Random Cluster Dynamics for the Ising model is Rapidly Mixing

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Joint work with Mark Jerrum

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The model and its dynamics
The random cluster model [Fortuin, Kasteleyn 1969]

Parameters $0 \leq p \leq 1$ (edge weight), $q \geq 0$ (cluster weight).

Given graph $G = (V, E)$, the measure on subgraph $r \subseteq E$ is defined as

$$\pi_{RC}(r) \propto p^{|r|} (1 - p)^{|E\setminus r|} q^{\kappa(r)},$$

where $\kappa(r)$ is the number of connected components in $(V, r)$.

(1 - p)^4 q^4

$p^2 (1 - p)^2 q^2$

$p^4 q$
The random cluster model [Fortuin, Kasteleyn 1969]

The partition function (normalizing factor):

\[ Z_{RC}(p, q) = \sum_{r \subseteq E} p^{|r|} (1 - p)^{|E \setminus r|} q^{\kappa(r)}. \]

Equivalent to the Tutte polynomial \( Z_{Tutte}(x, y) \):

\[ q = (x - 1)(y - 1) \quad \quad p = 1 - \frac{1}{y} \]
The random cluster model [Fortuin, Kasteleyn 1969]

\[ \pi_{RC}(r) \propto p^{r_1}(1 - p)^{|E \setminus r_1|} q^{K_r} \]

The motivation is to unify:

- Ising model
- Potts model
- Bond percolation
- Electrical network
The random cluster model [Fortuin, Kasteleyn 1969]

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- Electrical network \( q \to 0 \) (Spanning trees if \( p \to 0 \) and \( \frac{q}{p} \to 0 \))
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Glauber dynamics (single edge update) $P_{RC}$ (Metropolis):

Current state $x \subseteq E$

1. With prob. $1/2$ do nothing. (Lazy)

2. Otherwise, choose an edge $e$ u.a.r.

3. Move to $y = x \oplus \{e\}$ with prob. $\min\left\{ 1, \frac{\pi_{RC}(y)}{\pi_{RC}(x)} \right\}$.

Detailed balance:

$$\pi(x)P(x, y) = \pi(y)P(y, x) = \min\{\pi(x), \pi(y)\}$$
Glauber dynamics

Glauber dynamics (single edge update) $P_{RC}$ (Metropolis):

$$P_{RC}(x, y) = \begin{cases} 
\frac{1}{2m} \min \left\{ 1, \frac{\pi_{RC}(y)}{\pi_{RC}(x)} \right\} & \text{if } |x \oplus y| = 1; \\
1 - \frac{1}{2m} \sum_{e \in E} \min \left\{ 1, \frac{\pi_{RC}(x \oplus \{e\})}{\pi_{RC}(x)} \right\} & \text{if } x = y; \\
0 & \text{otherwise.}
\end{cases}$$

We are interested in the mixing time $\tau_\epsilon(P_{RC})$:

$$\tau_\epsilon(P_{RC}) = \min \left\{ t : \|P_{RC}^t(x_0, \cdot) - \pi\|_{TV} \leq \epsilon \right\}.$$
A simple example

Let $p < 1/2$.

\[
\min \left\{ 1, \frac{\pi_{RC}(X \cup \{e\})}{\pi_{RC}(X)} \right\}
\]

\[
= \begin{cases} 
  \frac{p}{1-p} & \text{if } e \text{ is not a cut edge} \\
  \frac{p}{q(1-p)} & \text{if } e \text{ is a cut edge}
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Brief History

Studied extensively for special graphs, such as the complete graph (mean-field) and the lattice $\mathbb{Z}^2$.

- **Mean-field:** [Gore, Jerrum 1999]
  
  [Blanca, Sinclair 2015]

- **$\mathbb{Z}^2$:** [Borgs et al. 1999]
  
  [Blanca, Sinclair 2016]
  
  [Gheissari, Lubetzky 2016]

$q > 2$: Slow mixing for the complete graph.

$0 \leq q \leq 2$: No known fast mixing bound for general graphs.
Main theorem

Theorem

For the random cluster model with parameters $0 < p < 1$ and $q = 2$,

$$\tau_{\epsilon}(P_{RC}) \leq 10n^4m^2(\ln \pi_{RC}(x_0)^{-1} + \ln \epsilon^{-1}).$$

For $q > 2$, there exists $p$ such that $P_{RC}$ is slow mixing on complete graphs. [Gore, Jerrum 1999] [Blanca, Sinclair 2015]

For $q > 2$ and $0 < p < 1$, it is $\#BIS$-hard to approximate $Z_{RC}(p, q)$. [Goldberg, Jerrum 2012]

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Swendsen-Wang algorithm
Ferromagnetic Ising model [Ising, Lenz 1925]

Parameter $\beta > 1$.

A configuration $\sigma : V \to \{+, -\}$.

$$\pi_{Ising}(\sigma) \propto \beta^{mono(\sigma)} = \beta^{m-cut(\sigma)}$$

Partition function $Z_{Ising}(\beta) = \sum_{\sigma} \beta^{mono(\sigma)}$
Equivalence at $q = 2$

Let $\beta = \frac{1}{1-p}$.

$$Z_{Ising}(\beta) = |E| \beta Z_{RC}(p, 2)$$

A global Markov chain to sample Ising configurations.

Current configuration $\sigma$

1. Mark all monochromatic edges under $\sigma$ as $M$
2. Remove each edge in $M$ with probability $\beta^{-1}$ (Recall $\beta^{-1} = 1 - p$)
3. Assign a random spin to each component of $(V, M)$

Practically very fast for the Ising model, but difficult to analyze.

Conjectured to be rapidly mixing for all graphs.

(Open problem since 90s.)
Another simple example

1. Activate mono edges
2. Re-randomize mono edges
3. Color components
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- Swendsen-Wang algorithm on the complete graph:
  
  [Gore, Jerrum 1999]
  
  [Cooper, Dyer, Frieze, Rue 2000]
  
  [Long, Nachimus, Ning, Peres 2011]
  
  [Borgs, Chayes, Tetali 2011]
  
  [Galanis, Štefankovič, Vigoda 2015]

Theorem (Ullrich 2014)

\[ \tau_\epsilon(P_{SW}) \leq \tau_\epsilon(P_{RC}) \]
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$$\tau_\varepsilon(P_{SW}) \leq \tau_\varepsilon(P_{RC})$$

Combine with our theorem:

the Swendsen-Wang algorithm is rapidly mixing at $q = 2$,
namely, for the ferromagnetic Ising model at any temperature.

- The Swendsen-Wang algorithm is conjectured to have a $n^{1/4}$ mixing time
  (by Peres and Sokal).
Consequence — Swendsen-Wang algorithm is rapidly mixing

**Theorem (Ullrich 2014)**

\[ \tau_\epsilon(P_{SW}) \leq \tau_\epsilon(P_{RC}) \]

Combine with our theorem:

the Swendsen-Wang algorithm is **rapidly mixing** at \( q = 2 \), namely, for the ferromagnetic Ising model at any temperature.

- The Swendsen-Wang algorithm is conjectured to have a \( n^{1/4} \) mixing time (by Peres and Sokal).
Even subgraphs
Another equivalent formulations at $q = 2$

**Even subgraphs**

Let $r \subseteq E$ such that every vertex in $(V, r)$ has an even degree.

$$\pi_{\text{even}}(r) \propto p^{|r|} (1 - p)^{|E \setminus r|}$$

**Partition function $Z_{\text{even}}(p)$**

- $(1 - p)^4$: \( \pi_{\text{even}}(r) \propto p^4 \) (even)
- NOT EVEN
- \( p^4 \)
Equivalence at \( q = 2 \)

Let \( \beta = \frac{1}{1-p} \).

\[
Z_{\text{Ising}}(\beta) = \beta^{\lvert E \rvert} Z_{RC}(p, 2) = 2^{\lvert V \rvert} \beta^{\lvert E \rvert} Z_{\text{even}} \left( \frac{p}{2} \right)
\]
Equivalence at $q = 2$

Random-cluster
$(p, 2)$

Ising model
$\beta = (1 - p)^{-1}$

Even subgraphs
$p/2$

[Fortuin, Kasteleyn 1969]

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Slow mixing
FPRAS
[Jerrum, Sinclair 93]
Equivalence at $q = 2$

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Random-cluster $(p, 2)$

Even subgraphs $p/2$

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[Heng Guo (QMUL) Random Cluster 2016/11/03 21 / 41]
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Even subgraphs $p/2$

Slow mixing

This talk

FPRAS [Jerrum, Sinclair 93]

[Grimmer, Janson 2009]

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Grimmett-Janson coupling

Given a graph $G$, draw a random even subgraph $S \subseteq E$ with $p \leq \frac{1}{2}$:

$$\Pr(S = s) = \pi_{even}(s).$$

Then we add every edge $e \notin S$ with probability $p' = \frac{p}{1-p}$.

Call this subgraph $R$.

**Theorem (Grimmett, Janson 2009)**

$$\Pr(R = r) = \pi_{RC;2p,2}(r).$$
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The Proof
A Markov chain is a random walk on its state space (exponentially large).

There are \(2^j\) configurations. Two configurations are adjacent if they differ by exactly one edge.

Rapidly mixing, the state space is very well connected.
A Markov chain is a random walk on its state space (exponentially large).

- There are \(2^{|E|}\) many configurations.
- Two configurations are adjacent if they differ by exactly one edge.
Bound the mixing time

- A Markov chain is a random walk on its state space (exponentially large).

- There are $2^{|E|}$ many configurations.

- Two configurations are adjacent if they differ by exactly one edge.

- Rapidly mixing $\iff$ The state space is very well connected.
Construct a set $\Gamma$ of canonical paths $\gamma_{xy}$ for all pairs of states $(x, y)$.

The key quantity of $\Gamma$ is its congestion:

$$\rho(\Gamma) := \max_{(z, z') \in \Omega^2, P(z, z') > 0} \frac{L}{\pi(z) P(z, z')} \sum_{x, y \in \Omega^2, \gamma_{xy} \ni (z, z')} w(\gamma_{xy}),$$

where

$$w(\gamma_{xy}) = \pi(x) \pi(y).$$

**Theorem (Sinclair 1992)**

$$\tau_\varepsilon(P) \leq \rho(\Gamma) (\ln \pi(x_0)^{-1} + \ln \varepsilon^{-1}).$$
Fix $\Gamma = \{\gamma_{xy}\}$ and an integer $k \leq L$.

1. Draw the initial and final states $I$ and $F$ independently according to $\pi(\cdot)$.

2. A random path $\gamma_{IF} \in \Gamma$.

$$\mu(\gamma_{IF}) = w(\gamma_{IF}) = \pi(I)\pi(F)$$

3. Let $Z_k$ be the $k$th state of $\gamma_{IF}$.

   (Assume all paths in $\Gamma$ have the same length $L$.)

The congestion $\rho(\Gamma)$ is polynomial related with $\max_k \frac{\Pr(Z_k=z)}{\pi(z)}$. 
Let $q = 1$. Then $\pi_{RC}(\cdot)$ is a product measure.
Alternative view in action

Let $q = 1$. Then $\pi_{RC}(\cdot)$ is a product measure.

\begin{equation}
G:
\end{equation}

\begin{align*}
I & \quad \rightarrow \quad F \\
\end{align*}
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\[
\begin{align*}
\Pr(Z_k = z) &= 1
\end{align*}
\]
Instead of one path from $x$ to $y$, we can have a random path from $x$ to $y$.

Flow $\Gamma$ is a collection of paths equipped with weights $w(\cdot)$ such that

$$
\sum_{\gamma \text{ is from } x \text{ to } y} w(\gamma) = \pi(x)\pi(y).
$$

$Z_k$ is defined similarly.

1. Random initial and final states $I$ and $F$
2. A random path $\gamma$ from $I$ to $F$ according to $w(\cdot)$.
3. $Z_k$ is the $k$th state of $\gamma$.

We will look at $\frac{\Pr(Z_k = z)}{\pi(z)}$. 
Lifting canonical paths

In an ideal world . . .

- Suppose we have canonical paths $\Gamma_{\text{even}}$ for even subgraphs with low congestion. (similar to [Jerrum, Sinclair 93])

- Then use Grimmert-Janson to lift $\Gamma_{\text{even}}$ to a flow for random cluster.

\[ l = W_0 \rightarrow W_1 \rightarrow W_2 \rightarrow \cdots \rightarrow W_{L-1} \rightarrow W_L = F \]

- \( w(\zeta) = w(\gamma) \Pr(\gamma \rightarrow \zeta) \)
Ideal lifting

If $W_k$ deviates from $\pi_{\text{even}}(\cdot)$ by at most polynomial, then so does $Z_k$ from $\pi_{\text{RC}}(\cdot)$.

$$
\frac{\Pr(W_k = w)}{\pi_{\text{even}}(W)} \leq n^{O(1)} \rho(\Gamma)
$$

$$
\Pr(Z_k = z) = \sum_{w \subseteq z, w \text{ even}} \Pr(W_k = w) \left( \frac{p}{1 - p} \right)^{|z \setminus w|} \left( \frac{1 - 2p}{1 - p} \right)^{|E \setminus z|}
$$

$$
\leq n^{O(1)} \rho(\Gamma) \sum_{w \subseteq z, w \text{ even}} \pi_{\text{even}}(w) \left( \frac{p}{1 - p} \right)^{|z \setminus w|} \left( \frac{1 - 2p}{1 - p} \right)^{|E \setminus z|}
$$

$$
= n^{O(1)} \rho(\Gamma) \pi_{\text{RC}}(Z)
$$

(by GJ)
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(by GJ)
In the real world . . .

Two issues:

1. We do not have good canonical paths for even subgraphs —
   **Jerrum-Sinclair** chain moves among all subgraphs!

2. **Grimmett-Janson** adds independent edges —
   
   $Z_i$ and $Z_{i+1}$ are not adjacent states!
   
   They may differ by a lot of edges.
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2. Grimmett-Janson adds independent edges — $Z_i$ and $Z_{i+1}$ are not adjacent states!
   They may differ by a lot of edges.
Issue 1: need canonical paths for even subgraphs.

Construct paths $\Gamma_{\text{even}} = \{\gamma_{xy}\}$ where $x$ and $y$ are both even subgraphs.

- $x \oplus y$ is also even.
  
  $x \oplus y$ can be covered by edge-disjoint cycles.

- Pick a canonical ordering of edges. Unwind each cycle:
  
  $W_0 = x, \ W_i = W_{i-1} \oplus e_i$

- Enlarge the state space to all even and near-even subgraphs.
  
  Every path is in the augmented space.

- $\Gamma_{\text{even}}$ has low congestion — same reason as [Jerrum, Sinclair 1993].
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Patch 1

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  - Pick a \textit{canonical} ordering of edges. \textbf{Unwind} each cycle:
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\[ Z_1 \]
\[ Z_2 \]
\[ Z_3 \]
\[ Z_4 \]
\[ Z_5 \]
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$\Gamma_{\text{even}}$ has low congestion — combinatorial encoding [Jerrum, Sinclair 1993].

For any $y_{xy} \ni (z, z')$, let $u = x \oplus y \oplus z$. This mapping is injective.
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One final problem for issue 1:

- $W_0$ and $W_L$ are both even,
  but intermediate $W_i$’s can be near-even.

A generalization of Grimmett-Janson:

- Give each near-even subgraph a penalty of $1/n^2$.
- Add independent edges with prob. $\frac{p}{1-p}$ as before.
  Call the resulting measure $\widehat{\pi}(\cdot)$.

$$\frac{\widehat{\pi}(x)}{\pi_{RC}(x)} = \Theta(1).$$
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RC (x)
Issue 2: $Z_i$ and $Z_{i+1}$ differ by more than 1 edge.

- An easy fix: insert intermediate states to change edges one by one in $Z_i \oplus Z_{i+1}$, which has a product measure on $E \setminus (W_i \cup W_{i+1})$.

- The distribution of $Z_i^j$ is the same as that of $Z_i$ ($j < m$).

- Total length is $mL$. 
Better patch 2

Issue 2: $Z_i$ and $Z_{i+1}$ differ by more than 1 edge.

- Lift $W_{i+1}$ to $Z_{i+1}$ conditional on $Z_i$ such that $Z_{i+1}$ and $Z_i$ are adjacent and the marginal of $Z_{i+1}$ is correct.

- The marginal distributions of $Z_0$ and $Z_L$ are correct, but their joint distribution is not — $Z_0$ and $Z_L$ are correlated.

- Append a tail on the path after $Z_L$ to re-randomize edges that are not in $W_L$. This removes the correlation.

- Total length is at most $L + m$. 
Putting everything together

\[ Z_0 \quad \overset{\text{Re-randomization}}{\longrightarrow} \quad Z_L + m \]
Putting everything together

$$\begin{align*}
  Z_0 &= \text{Grimmett-Janson} \\
  W_0 &= \\
  Z_1 &= \text{prob. } p' \\
  Z_2 &= \text{prob. } 1 - p' \\
  Z_{L+m} &= \text{Grimmett-Janson} \\
  W_L &= 
\end{align*}$$
Putting everything together

\[ Z_0 \]

\[ W_0 \rightarrow W_1 \rightarrow W_2 \rightarrow \cdots \rightarrow W_L \]

\[ Z_{L+m} \]

Grimmett-Janson

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Putting everything together

\[ W_1 = W_0 \cup \{e\} \quad \Rightarrow \quad Z_1 = Z_0 \cup \{e\} \]
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\[ W_2 = W_1 \setminus \{e'\} \quad \Rightarrow \quad Z_2 = \begin{cases} Z_1 & \text{prob. } p' \\ Z_1 \setminus \{e'\} & \text{prob. } 1 - p' \end{cases} \]
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Z_1 & \text{prob. } p' \\
Z_1 \setminus \{e'\} & \text{prob. } 1 - p'
\end{cases} \]
Future directions
Tutte polynomial \([Goldberg, Jerrum 08,12,14]\)

\[ q = (x - 1)(y - 1) \]

- **Tractable**
- **FPRAS**
- **NP-hard**
  - (most \#P-hard)
- **\#PM-equivalent**
- **\#BIS-hard**

**Open:**
- All white
- \(0 \leq q < 1\)
- \(1 < q < 2\)
Recap

**Theorem**

At $q = 2$, $\tau_\epsilon(P_{RC}) \leq 10n^4m^2(\ln \pi_{RC}(x_0)^{-1} + \ln \epsilon^{-1})$.

- $q = 2$ tighter mixing time bound?
- $1 < q < 2$ (monotone) *fast mixing*?
- $0 \leq q < 1$ (e.g. Tutte(2,1) = #Forests) *fast mixing***?
Recap

Theorem

At $q = 2$, $\tau_\epsilon(P_{RC}) \leq 10n^4m^2(\ln \pi_{RC}(x_0)^{-1} + \ln \epsilon^{-1})$.

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Thank You!

Paper available: arxiv.org/abs/1605.00139