

# Applications of the duality method to generalizations of the Jordan canonical form

Olga Holtz

Department of Mathematics

University of Wisconsin

Madison, Wisconsin 53706 U.S.A.

holtz@math.wisc.edu

We show how Ptak's duality method leads to short proofs of two extensions of the Jordan canonical form, viz. the normal form for a matrix over an arbitrary (not necessarily algebraically closed) field under similarity and the canonical form for a pair of matrices under contragredient equivalence.

The duality method is summarized in the following.

**Lemma.** *Let  $V$  be a finite-dimensional space over a field  $F$ , let  $A : V \rightarrow V$  be a linear map, and  $S \subset V$  be an  $A$ -invariant subspace of  $V$ . If  $T \subset V^*$  is an  $A^*$ -invariant subspace of the dual  $V^*$  of  $V$  such that*

$$s \in S, \quad \langle s, t \rangle = 0 \quad \forall t \in T \implies s = 0, \tag{1}$$

$$t \in T, \quad \langle s, t \rangle = 0 \quad \forall s \in S \implies t = 0, \tag{2}$$

then  $V = S \dot{+} \text{ann}(T)$  is an  $A$ -invariant direct sum decomposition of  $V$ , with  $\text{ann}(T) := \{v \in V : \langle v, t \rangle = 0 \quad \forall t \in T\}$  the annihilator of  $T$ .

We give a proof for the sake of completeness.

**Proof.** The condition (1) implies that the sum  $S + \text{ann}(T)$  is direct. If  $\dim T \geq \dim S$  and  $\{t_j\}_{j=1}^{\dim T}$  ( $\{s_j\}_{j=1}^{\dim S}$ ) is a basis of  $T$  ( $S$ ), then the matrix  $G := (\langle s_i, t_j \rangle : i = 1, \dots, \dim S, j = 1, \dots, \dim T)$  has fewer rows than columns, hence the equation  $Gx = 0$  has a nontrivial solution, so (2) fails. In other words, (2) implies that  $\dim T \leq \dim S$ , hence  $\dim \text{ann}(T) \geq \dim V - \dim S$ . Thus,  $V = S \dot{+} \text{ann}(T)$ . Since  $T$  is  $A^*$ -invariant,  $\text{ann}(T)$  is  $A$ -invariant, which completes the proof.  $\square$

## 1 The analogue of the Jordan form for an arbitrary field

**Theorem 1.** *Let  $V$  be a finite-dimensional linear space over a field  $F$  and let  $A : V \rightarrow V$  be a linear map. There exists a basis of  $V$  such that the representation of  $A$  with respect to that basis has the form*

$$\text{diag}(A_1, \dots, A_p), \tag{3}$$

where

$$A_i = \begin{pmatrix} C_i & 0 & \cdots & 0 & 0 \\ B_i & C_i & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & C_i & 0 \\ 0 & 0 & \cdots & B_i & C_i \end{pmatrix}, \quad B_i = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}_{d_i \times d_i}$$

$$C_i = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & a_{d_i} \\ 1 & 0 & 0 & \cdots & 0 & 0 & a_{d_i-1} \\ 0 & 1 & 0 & \cdots & 0 & 0 & a_{d_i-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & a_3 \\ 0 & 0 & 0 & \cdots & 1 & 0 & a_2 \\ 0 & 0 & 0 & \cdots & 0 & 1 & a_1 \end{pmatrix}_{d_i \times d_i},$$

$x^{d_i} - a_1x^{d_i-1} - \cdots - a_{d_i}$  is a prime in  $F[x]$ .

This form is unique up to reordering of the blocks  $A_1, \dots, A_p$ .

**Proof.** Since the space of all linear maps on  $V$  is finite-dimensional, there exists  $k \in \mathbb{N}$  such that  $A^k \in \text{span}\{I, A, \dots, A^{k-1}\}$ , so  $f(A) = \{0\}$ , hence some monic polynomial in  $F[x]$  annihilates  $A$ .

Let  $f \in F[x]$  be the monic polynomial of minimal degree such that  $f(A) = \{0\}$  and let  $f = (f_1)^{k_1} \cdots (f_r)^{k_r}$  be its decomposition into powers of distinct (monic) primes  $f_i$ ,  $i = 1, \dots, r$ . Let  $g_i := \prod_{j=1, j \neq i}^r (f_j)^{k_j}$ . Since  $F[x]$  is a Euclidean domain and  $\gcd(g_1, \dots, g_r) = 1$ , it follows that  $g_1 h_1 + \cdots + g_r h_r = 1$  for some  $h_1, \dots, h_r \in F[x]$ , hence  $v = h_1(A)g_1(A)v + \cdots + h_r(A)g_r(A)v$  for any  $v \in V$ . But  $h_i(A)g_i(A)V \subseteq V_i := \ker(f_i(A))^{k_i}$ , so  $V = V_1 + \cdots + V_r$ . Suppose  $v \in V_i \cap \sum_{j \neq i} V_j$ . As the polynomials  $(f_i)^{k_i}$  and  $\prod_{j \neq i} (f_j)^{k_j}$  are relatively prime, there exist  $s_{i,1}, s_{i,2} \in F[x]$  such that  $s_{i,1}(f_i)^{k_i} + s_{i,2} \prod_{j \neq i} (f_j)^{k_j} = 1$ , hence  $v = s_{i,1}(A)(f_i(A))^{k_i}v + s_{i,2}(A) \prod_{j \neq i} (f_j(A))^{k_j}v = 0$ , since  $V_i = \ker(f_i(A))^{k_i}$  and  $\sum_{j \neq i} V_j \subseteq \ker \prod_{j \neq i} (f_j(A))^{k_j}$ . So,  $V = V_1 + \cdots + V_r$  is a(n  $A$ -invariant) direct sum decomposition of  $V$ . The arguments given so far are standard.

Now show how to split the subspaces  $V_i$ . Let  $\tilde{V}$  stand for  $V_1$ ,  $\tilde{A}$  for  $A|_{V_1}$ ,  $\tilde{f}$  for  $f_1$ ,  $k$  for  $k_1$ ,  $d$  for  $\deg f_1$ . Since  $f$  is the minimal polynomial annihilating  $A$ ,  $\tilde{f}^k$  is the minimal polynomial annihilating  $\tilde{A}$ , so there exists  $v \in \tilde{V}$  such that  $w := (\tilde{f}(\tilde{A}))^{k-1} \tilde{A}^{d-1} v \neq 0$ .

We claim that  $w \notin \text{span}\{(\tilde{f}(\tilde{A}))^{k-1} \tilde{A}^j v : j = 0, \dots, d-2\}$ . Indeed, if  $w$  were in that span, it would imply  $h(\tilde{A})(\tilde{f}(\tilde{A}))^{k-1} v = 0$  for some polynomial  $h$  of degree  $d-1$ . But any polynomial of degree  $d-1$  is coprime to  $f$ , so there would exist a combination of  $h$  and  $f$  (with coefficients from  $F[x]$ ) equal to 1, which would yield  $(f(\tilde{A}))^{k-1} v = 0$ , contradicting  $w \neq 0$ . Hence the claim follows.

So, there exists  $v' \in \tilde{V}^*$  such that

$$\langle (f(\tilde{A}))^{k-1} \tilde{A}^j v, v' \rangle \begin{cases} = 0 & \text{if } j = 0, \dots, d-2 \\ \neq 0 & \text{if } j = d-1. \end{cases}$$

Let

$$W_1 := \text{span}\{(f(\tilde{A}))^{i_1-1} \tilde{A}^{i_2-1} v : i_1 = 1, \dots, k, \quad i_2 = 1, \dots, d\},$$

$$W'_1 := \text{span}\{(f(\tilde{A}^*))^{i_1-1} (\tilde{A}^*)^{i_2-1} v' : i_1 = 1, \dots, k, \quad i_2 = 1, \dots, d\}.$$

Notice that

$$g_{(i_1, i_2), (j_1, j_2)} := \langle (f(\tilde{A}))^{i_1-1} \tilde{A}^{d-i_2} v, (f(\tilde{A}^*))^{k-j_1} (\tilde{A}^*)^{j_2-1} v' \rangle \neq 0$$

only if  $(i_1, i_2) \preceq (j_1, j_2)$  (in lexicographic order). So, the  $kd \times kd$ -matrix  $(g_{(i_1, i_2), (j_1, j_2)} : i_1, j_1 = 1, \dots, k, i_2, j_2 = 1, \dots, d)$  is upper triangular with nonzero diagonal elements, hence, by the Lemma,  $\tilde{V} = \dot{+} \text{ann}(W'_1)$  is an  $\tilde{A}$ -invariant direct sum decomposition of  $\tilde{V}$ . The matrix representation of  $\tilde{A}|_{W'_1}$  with respect to the basis  $((f(\tilde{A}))^{i_1-1} \tilde{A}^{i_2-1} v : i_1 = 1, \dots, k, i_2 = 1, \dots, d)$  ordered lexicographically is one of the diagonal blocks in (3) with  $d_i = d$  and  $\tilde{f}(x) = x^d - a_1 x^{d-1} - \dots - a_d$ .

Splitting the spaces  $\text{ann}(W'_1), V_2, \dots, V_r$  in the same way as above, we obtain a direct sum  $V = W_1 \dot{+} \dots \dot{+} W_p$  of  $A$ -invariant indecomposable subspaces and a basis in each so that the matrix representation of  $A$  with respect to the concatenation of the bases of  $W_i$ 's has the form (3).

Since the minimal polynomial  $f$  of  $A$  is unique, the (monic) prime factors  $f_i$  and the powers  $k_i$  with which they occur in  $f$  are determined uniquely. Let

$$n_j^i := \dim \ker(f_i(A))^j = \sum_{W_l \subseteq \ker(f_i(A))^{k_i}} \min(\dim W_l, j \cdot \deg f_i), \quad i = 1, \dots, r, \quad j = 1, \dots, k_i.$$

Then  $\Delta n_j^i := n_{j+1}^i - n_j^i$  is the number of blocks for  $f_i$  of order greater than  $j \cdot \deg f_i$ , so the number of blocks of order  $j \cdot \deg f_i$  equals  $-\Delta^2 n_{j-1}^i / \deg f_i = (\Delta n_{j-1}^i - \Delta n_j^i) / \deg f_i$ . Since the numbers  $n_j^i$  are uniquely determined by the map  $A$ , this completes the proof of the uniqueness of (3).  $\square$

**Remarks.** 1. The arguments in the two preceding paragraphs are variations of those due to de Boor [1]. 2. If  $F$  is algebraically closed, the polynomials  $f_i$  are of degree 1, so (3) becomes the Jordan normal form of  $A$ . 3. In the proof above, all the factors of the minimal polynomial are treated in the same way in contrast to the proof in [7] where the canonical splitting is first given for the nilpotent part of  $A$  and then follows for all other parts by shifting  $A$  by an eigenvalue  $\lambda$  (for that completion of the proof in [7], see [1]). 4. Theorem 1 is classical and can be found, e.g., in [5, pp. 92–97]. In the sequel, we refer to a matrix in the form (3) as being in the *Jordan normal form for the field  $F$* , and as the Jordan normal form of the operator  $A$ .

## 2 The canonical form under contragredient equivalence

Two pairs of matrices,  $(A, B)$  and  $(C, D)$ , are called contragrediently equivalent if  $A, C \in F^{m \times n}$ ,  $B, D \in F^{n \times m}$ , and  $A = SCT^{-1}$ ,  $B = TDS^{-1}$  for some invertible  $S \in F^{m \times m}$ ,  $T \in F^{n \times n}$ .

The problem of classification of pairs of matrices under contragredient equivalence can be restated as follows. Given an  $n$ -dimensional linear space  $V$  and an  $m$ -dimensional linear space  $W$  and linear maps  $A : V \rightarrow W$ ,  $B : W \rightarrow V$ , choose bases of  $V$  and  $W$  so that the pair  $(A, B)$  has a simple representation with respect to these bases.

**Theorem 2.** *Let  $V, W$  be finite-dimensional linear spaces over a field  $F$  and let  $A : V \rightarrow W$ ,  $B : W \rightarrow V$  be linear maps. There exist bases of  $V$  and  $W$  such that, with respect to those bases, the pair  $(A, B)$  has the representation*

$$(\text{diag}(I, A_1, \dots, A_p, 0), \quad \text{diag}(J_{AB}, B_1, \dots, B_p, 0)) \quad (4)$$

where  $J_{AB}$  is the nonsingular part of the Jordan form of  $AB$ ,  $A_i, B_i \in F^{m_i \times n_i}$ ,  $|m_i - n_i| \leq 1$ , and

$$(A_i, B_i) \in \left\{ \left( \begin{pmatrix} I_{m_i-1} & 0 \\ & 0 \end{pmatrix}, \begin{pmatrix} 0 \\ I_{m_i-1} \end{pmatrix} \right), \left( \begin{pmatrix} 0 \\ I_{m_i-1} \end{pmatrix}, \begin{pmatrix} I_{m_i-1} & 0 \\ & 0 \end{pmatrix} \right), (I_{m_i}, J_{m_i}), (J_{m_i}, I_{m_i}) \right\}$$

where  $J_k$  denotes the  $k \times k$ -matrix with ones on the first subdiagonal and zeros elsewhere. The representation (4) is unique up to the order of the pairs of blocks  $(A_i, B_i)$ ,  $i = 1, \dots, p$ . Two pairs  $(A, B)$  and  $(C, D)$  are contragrediently equivalent if and only if  $AB$  is similar to  $CD$  and

$$\begin{aligned} \text{rank } A &= \text{rank } C, \text{ rank } BA = \text{rank } DC, \dots, \text{rank}(BA)^t = \text{rank}(DC)^t, \\ \text{rank } B &= \text{rank } D, \text{rank } AB = \text{rank } CD, \dots, \text{rank}(AB)^t = \text{rank}(CD)^t, \\ t &= \min\{m, n\}. \end{aligned} \quad (5)$$

**Proof. Step 1.** By Theorem 1 of [7] (whose proof holds over an arbitrary field), there exist  $V_1$  ( $W_1$ ) and  $V_2$  ( $W_2$ ) such that  $BA$  ( $AB$ ) is invertible on  $V_1$  ( $W_1$ ) and nilpotent on  $V_2$  ( $W_2$ ) and  $V = V_1 \dot{+} V_2$  ( $W = W_1 \dot{+} W_2$ ). Moreover,  $V_1 = \text{range}(BA)^r$ ,  $V_2 = \ker(BA)^r$ ,  $W_1 = (AB)^r$ ,  $W_2 = \ker(AB)^r$  for some  $r \in \mathbb{N}$ . If  $x \in V_1$ , then  $x = (BA)^r y$  for some  $y \in V$ , hence  $(AB)^r Ay = Ax$ , that is,  $Ax \in W_1$ . Analogously,  $Bx \in V_1$  whenever  $x \in W_1$ . So,  $V = V_1 \dot{+} V_2$ ,  $W = W_1 \dot{+} W_2$ ,  $A$  maps  $V_i$  to  $W_i$ ,  $B$  maps  $W_i$  to  $V_i$  for  $i = 1, 2$ .

If  $x \in V_2$ , then  $(AB)^r Ax = 0$ , so  $Ax \in W_2$ . If  $x \in V_1$  and  $Ax = 0$ , then  $BAx = 0$ , therefore,  $x = 0$ , since  $BA$  is invertible on  $V_1$ . So,  $A$  induces a one-one map from  $V_1$  to  $W_1$ . Likewise,  $B$  induces a one-one map from  $W_1$  to  $V_1$ . So,  $V_1$  and  $W_1$  have the same dimension and the induced maps are also onto.

This step of the proof not only uses Theorem 1 of [7], but also parallels it.

Now one can choose bases of  $V_1$  and  $W_1$  so that  $A|_{V_1}$  is the identity matrix and  $B|_{W_1}$  is in Jordan normal form (which is the nonsingular part of the Jordan normal form of  $AB$ ).

**Step 2.** The spaces  $V_2$  and  $W_2$  are further split as follows. Let  $l$  be the length of the longest nonzero product of the form  $\dots ABA$  or  $\dots BAB$ . Call such a product  $C$  and suppose it ends in  $A$ . Pick  $x \in V_2$  so that  $Cx \neq 0$  and form the sequence  $x, Ax, BAx, \dots, Cx$ , whose elements are alternately in  $V_2$  and  $W_2$ . Let  $V_3$  ( $W_3$ ) be the span of the elements of the sequence belonging to  $V_2$  ( $W_2$ ).

If  $l$  is even, then  $\dim V_3 = \dim W_3 + 1 = 1 + l/2$ . Pick  $x' \in V_2^*$  so that  $\langle Cx, x' \rangle \neq 0$ . Form the sequence  $x', B^*x', \dots, A^*B^*x', \dots, C^*x'$ . Let  $V_4$  ( $W_4$ ) be the annihilator in  $V_2$  (in  $W_2$ ) of the elements of the sequence that lie in  $V_2^*$  ( $W_2^*$ ). The  $(1 + l/2) \times (1 + l/2)$ -matrix  $\langle \langle (BA)^{i-1}x, (A^*B^*)^{1+l/2-j}x' \rangle : i, j = 1, \dots, 1 + l/2 \rangle$  is upper triangular with nonzero diagonal entries, hence, by the Lemma,  $V_2 = V_3 \dot{+} V_4$ . This argument is exactly the same as the corresponding argument in [1].

Analogously,  $W_2 = W_3 \dot{+} W_4$ . Moreover,  $A$  maps  $V_i$  to  $W_i$ ,  $B$  maps  $W_i$  to  $V_i$ ,  $i = 3, 4$ , and the pair  $(A|_{V_3}, B|_{W_3})$  has the form

$$\left( \begin{pmatrix} I_{l/2} & 0 \\ & 0 \end{pmatrix}, \begin{pmatrix} 0 \\ I_{l/2} \end{pmatrix} \right).$$

If  $l$  is odd, then  $\dim V_3 = \dim W_3 = (1 + l)/2$ , and the above construction gives  $V_2 = V_3 \dot{+} V_4$ ,  $W_2 = W_3 \dot{+} W_4$  with  $A$  mapping  $V_i$  to  $W_i$ ,  $B$  mapping  $W_i$  to  $V_i$ ,  $i = 3, 4$ , the pair  $(A|_{V_3}, B|_{W_3})$  having the form  $(I_{(1+l)/2}, J_{(1+l)/2})$ .

If  $C$  ends in  $B$ , then  $(A|_{V_3}, B|_{W_3})$  has the form

$$\left( \begin{pmatrix} 0 \\ I_{l/2} \end{pmatrix}, \begin{pmatrix} I_{l/2} & 0 \end{pmatrix} \right) \quad \text{or} \quad (J_{(1+l)/2}, I_{(1+l)/2}).$$

This step of the proof parallels, with necessary modifications, Theorem 2 of [7].

The problem is now reduced to splitting  $V_4$  and  $W_4$  in the same way. The splitting process ends at the  $j$ -th stage if  $A|_{V_{2j}} = 0$  and  $B|_{W_{2j}} = 0$ .

Thus one obtains the canonical form (4). It is completely determined by the nonsingular part of the Jordan form of  $AB$  and the ranks  $\text{rank}(A)$ ,  $\text{rank}(BA)$ ,  $\text{rank}(ABA)$ ,  $\dots$ ,  $\text{rank}(B)$ ,  $\text{rank}(AB)$ ,  $\text{rank}(BAB)$ ,  $\dots$ . Since the rank of any such product equals the size of  $J_{AB}$  if the length of the product exceeds  $2 \min\{m, n\}$ , the infinite sequences above can be terminated at  $(BA)^{\min\{m, n\}}$ ,  $(AB)^{\min\{m, n\}}$ . It follows that 1) the representation (4) is unique up to the order of the pairs of blocks and that 2) two pairs  $(A, B)$  and  $(C, D)$  are contragrediently equivalent if and only if  $AB$  is similar to  $CD$  and (5) holds.  $\square$

**Remarks.** 1. Ptak's duality method was rediscovered by I. Kaplansky [6], who also described how to derive the canonical form (4). The same form was first published by N. T. Dobrovolskaya and V. A. Ponomarev [2]. J. Gelonch and P. Rubi3 i Diaz [3, Theorem 2] proved that the pair  $(A, B)$  can be represented as

$$(\text{diag}(A_1, \dots, A_q), \quad \text{diag}(B_1, \dots, B_q))$$

where  $A_i$  and  $B_i^*$  are of the same size and

$$(\dim \ker A_i, \dim \ker B_i) \in \{(0, 1), (1, 0)\} \quad \text{unless} \quad A_i = 0, B_i = 0.$$

R. Horn and D. Merino derived the canonical form (4) in [4, Theorem 5]. All the derivations (in [2], [6], [3], and [4]) were for the field  $\mathbb{C}$ . 2. Observe that the canonical form of the pair  $(I, A)$  under contragredient equivalence is  $(I, J_A)$ , where  $J_A$  is the Jordan normal form of  $A$ . This and many other applications of the canonical form (4) are discussed in [4].

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