Complexity of the Cover Polynomial

Markus Bläser and Holger Dell

Computational Complexity Group
Saarland University, Germany
{mblaeser,hdell}@cs.uni-sb.de

Abstract. The cover polynomial introduced by Chung and Graham is a two-variate graph polynomial for directed graphs. It counts the (weighted) number of ways to cover a graph with disjoint directed cycles and paths, it is an interpolation between determinant and permanent, and it is believed to be a directed analogue of the Tutte polynomial. Jaeger, Vertigan, and Welsh showed that the Tutte polynomial is #P-hard to evaluate at all but a few special points and curves. It turns out that the same holds for the cover polynomial: We prove that, in almost the whole plane, the problem of evaluating the cover polynomial is #P-hard under polynomial-time Turing reductions, while only three points are easy. Our construction uses a gadget which is easier to analyze and more general than the XOR-gadget used by Valiant in his proof that the permanent is #P-complete.

1 Introduction

Graph polynomials map directed or undirected graphs to polynomials in one or more variables, such that this mapping is invariant under graph isomorphisms. Probably the most famous graph polynomials are the chromatic polynomial or its generalization, the Tutte polynomial. The chromatic polynomial is the polynomial in the variable λ that counts the number of valid λ-colourings of a given undirected graph. The Tutte polynomial T in two variables x and y has interpretations from different fields of combinatorics. For example, $T(G, 1, 1)$ is the number of spanning trees, $T(G, 1, 2)$ is the number of spanning subgraphs of an undirected graph G, and also the number of nowhere-zero flows or the Jones polynomial of an alternating link are contained in the Tutte polynomial.

While the Tutte polynomial has been established for *undirected* graphs, the cover polynomial by Chung and Graham [1] is an analogue for the directed case. Both graph polynomials satisfy similar identities such as a contraction-deletion identity and product rule, but the exact relation between the Tutte and the cover polynomial is not yet known. The cover polynomial has connections to rook polynomials and drop polynomials, but from a complexity theoretic point of view, we tend to see it as a generalization of the permanent and the determinant of a graph. The cover polynomial of a graph is the weighted sum of all of its spanning subgraphs that consist of disjoint, directed, and simple cycles and paths. As it is the case for most graph polynomials, the cover polynomial is of interest because it combines a variety of combinatorial problems into one generalized theoretical framework.

Previous Results

Jaeger, Vertigan, and Welsh [2] have shown that, except along one hyperbola and at nine special points, computing the Tutte polynomial is #P-hard. In recent years, the complexity and approximability of the Tutte polynomial has received increasing attention: Lotz and Makowsky [3] prove that the coloured Tutte polynomial by Bollobás and Riordan [4] is complete for Valiant's algebraic complexity class VNP, Giménez and Noy [5] show that evaluating the Tutte polynomial is #P-hard even for the rather restricted class of bicircular matroids, and Goldberg and Jerrum [6] show that the Tutte polynomial is mostly inapproximable.

A different graph invariant that seems related to the Tutte polynomial is the weighted sum of graph homomorphisms to a fixed graph H, so basically, it is the number of H-colourings. Bulatov and Grohe [7] and Dyer, Goldberg, and Paterson [8] prove that the complexity of computing this sum is #P-hard for most graphs H.

Our Contribution

In this paper, we show that the problem of evaluating the cover polynomial is #P-hard at all evaluation points except for three points where this is easy.

The big picture of the proof is as follows: We use elementary identities of the cover polynomial in order to construct simple reductions along horizontal lines. Furthermore, we establish an interesting identity along the y-axis. As it turns out, there is quite a strong connection between cover polynomial and permanent, and our construction uses an equality gadget with a similar effect as the XOR-gadget which Valiant [9] uses in his proof that the permanent is #P-complete. Our gadget, however, is simpler to analyze and more general, in the sense that it satisfies additional properties about the number of cycles contributed by our gadget to each cycle cover.

In addition, we carry over the hardness result to the geometric cover polynomial introduced by D'Antona and Munarini [10].

2 Preliminaries

Let $\mathbb{N} = \{0, 1, \dots\}$. The graphs in this paper are directed *multigraphs* $D = (V, E)$ with parallel edges and loops allowed. We denote by $\mathcal{G}$ the set of all such graphs. We write n for the number of vertices, and m for the number of edges. Two graphs are called *isomorphic* if there is a bijective mapping on the vertices that transforms one graph into the other.

A *graph invariant* is a function $f : \mathcal{G} \to F$, mapping elements from $\mathcal{G}$ to some set F, such that all pairs of isomorphic graphs G and G' have the same image under f. In the case that F is a polynomial ring, f is called *graph polynomial*.

Counting Complexity Basics. The class #P consists of all functions $f : \{0,1\}^* \to \mathbb{N}$ for which there is a non-deterministic polynomial-time bounded Turing machine M which has exactly $f(x)$ accepting paths on input x. For two counting problems $f, g : \{0,1\}^* \to \mathbb{Q}$ not necessarily in #P, we say f *Turing-reduces* to g in polynomial time ($f \preccurlyeq^{\mathrm{p}}_{\mathrm{T}} g$), if there is a deterministic oracle Turing machine M which computes f in polynomial time with oracle access to g. If the oracle is used only once, we say f *many-one reduces* to g ($f \preccurlyeq^{\mathrm{p}}_{\mathrm{m}} g$), and if the oracle output is the output of the reduction, we speak of a *parsimonious* many-one reduction ($f \preccurlyeq^{\mathrm{p}} g$). The notions of #P-hardness and #P-completeness (under polynomial-time Turing reductions) are defined in the usual way.

Polynomials. Polynomials $p(x_1, \ldots, x_m)$ are elements of the polynomial ring $\mathbb{Q}[x_1, \ldots, x_m]$, and, in this context, the variables are abstract objects. Lagrange interpolation can be used to compute the following problem in polynomial time.

Input Point-value pairs $(a_1, p_1), \ldots, (a_d, p_d) \in \mathbb{Q}^2$, encoded in binary.
Output The coefficients of the polynomial $p(x)$ with $\deg(p) \le d$ and $p(a_j) = p_j$.

The Cover Polynomial. The cover polynomial basically counts a relaxed form of cycle covers, namely *path-cycle covers*. For a directed graph $D = (V, E)$ and some subset $C \subseteq E$, we denote the subgraph (V, C) again by C. A path-cycle cover of D is a set $C \subseteq E$, such that in C every vertex $v \in V$ has an indegree and an outdegree of at most 1. A path-cycle cover thus consists of disjoint simple paths and simple cycles. Note that also an independent vertex counts as a path, and an independent loop counts as a cycle.

By the graph invariant $c_D(i, j)$, we denote the number of path-cycle covers of D that have exactly i paths and j cycles. It is not hard to prove that the function $D \mapsto c_D(i, j)$ is #P-complete (cf. [11]). The cover polynomial by Chung and Graham [1] is a graph polynomial in the variables x and y, and it is defined by the equation

$$C(D, x, y) := \sum_{i=0}^{m} \sum_{j=0}^{m} c_D(i, j) x^{\underline{i}} y^j, \tag{1}$$

where $x^{\underline{i}} := x(x-1)\ldots(x-i+1)$ denotes the falling factorial.

Writing $i(C)$ and $j(C)$ for the number of paths and cycles of a path-cycle cover C, the cover polynomial is the weighted sum over all covers of D:

$$C(D, x, y) = \sum_{\substack{\text{path-cycle} \\ \text{cover } C}} x^{\underline{i(C)}} y^{j(C)}.$$

3 Overview

The problem of evaluating the cover polynomial, written $C(a, b)$, is parameterized by the coordinates (a, b). Formally, $C(a, b)$ is the function from $\mathcal{G} \to \mathbb{Q}$ with $D \mapsto C(D, a, b)$ where we assume graphs and rationals to be represented explicitly in a standard way. Our main theorem is the following.

Main Theorem. *Let* $(a, b) \in \mathbb{Q}^2$. *It holds:*
If $(a, b) \notin \{(0, 0), (0, -1), (1, -1)\}$, *then* $C(a, b)$ *is* #P*-hard.*
Otherwise, $C(a, b)$ *is computable in polynomial time.*

Proof (outline). The proof is in several steps (cf. Fig. 1). We begin by classifying the polynomial-time computable points in Section 4. Furthermore, we point out that $C(0, 1)$ is the permanent and $C(1, 0)$ is the number of Hamiltonian paths, which both are #P-complete counting problems.

In Section 5, using elementary identities of the cover polynomial and interpolation, we reduce along horizontal lines, that means we prove $C(0, b) \preccurlyeq^{\mathrm{p}}_{\mathrm{T}} C(a, b)$ for all a, b and $C(1, 0) \preccurlyeq^{\mathrm{p}}_{\mathrm{T}} C(a, 0)$ for all $a \neq 0$. This implies the hardness of $C(a, 1)$ for all a and of $C(a, 0)$ for $a \neq 0$.

To prove the remaining hardness part where $b \notin \{-1, 0, 1\}$, we reduce the permanent to $C(0, b)$. Section 6 is the core part of our proof, establishing this reduction $C(0, 1) \preccurlyeq^{\mathrm{p}}_{\mathrm{T}} C(0, b)$ along the y-axis. There we introduce and analyze the equality gadget, use it to establish a new identity for the *weighted* cover polynomial, and show how to derive a reduction for the standard cover polynomial from this. ☐

In Section 7, we briefly show how to carry over our result to the geometric version of the cover polynomial.

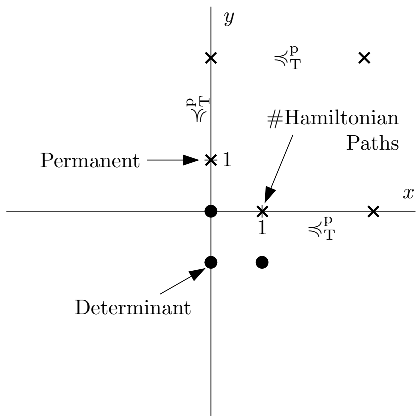

Fig. 1. The big picture: three points (*black discs*) are easy to evaluate, and the rest of the plane is #P-hard. The *crosses* indicate the points for the reductions.

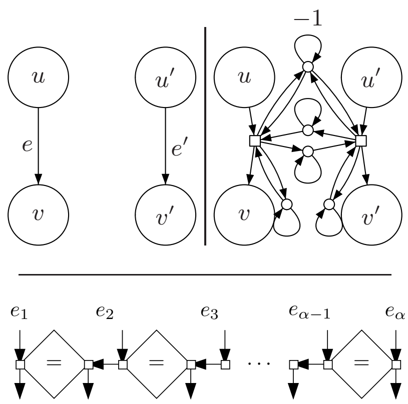

Fig. 2. Top: Shows how the equality gadget connects two edges e, e'. Bottom: Shows how the multiequality gadget connects α edges with equality gadgets from the top (*large diamonds*).

4 Special Points

A *Hamiltonian path* is a path-cycle cover with exactly one path and zero cycles, and a *cycle cover* is a path-cycle cover without paths. The *permanent* $\mathrm{Perm}(D)$ is the permanent of the adjacency matrix of D, and it counts the number of cycle covers of D. The *determinant* $\det(D)$ is the determinant $\det(A)$. Remarkably, both the determinant and the permanent can be found in the cover polynomial.

Lemma 1. *For any nonempty directed graph D, we have*

(i) $C(D, 0, 0) = 0$,
(ii) $C(D, 1, 0) = \#\textsc{HamiltonianPaths}(D)$,
(iii) $C(D, 0, 1) = \mathrm{Perm}(D)$,
(iv) $C(D, 0, -1) = (-1)^n \det(D)$,
(v) $C(D, 1, -1) = C(D, 0, -1) - C(D', 0, -1)$ *where D' is derived from D by adding a new vertex v_0 with all edges to and from the nodes of D,*

Proof (sketch). (For a detailed proof, cf. [11])

The proof of the first three claims is simple. The proof of the fourth claim uses Laplace expansion and the multilinearity of the determinant to show that the contraction-deletion identity from (1)–(3) in [1] is satisfied by the function $D \mapsto (-1)^{n(D)} \det(D)$. Since the contraction-deletion identities are defining equations of the cover polynomial, the claim follows.

For the last claim, notice that $C(D, 1, -1)$ counts all path-cycle covers with at most one path (weighted with $(-1)^{j(C)}$), while the determinant $C(D, 0, -1)$ counts only cycle covers. The idea is that $C(D, 1, -1) - C(D, 0, -1)$ is the number of covers of D with exactly one path, and can be expressed by $C(D', 0, -1)$, the number of cycle covers of D'. This is because every path-cycle cover of D with one path becomes a cycle cover in D' where the path gets closed by the v_0-edges to form a cycle. □

As a consequence, $C(0, 1)$ and $C(1, 0)$ are #P-hard [9, 12], while $C(0, 0)$, $C(0, -1)$ and $C(1, -1)$ are polynomial-time computable. The #P-hardness of $C(2, -1)$ follows from the following lemma and requires a sophisticated proof.

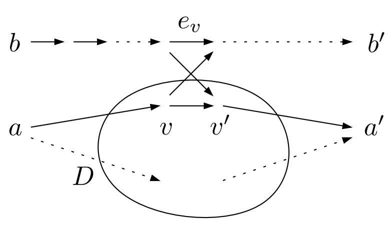

Fig. 3. Shows the graph D' constructed in the proof of Lemma 2. The two edges $e_v, (v, v')$ together with the corresponding crossing edges form the *crossing gadget*.

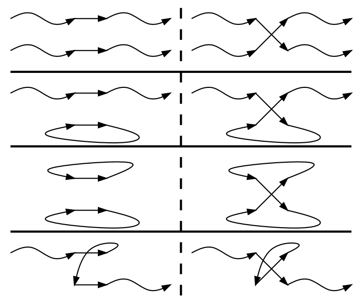

Fig. 4. Shows the change in the number of paths and cycles in a path-cycle cover if we switch one crossing gadget.

Lemma 2. *It holds* $\#\textsc{HamiltonianPaths} \preccurlyeq^{\mathrm{p}}_{\mathrm{m}} C(2, -1)$.

Proof. Let D be a directed graph with vertex set $V = \{1, \ldots, n\}$. We construct a graph D' from D as follows (cf. Fig. 3).

– uncontract all vertices v from D, into two vertices v, v' with one edge (v, v') in between, and move all outgoing edges (v, w) to (v', w),

– add fresh vertices a, a' to D, and add the edges (a, v) and (v', a') for all v,
– add an independent directed path of length n with edges $e_1, \ldots, e_n$, and
– add the *crossing edges* $(v, w), (u, v')$ for all (v, v') and $e_v = (u, w)$.

Note that uncontracting edges as above does not change the structure of the path-cycle covers. Therefore, we refer to the graph induced by the vertices v and v' again by D. Let b be the start vertex of e_1, and let b' be the end vertex of e_n. Since a and b have no incoming edges, a and b are starting points of paths, and similarly a' and b' are ending points of paths in every path-cycle cover.

For the cover polynomial of D', we have

$$C(D', 2, -1) = 2 \cdot \sum_C (-1)^{j(C)},$$

where the sum is only over those path-cycle covers of D' that have exactly two paths and an arbitrary number of cycles.

In the following, we prove $C(D', 2, -1) = 2^{n+1} \cdot \#\textsc{HamiltonianPaths}(D)$.

From a given path-cycle cover C, related path-cycle covers can be constructed by *switching* the presence of the edges $e_v, (v, v')$ and their corresponding crossing edges. Let $\mathcal{C}_0$ be the set of path-cycle covers C_0 of D' that have exactly two paths, use no crossing edge, and have at least one cycle. We define the set of *bad* cycle covers $\mathcal{C}_b$ as the closure of $\mathcal{C}_0$ under switching arbitrary crossing gadgets.

Let $C \in \mathcal{C}_b$ be arbitrary. By switching, we can uniquely turn C into a path-cycle cover $C_0 \in \mathcal{C}_0$. Let v be the smallest vertex of D that occurs in a cycle of C_0. We define the partner of C as the path-cycle cover $p(C) \in \mathcal{C}_b$ which is derived from C by switching the crossing gadget $e_v, (v, v')$. As depicted in Fig. 4, the numbers of cycles in C and in $p(C)$ differ by exactly 1.

Since p is a permutation on $\mathcal{C}_b$ with the property $(-1)^{j(C)} = -(-1)^{j(p(C))}$, the weights $(-1)^{j(C)}$ of the bad cycle covers sum up to 0 in $C(D', 2, -1)$. This means that $C(D', 2, -1)$ is just 2 times the number of path covers of D' with exactly two paths. Any such 2-path cover C of D' translates to an Hamiltonian path of D (by switching all gadgets to C_0, recontracting the edges (v, v'), and removing a, a' and the b-b'-path), and this procedure does not add any cycles. Since there are 2^n possible gadget states, we get

$$C(D', 2, -1) = 2 \cdot 2^n \cdot \#\textsc{HamiltonianPaths}(D). \quad \square$$

Note that in the proof above, we basically examined an operation necessary for Gaussian elimination: Exchanging two rows or columns in the adjacency matrix switches the sign of the determinant, but as soon as we allow more than one path, this is no longer true for the cover polynomial.

5 Horizontal Reductions

Let us consider reductions along the horizontal lines $L_b := \{(a, b) : a \in \mathbb{Q}\}$. For a directed graph D, let $D^{(r)}$ be the graph obtained by adding r independent vertices. Corollary 4 in [1] is the core part of the horizontal-line reductions:

$$C(D^{(r)}, x, y) = x^{\underline{r}} C(D, x - r, y). \tag{2}$$

From this equation, it is not hard to prove the next lemma (also cf. [11]).

Lemma 3. *For all* $(a, b) \in \mathbb{Q}^2$*, we have* $C(0, b) \preccurlyeq_T^p C(a, b)$*.*

Proof (sketch). For $a \in \mathbb{N}$, it follows directly. For $a \notin \mathbb{N}$, we can compute the values $C(D, a-1, b), \ldots, C(D, a-m, b)$ and interpolate to get $C(D, x, b)$. □

In a similar fashion, one can also prove $C(1, 0) \preccurlyeq_T^p C(a, 0)$ for $a \neq 0$ and $C(2, -1) \preccurlyeq_T^p C(a, 0)$ for $a \neq 0, 1$. Please note that $C(a, b)$ is now known to be #P-hard for every point (a, b) on the lines L_1, L_0, and L_{-1}, except for the three easy points $(0, 0)$, $(0, -1)$, and $(1, -1)$.

6 Vertical Reduction

In this section, we reduce the permanent along the y-axis:

Theorem 1. *Let* $b \in \mathbb{Q}$ *with* $-1 \neq b \neq 0$*. It holds* $C(0, 1) \preccurlyeq_T^p C(0, b)$*.*

Proof (outline). For some graph D, we compute $C(D, 0, 1)$ with oracle access to $C(0, b)$, and we use interpolation to do so.

In order to interpolate the polynomial $C(D, 0, y)$, we need to compute some values $C(D, 0, b_1), \ldots, C(D, 0, b_m)$. This can be done by using the oracle for some values $C(D_1, 0, b), \ldots, C(D_m, 0, b)$ instead. More specifically, we construct graphs D^α containing α copies of a graph D, such that there is a polynomial-time computable relation between $C(D, 0, b^\alpha)$ and $C(D^\alpha, 0, b)$. Computing $C(D, 0, b^\alpha)$ for $\alpha = 1, \ldots, m$ and applying interpolation, we get $C(D, 0, y)$.

Construction details are spelled out in the remainder of this section. □

The constructed graph D^α is a graph in which every cycle cover ideally has α times the number of cycles a corresponding cycle cover of D would have. This way, the terms y^j in the cover polynomial ideally become $y^{\alpha j}$, and some easily computable relation between $C(D, 0, b^\alpha)$ and $C(D^\alpha, 0, b)$ can be established.

In the construction, we therefore duplicate the graph α times, and we connect the duplicates by equality gadgets. These equality gadgets make sure that every cycle cover of D^α is a cycle cover of D copied α times, and thus has roughly α times the number of cycles. Let us construct the graph D^α explicitly.

- Start with the input graph D.
- Create α copies $D_1, \ldots, D_\alpha$ of D.
- Let e_i be the copy of e in the graph D_i. Replace every tuple of edges $(e_1, \ldots, e_\alpha)$ by the multiequality gadget on α edges, which is drawn in Fig. 2.

The D_1-part of every cycle cover of D^α is isomorphic to a cycle cover of D and has to be imitated by the other subgraphs D_i because of the equality gadgets. What we are left with is to prove that the equality gadget indeed ensures that, in every cycle cover, two edges are either both present or both absent.

Note that some edges in the multiequality gadget become paths of length three. This is because we add the equality gadget first to, say, e_1 and e_2 which gives two new edges each. Next we apply it to one of the new edges for e_2 and to e_3 which explains why e_2 now has three edges. Please also note that the equality gadget can be adapted easily to work as an XOR-gadget [11], easier to analyze than that found in [9].

6.1 The Weighted Cover Polynomial

Unfortunately, the equality gadget cannot[1] enforce equality in every possible cycle cover, and we call those cycle covers *good* that satisfy equality (we later change this notion slightly). As you noticed, we introduce weights $w_e \in \{-1, 1\}$ on the edges. These weights make sure that, in the *weighted* cover polynomial

$$C^w(D, 0, y) := \sum_{\text{cycle cover } C} w(C) y^{j(C)} := \sum_{\text{cycle cover } C} y^{j(C)} \prod_{e \in C} w_e,$$

the *bad* cycle covers sum up to 0, so *effectively* only the good cycle covers are visible. We define the evaluation function $C^w(a, b)$ in the weighted case again as $D \mapsto C^w(D, a, b)$ and show that both evaluation complexities are equal.

Lemma 4. *For all $b \in \mathbb{Q}$, it holds $C(0, b) \preccurlyeq^{\mathrm{p}}_{\mathrm{m}} C^w(0, b)$ and $C^w(0, b) \preccurlyeq^{\mathrm{p}}_{\mathrm{m}} C(0, b)$.*

Proof (sketch). The first claim is trivial. For the second claim we adapt the proof of Valiant [9] to the cover polynomial: We replace the -1-edges by an unweighted N-gadget that simulates a sufficiently large weight N, and we compute modulo $N + 1$. However, our N-gadget is different from Valiant's since it has to contribute a constant amount of cycles to cycle covers. To handle rational values, we multiply with the common denominator, and for negative values, we choose N even larger. (for details, cf. [11]) □

6.2 Partner Elimination

In Fig. 5, we have drawn all possible ways for an equality gadget to be covered by cycles if the surrounding graph allows it. The *state* of the gadget is the set of edges chosen to cover it. Notice that the GOOD states as well as $\text{STATE}_1^{\pm}$ and $\text{STATE}_2^{\pm}$ have the equality property, that is, the left path is completely present or absent if and only if the same holds for the right one.

As stated above, the problem is that the equality gadget does *not* prevent states that do not have the equality property. Therefore, using Lemma 4, we switch to the weighted cover polynomial for which our equality gadget is constructed in such a way, that the good cycle covers contribute a weight $\neq 0$ to the sum $C^w(D^\alpha, 0, b)$ while the bad cycle covers do not contribute to the sum, that is, the weights of the bad cycle covers sum up to zero. This is achieved by the fact that every bad cycle cover of weight $+1$ has a unique bad cycle cover of weight -1 as a partner. More specifically, every cycle cover C has a partner C' such that the corresponding summands in the weighted cover polynomial vanish, that is,

$$w(C) y^{j(C)} + w(C') y^{j(C')} = 0.$$

Note that not only the weights must be of different sign, but also the numbers of cycles must be equal! This is the crucial factor why we cannot simply adapt the XOR-gadget of Valiant to form an equality gadget for the cover polynomial.

[1] *Fortunately*! If it were possible without weights, we could adapt the gadget for a parsimonious reduction from #SAT to the permanent and thereby prove P = NP.

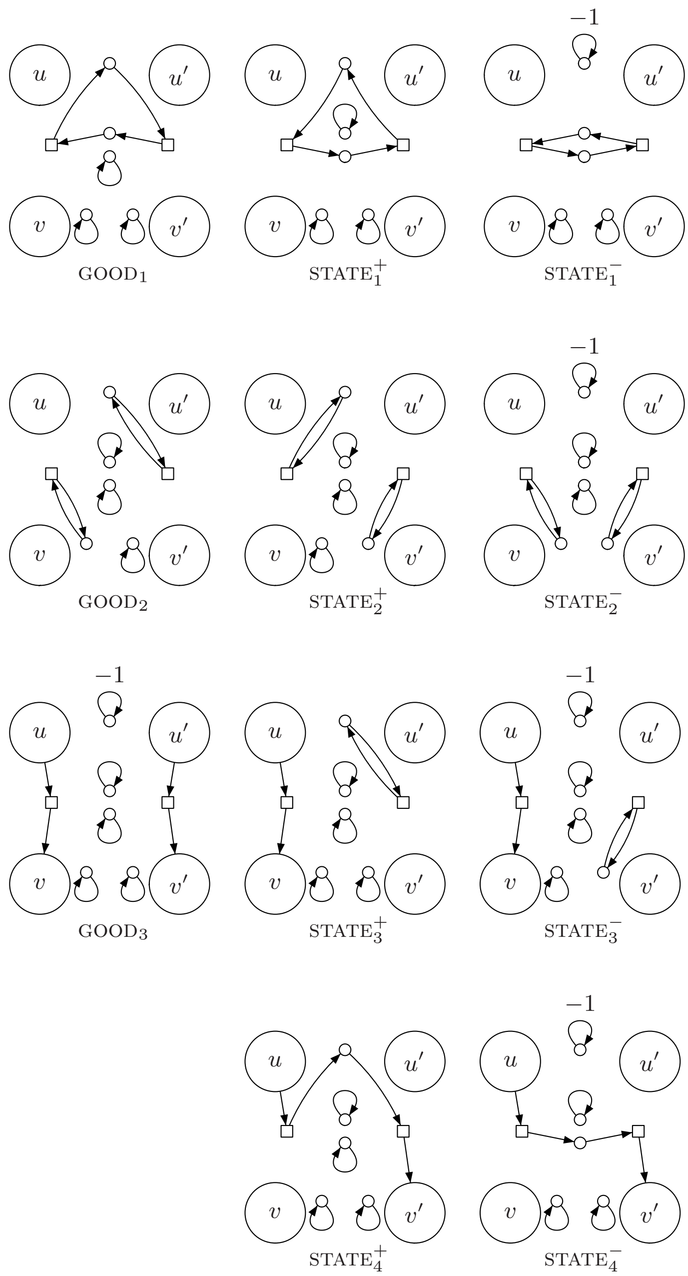

Fig. 5. All possible states (=partial cycle covers) of the equality gadget except that the states STATE$_3^{\pm}$ and STATE$_4^{\pm}$ also have symmetric cases, which we have not drawn. We call GOOD$_1$, GOOD$_2$, and GOOD$_3$ *good* states as they have no partners and satisfy the equality property. Note that the choice of the good states is arbitrary as long as the equality property is satisfied and all STATE$_i^{\pm}$ have partners.

The number of cycles contributed by his XOR-gadget varies a lot, and thus the summands corresponding to the bad cycle covers do not cancel out. (If we plug in $y = 1$ to get the permanent, the condition on the cycles is not needed, and the XOR-gadget works, of course.)

Now let us quickly summarize and prove the properties of the equality gadget. We now call a cycle cover *bad* if an equality gadget is in a state $\text{STATE}_i^{\pm}$.

Lemma 5. *Every bad cycle cover C of D^α has a partner C' with the properties*

(i) C' is again bad, and its partner is C,
(ii) for the weights, it holds $w(C') = -w(C)$, and
(iii) for the number of cycles, it holds $j(C') = j(C)$.

Proof. We choose an arbitrary ordering on the equality gadgets of D^α. Let C be a bad cycle cover and g be its smallest gadget in state $\text{STATE}_i^{\pm}$. We define its partner C' to be the same cycle cover but with gadget g in state $\text{STATE}_i^{\mp}$ instead. Verifying the three properties proves the claim. □

It immediately follows that only the good cycle covers of D^α remain in $C^w(D^\alpha)$:

$$C^w(D^\alpha, 0, y) = \sum_{\substack{\text{cycle cover } C, \\ C \text{ is good}}} w(C) y^{j(C)}.$$

6.3 Counting the Good Cycle Covers

In order to finish the proof of Theorem 1, and thus also the proof of the Main Theorem, it remains to express $C^w(D^\alpha)$ in an appropriate way in terms of $C^w(D)$.

Lemma 6. *It holds $C^w(D^\alpha, 0, y) = \left(y^{n+4m}(1+y)^{m-n}\right)^{\alpha-1} C^w(D, 0, y^\alpha)$.*

Proof (sketch). Every cycle cover C of D induces several possible good cycle covers C' of D^α. Since $|C| = n$, a number of $(\alpha - 1)n$ gadgets have state GOOD_3 in C'. The other $m - n$ gadget have the free choice between the states GOOD_1 and GOOD_2. In addition to the $\alpha j(C)$ large cycles, every gadget contributes small cycles to C'. More precisely, $j(C') = \alpha j(C) + 4 \cdot \#\text{GOOD}_1 + 5 \cdot \#\text{GOOD}_2 + 5 \cdot \#\text{GOOD}_3$. Using the fact that there are $\binom{m-n}{i}$ possible C' corresponding to C and with the property $\#\text{GOOD}_1(C') = i$, the claim can be verified by a simple computation (cf. [11]). □

The easily computable relation between $C^w(D^\alpha, 0, y)$ and $C^w(D, 0, y^\alpha)$ from the last lemma proves (a weighted formulation of) Theorem 1.

Finally, the Main Theorem follows, for $-1 \neq b \neq 0$, from the reduction chain

$$C(0,1) \preccurlyeq_{\mathrm{T}}^{\mathrm{p}} C^w(0,1) \preccurlyeq_{\mathrm{T}}^{\mathrm{p}} C^w(0,b) \preccurlyeq_{\mathrm{T}}^{\mathrm{p}} C(0,b) \preccurlyeq_{\mathrm{T}}^{\mathrm{p}} C(a,b).$$

7 The Geometric Cover Polynomial

The geometric cover polynomial $C_{\mathrm{geom}}(D, x, y)$ introduced by D'Antona and Munarini [10] is the geometric version of the cover polynomial, that is, the falling factorial $x^{\underline{i}}$ is replaced by the usual power x^i in (1). We are able to give complexity results also for this graph polynomial, except for the line L_{-1}.

Theorem 2. *Let $(a, b) \in \mathbb{Q}^2$. It holds:*
If $(a, b) \notin \{(0, 0), (0, -1)\}$, then $C_{geom}(a, b)$ is #P-hard.
Otherwise, $C_{geom}(a, b)$ is computable in polynomial time.

Proof (sketch). The results on the y-axis follow from $C(D, 0, y) = C_{\mathrm{geom}}(D, 0, y)$. For the other points, a new horizontal reduction must be established. This can be done by considering the α-fattening of the graph, in which every edge is replaced by α parallel edges (for details, cf. [11]). □

8 Conclusion and Further Work

In this paper, we completely characterized the complexity of evaluating the cover polynomial in the rational plane. Our reductions should also work for complex or algebraic numbers, but we do not have any interpretation for such points yet.

The next natural step along the path is to characterize the complexity of approximately evaluating the cover polynomial, in analogy to recent results by Goldberg and Jerrum [6] for the Tutte polynomial. Very recently, Fouz had success in characterizing the approximability of the geometric cover polynomial in large regions of the plane.

One might conjecture that the connection between the Tutte and the cover polynomial is solved because both are #P-hard to evaluate at most points, so they can be computed from each other (where they are in #P). From a combinatorial point of view, however, #P-reductions are not really satisfying steps towards a deeper understanding of the connection between the two graph polynomials.

One promising approach might be to consider generalizations of the cover polynomial. Very recently, Courcelle analyzed general contraction-deletion identities analogous to those of the coloured Tutte polynomial, and he found conditions for the variables which are necessary and sufficient for the well-definedness of the polynomial. Unfortunately, these conditions seem too restrictive.

The contraction-deletion identities of Tutte and cover polynomial look astonishingly similar, but contractions work differently for directed and for undirected graphs. Explaining the connection between Tutte and cover polynomial means overcoming this difference.

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