gamma\) and \(\lambda = 1\), which admits an FPRAS.
Open Problems

- Anti-ferromagnetic ($\beta \gamma < 1$) 2-spin system.
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  - On the other hand it is known to be #BIS-easy for any parameters even in general graphs [Goldberg, Jerrum 07].
  - Recent progress on #BIS-hardness has been made based on our results [Liu, Lu, Zhang 14].
Thank You!

Papers and slides available on my homepage:

www.cs.wisc.edu/~hguo/