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On orthogonal polynomial approximation with the dimensional expanding technique for precise time integration in transient analysis ☆

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Abstract

We use four orthogonal polynomial series, Legendre, Chebyshev, Hermite and Laguerre series, to approximate the non-homogeneous term for the precise time integration and incorporate them with the dimensional expanding technique. They are applied to various structures subjected to transient dynamic loading together with Fourier and Taylor approximation proposed in previous works. Numerical examples show that all six methods are efficient and have reasonable precision. In particular, Legendre approximation has much higher precision and better convergence; Chebyshev approximation is also good, but only slightly inferior to Legendre approximation. The other four approximation methods usually produce results with errors hundreds of thousands of times larger. Hermite and Laguerre approximation may be useful for some special non-homogeneous terms, but do not work sufficiently well in our numerical examples. Other contributions of this paper include, a Dynamic Programming scheme for computing series coefficients, a general formula to find the assistant matrix for any polynomial series.

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1. Introduction

In transient analysis, ODEs (Ordinary Differential Equations) in the following form is common and important:

$$\dot{v} = Av + f, \quad v(0) = v_0 \tag{1}$$

where $v(t)$ is an n-dimensional vector function to be determined, A is a given $n \times n$ constant matrix, and $f(t)$ is a given n-dimensional vector function.

For example, the equations of these problems may be transformed into Eq. (1): Dynamic responses of structures subjected to transient loading [1–4], multibody system dynamic models [5], some transient heat transfer problems [6–9] and matrix Riccati differential equations [10,11]. Among such transform, the variational principle is usually used to turn higher order ODEs into first order ones.

In most cases, finding an analytical solution to Eq. (1) is very difficult or even impossible. Thus numerical solution becomes a common alternative. Zhong proposed the Precise Time Integration (PTI) method [3], which can produce numerical results with extremely high precision for Eq. (1) without the non-homogeneous term $f(t)$. Additionally, the PTI was proved to be unconditionally stable and demonstrated adaptivity to stiff problems [1–3].

In order to solve Eq. (1), there are generally two ways: The original one is using the solution theory for ODEs, which first finds one particular solution and then computes the general solution. It requires computing inverse matrices, which is inefficient and induces relatively large errors. Specifically, if the matrix to be inversed is singular or approximately singular, the errors induced are considerable. Therefore the dimensional expanding technique (DET), that transforms non-homogenous ODEs into homogenous ones, is preferable, and adopted by recent works [12] and this paper, which was first proposed by Gu et al. [13].

Function approximation to the non-homogeneous term $f(t)$ is necessary in many cases, because it is usually very difficult or even impossible to find one particular solution if adopting the solution theory for ODEs, and the precondition of the DET i.e. the derivative of the expanded state vector p can be expressed as a linear combination of p itself is also impossible to achieve. Hence efforts have been devoted into this since the very beginning: Originally, Zhong adopted a linear approximation method [3]; Afterwards, Lin et al. applied Fourier approximation [1]; then Zhou et al. carried out Taylor approximation [12].

In this paper, four orthogonal polynomial series, Legendre, Chebyshev, Hermite and Laguerre series, are used to achieve better approximation to $f(t)$ in Eq. (1). The rest of the manuscript is arranged as follows: In Section 2 the PTI for the homogeneous form of Eq. (1) is presented; Sections 3 and 4 discuss the techniques to approximate $f(t)$ and incorporate the DET with the four orthogonal polynomial series respectively; Section 5 is numerical examples where our method is compared with Lin's Fourier approximation [1] and Zhou's Taylor approximation [12]; Section 6 provides some concluding remarks.

2. Precise time integration for the homogeneous form of Eq. (1)

Generally, the homogeneous form of Eq. (1) i.e. with the non-homogenous term $f(t)$ removed should be solved first. Its solution is

$$v = \exp(A \cdot t) \cdot v_0 \tag{2}$$

The time step size is denoted as τ satisfying $t_0 = 0,\ t_1 = \tau, \ldots, t_k = k \cdot \tau, \ldots$ Thus we have the following recursive steps:

$$v_1 = Tv_0, \quad v_2 = Tv_1, \ldots, \quad v_{k+1} = Tv_k, \ldots \tag{3}$$

where $T = \exp(A \cdot \tau)$.

It is seen that, computing the matrix exponential T is the essential, which is illustrated below.

Split the time interval τ into smaller ones. Define $\Delta t = \tau/2^N$, in which N is a natural number. When N is 20, $\Delta t = \tau/1048576$ is very small. Certainly N can be even larger. Then

$$T = [\exp(A \cdot \Delta t)]^{2^N} = (I + T_{a0})^{2^N} = [(I + T_{a0}) \times (I + T_{a0})]^{2^{N-1}} = (I + T_{a1})^{2^{N-1}} \tag{4}$$

Such factorization should be iterated N times. This results in a series $\{T_{ai}\}$ in which $i = 0, 1, 2, \ldots, N$. And initially an Mth order Taylor expansion is carried out

$$I + T_{a0} = \exp(A \cdot \Delta t) \approx I + A\Delta t + (A\Delta t)^2/2 + (A\Delta t)^3/3! + \cdots + (A\Delta t)^M/M! \tag{5}$$

Note that T_{a0} is extremely small. So it will almost be rounded-off if added to the unit matrix I directly. The way to avoid this is to operate on the incremental part $\{T_{ai}\}$ rather than on the total value $\{I + T_{ai}\}$. $(I + T_{ai})^2 = I + 2T_{ai} + T_{ai} \times T_{ai} = I + T_{a(i+1)}$, this yields the recurrence as follows:

$$T_{a(i+1)} = 2T_{ai} + T_{ai} \times T_{ai} \quad \text{for every } i = 0, 1, \ldots, N-1 \tag{6}$$

T_{aN} is no longer small and so the addition can be done now

$$T = I + T_{aN} \tag{7}$$

Eqs. (4)–(7) constitute the main PTI algorithm.

3. Orthogonal polynomial approximation to the non-homogeneous term *f*(*t*)

3.1. The generalized form of polynomial approximation

If the non-homogenous term $f(t)$ is approximated by a qth order polynomial $L_q(t)$, Eq. (1) becomes

$$\dot{v} = Av + L_q(t) = Av + Cp \tag{8}$$

where $p = \{P_0(t), P_1(t), \ldots, P_q(t)\}^{\mathrm{T}}$ is a $(q+1)$-dimensional vector consisting of a polynomial sequence, and the coefficients matrix $C = \{c_0, c_1, \ldots, c_q\}$ is an $n \times (q+1)$ matrix. Cp is a generalized Fourier series. It is worthy of notice that since $f(t)$ is an n-dimensional vector function, $c_0, c_1, \ldots, c_q$ are all n-dimensional vectors.

3.2. The computation of orthogonal polynomial series and series coefficients

Despite of their various complicated definitions and computing schemes, the four orthogonal polynomial series can be easily obtained through the recurrence relations as follows:

$$\text{Legendre series } P_i : \begin{cases} (i+1)P_{i+1}(x) = (2i+1)xP_i(x) - iP_{i-1}(x) & \text{for all } i > 1 \\ P_0(x) = 1 \\ P_1(x) = x \end{cases} \tag{9a}$$

$$\text{Chebyshev series } T_i : \begin{cases} T_{i+1}(x) = 2xT_i(x) - T_{i-1}(x) & \text{for all } i > 1 \\ T_0(x) = 1 \\ T_1(x) = x \end{cases} \tag{9b}$$

$$\text{Hermite series } H_i : \begin{cases} H_{i+1}(x) = 2xH_i(x) - 2iH_{i-1}(x) & \text{for all } i > 1 \\ H_0(x) = 1 \\ H_1(x) = 2x \end{cases} \tag{9c}$$

$$\text{Laguerre series } L_i : \begin{cases} (i+1)L_{i+1}(x) = (2i+1-x)L_i(x) - iL_{i-1}(x) & \text{for all } i > 1 \\ L_0(x) = 1 \\ L_1(x) = -x + 1 \end{cases} \tag{9d}$$

With their orthogonality, the formulae for series coefficients computation are

$$c_i \text{ for Legendre series } P_i : c_i = \frac{2i+1}{2} \int_{-1}^{1} f(x)P_i(x)\,\mathrm{d}x \tag{10a}$$

$$c_i \text{ for Chebyshev series } T_i : c_i = \begin{cases} \frac{1}{\pi} \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} f(x)T_i(x)\,\mathrm{d}x & \text{for } i = 0 \\ \frac{2}{\pi} \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} f(x)T_i(x)\,\mathrm{d}x & \text{for all } i > 0 \end{cases} \tag{10b}$$

$$c_i \text{ for Hermite series } H_i : c_i = \frac{1}{2^i i! \sqrt{\pi}} \int_{-\infty}^{\infty} e^{-x^2} f(x) H_i(x)\,dx \tag{10c}$$

$$c_i \text{ for Laguerre series } L_i : c_i = \int_0^{\infty} e^{-x} f(x) L_i(x)\,dx \tag{10d}$$

3.3. Linear transformation

With Eqs. (9a), (10a) or Eqs. (9b), (10b), if let $x = t$ directly, least square or optimal uniform approximation to $f(t)$ over $[-1,+1]$ is performed in terms of Legendre or Chebyshev polynomials, i.e. minimizing every element of the n-dimensional vector $\int_{-1}^{1} |f(t) - Cp|^2\,dt$ or $\max_{-1 \leqslant t \leqslant 1} |f(t) - Cp|$ respectively. Thus the domain of $f(t)$ is confined to $[-1,+1]$. To overcome this restriction, a linear transform is applied to map $x \in [a,b]$ (in both cases, $a = -1$ and $b = 1$) onto $t \in [c,d]$ isometrically

$$x = \frac{b-a}{d-c} t - \frac{b-a}{d-c} c + a \tag{11}$$

Since the definite integration in Eqs. (10c) and (10d) ranges upon $[-\infty,\infty]$ and $[0,\infty]$, the linear transform Eq. (11) is not directly applicable. Hence a constant χ is used to substitute ∞ and Eqs. (10c) and (10d) become

$$c_i = \frac{1}{2^i i! \sqrt{\pi}} \int_{-\chi}^{\chi} e^{-x^2} f(x) H_i(x)\,dx \tag{10c$'$}$$

$$c_i = \int_0^{\chi} e^{-x} f(x) L_i(x)\,dx \tag{10d$'$}$$

Surprisingly, numerical examples show that χ need not and should not to be very large, because the weighting functions of Hermite and Laguerre series, $\exp(-x^2)$ and $\exp(-x)$, decay sharply as $|x|$ increases. The selection of χ is important for producing accurate results.

3.4. A dynamic programming scheme for computing series coefficients

When computing the integration in Eqs. (10), the Dynamic Programming (DP) technique can be used to reduce computational complexity from $O(q^2)$ to $O(q)$ and simplify the integrand.

The general form for computing orthogonal series coefficients over $t \in [c,d]$ is

$$c_i = \phi_i \int_c^d P_i(t) w(t) f(t)\,dt \tag{12}$$

where ϕ_i is a known constant, $w(t)$ is a certain weighting function of the orthogonal series, and P_i is an ith order polynomial expressed in the following form:

$$P_i(t) = k_{i0} + k_{i1} t + \cdots + k_{ii} t^i \tag{13}$$

Substituting Eq. (13) into Eq. (12) gives

$$c_i = \phi_i \left(k_{i0} \int_c^d w(t) f(t)\,dt + k_{i1} \int_c^d w(t) f(t) t\,dt + \cdots + k_{ii} \int_c^d w(t) f(t) t^i\,dt \right) \tag{14}$$

From Eq. (14) we find that, the value of $\int_c^d w(t) f(t) t^j\,dt$ is actually re-computed for $(q-j)$ times $(0 \leqslant j \leqslant q)$ if using Eq. (12) directly. Therefore we may utilize the thought of DP i.e. store and reuse earlier computing results: Let $s_j = \int_c^d w(t) f(t) t^j\,dt$, pre-compute and store s_j in an initialization step with traditional numerical integration methods such as Richardson extrapolation and Romberg integration. Then

$$c_i = \phi_i \int_c^d P_i(t) w(t) f(t)\,dt = \phi_i (k_{i0} s_0 + k_{i1} s_1 + \cdots + k_{ii} s_i) = \phi_i \alpha_i S^{\mathrm{T}} \tag{15}$$

where $S = \{s_0, s_1, \ldots, s_q\}$ and $\alpha_i = \{k_{i0}, k_{i1}, \ldots, k_{iq}\}$ $(k_{ij} = 0 \text{ for } i < j \leqslant q)$.

Thus, the number of times for computing the integration s_j $(0 \leqslant j \leqslant q)$ is reduced from $(q+2)(q+1)/2$ to $(q+1)$. And the original Eq. (12) is decomposed into simplified ones.

4. Incorporating the dimensional expanding technique

4.1. Solving Eq. (8) with the solution theory for ODEs

As a comparison of the DET, we first use the solution theory for ODEs to solve Eq. (8).

Expanding the term Cp in Eq. (8) gives

$$\dot{v} = Av + \sum_{i=0}^{q} c_i P_i(t) = Av + \sum_{i=0}^{q} \xi_i t^i \tag{16}$$

One particular solution is

$$v_p(t) = -\sum_{i=0}^{q} \xi_i \left[\sum_{j=0}^{i} (i!/j!) t^j A^{-(i+1-j)} \right] \tag{17}$$

So the general solution is

$$v(t) = T(v_0 - v_p(0)) + v_p(t) = \exp(A \cdot t) \cdot \left(v_0 + \sum_{i=0}^{q} \xi_i i! A^{-(i+1)} \right) - \sum_{i=0}^{q} \xi_i \left[\sum_{j=0}^{i} (i!/j!) t^j A^{-(i+1-j)} \right] \tag*{(18)[3]}$$

4.2. Solving Eq. (8) with the dimensional expanding technique

If using the solution theory for ODEs in Section 4.1, it is required to compute inverse matrices. For the sake of eliminating inverse matrix calculation, the DET is introduced, which transforms non-homogenous ODEs into homogenous ones. The cost for such transformation is a dimensional increase.

The crux of the DET is that, the derivative of the expanded state vector p can be expressed as a linear combination of p itself, i.e. there exists an assistant matrix D such that

$$\dot{p} = Dp \tag{19}$$

Eqs. (8) and (19) yield the homogenous ODEs

$$\dot{v}^* = \begin{Bmatrix} \dot{v} \\ \dot{p} \end{Bmatrix} = \begin{bmatrix} A & C \\ 0 & D \end{bmatrix} \begin{Bmatrix} v \\ p \end{Bmatrix} = Hv^* \tag{20}$$

with initial condition

$$v^*(0) = \begin{Bmatrix} v(0) \\ p(0) \end{Bmatrix} = \begin{Bmatrix} v_0 \\ p_0 \end{Bmatrix} \tag{21}$$

In which

$$p_0 = \{P_0(0), P_1(0), \ldots, P_q(0)\}^{\mathrm{T}} \tag{22}$$

For Legendre and Chebyshev series, $p_0 = \{1, -1, 1, -1, \ldots, (-1)^q\}^{\mathrm{T}}$; for Hermite series before the transform Eq. (11), $P_i(0) = (-2)^{i/2} \cdot 1 \cdot 3 \cdot 5 \cdots (i-1)$ when i is even and $P_i(0) = 0$ when i is odd; and for Laguerre series $p_0 = \{1, 1, 1, \ldots, 1\}^{\mathrm{T}}$.

Thus, the PTI algorithm in Section 2 can be applied to solve Eqs. (20) adn (21).

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[3] Eq. (23) in [1] is one of the degenerated forms of Eq. (18) in this paper. It should be indicated that Eq. (23) in [1] contains a typo, and should be $\{v(t_{k+1})\} = [T(\tau_k)] \times [v(t_k) + [H]^{-1}(\{r_0\} + [H]^{-2}\{r_1\})] - [H]^{-1}(\{r_0\} + [H]^{-1}\{r_1\} + \{r_1\} \times \tau_k)$.

4.3. The computation of the assistant matrix D

As aforementioned, finding an assistant matrix D that satisfies Eq. (19) is the key to apply the DET. A general formula to find D for any polynomial series is first proposed in this paper.

Define U_q as the set of all expanded state vectors consisting of a qth order polynomial sequence. $\forall p \in U_q$, denote $p = \{P_0(t), P_1(t), \ldots, P_q(t)\}^{\mathrm{T}}$, in which P_i $(0 \leqslant i \leqslant q)$ is an ith orderpolynomial and can be expressed as

$$P_i(t) = \sum_{j=0}^{q} k_{ij} t^j \tag{23}$$

where k_{ij} $(0 \leqslant j \leqslant q)$ is a given constant satisfying $k_{ii} \neq 0$, $k_{ij} = 0$ for $i < j \leqslant q$.

If the expanded state vector of the basic polynomial sequence $\{1, t, \ldots, t^i, \ldots, t^q\}^{\mathrm{T}}$ is denoted as p_s, and $\{k_{i0}, k_{i1}, \ldots, k_{iq}\}$ is denoted as a vector α_i, then

$$P_i(t) = \alpha_i p_s \tag{24}$$

So

$$p = E p_s \tag{25}$$

where

$$E = \{\alpha_0, \alpha_1, \ldots, \alpha_q\}^{\mathrm{T}} \tag{26}$$

It can be proved that E is existent, unique and invertible.

The derivative of p_s is easy to compute:

$$\dot{p}_s = D_s p_s, \quad \text{where } D_s = \begin{bmatrix} 0 & 0 & \ldots & 0 & 0 \\ 1 & 0 & \ldots & 0 & 0 \\ 0 & 2 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & q-1 & 0 \end{bmatrix} \tag{27}$$

Combining with Eq. (25) we have

$$\dot{p} = E\dot{p}_s = ED_s p_s = ED_s E^{-1} p \tag{28}$$

$$D = ED_s E^{-1} \tag{29}$$

The existence and uniqueness of E leads to the corresponding existence and uniqueness of D.

With Eq. (29), the formulae to find the assistant matrix D for the four orthogonal polynomial series can be obtained. Due to the limited space, only the D_s when q is odd are listed:

$$D_{\text{Legendre}} = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 1 & 0 & 5 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 3 & 0 & 7 & \cdots & 0 & 0 & 0 & 0 \\ 1 & 0 & 5 & 0 & \cdots & 2q-5 & 0 & 0 & 0 \\ 0 & 3 & 0 & 7 & \cdots & 0 & 2q-3 & 0 & 0 \\ 1 & 0 & 5 & 0 & \cdots & 2q-5 & 0 & 2q-1 & 0 \end{bmatrix} \tag{30}$$

$$D_{\text{Chebyshev}} = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 3 & 0 & 6 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 2q-6 & 0 & 2q-6 & \cdots & 0 & 0 & 0 & 0 \\ q-2 & 0 & 2q-4 & 0 & \cdots & 2q-4 & 0 & 0 & 0 \\ 0 & 2q-2 & 0 & 2q-2 & \cdots & 0 & 2q-2 & 0 & 0 \\ q & 0 & 2q & 0 & \cdots & 2q & 0 & 2q & 0 \end{bmatrix} \tag{31}$$

$$D_{\text{Hermite}} = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 2 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 4 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 6 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 8 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 2q & 0 \end{bmatrix} \tag{32}$$

$$D_{\text{Laguerre}} = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ -1 & 0 & 0 & \cdots & 0 & 0 \\ -1 & -1 & 0 & \cdots & 0 & 0 \\ -1 & -1 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & -1 & -1 & \cdots & -1 & 0 \end{bmatrix} \tag{33}$$

4.4. *D* after the linear transform Eq. (11)

After the transform Eq. (11)

$$\tilde{D} = \frac{d-c}{2} D \tag{34}$$

Because

$$\dot{\tilde{p}} = \frac{\mathrm{d}\tilde{p}}{\mathrm{d}t} = \frac{\mathrm{d}\tilde{p}}{\mathrm{d}x} \cdot \frac{\mathrm{d}x}{\mathrm{d}t} = D\tilde{p} \cdot \frac{d-c}{2} \tag{35}$$

where $\tilde{D}$ and $\tilde{p}$ are D and p after the transform Eq. (11) respectively.

5. Numerical examples

5.1. General annotation for numerical examples

① We introduce a new parameter σ to control the integration process: There is an underlying thinking pattern in previous PTI literature such that either the whole domain $[c, d]$ or each time interval $[t_k, t_{k+1}]$ of $f(t)$ is approximated each time. Actually it is also feasible to approximate several time intervals together (whose length is denoted as approximating step size σ), provided that $\tau|\sigma$ and $\sigma|d-c$. The introduction of σ may bring in some interesting results as shown later.

② For ease of notation, the nomenclatures, abbreviations and denotations below are made:

Nomenclatures, abbreviations and denotations

q	order of the generalized Fourier series Cp in Eq. (8)
Testcase	test of running the PTI algorithm with given A, v_0, $f(t)$, σ, q and approximation method
Instance	a group of testcases with the same A, v_0, $f(t)$ and σ but different q and approximation methods
Case	a group of instances with the same A and v_0 but different $f(t)$, σ, q and approximation methods
Max Err.	maximum of the absolute value of relative errors among all time points in a testcase
Ave Err.	average of the absolute value of relative errors over all time points in a testcase
0	a testcase with 0 *Max Err.* and *Ave Err.* indicates that, all computing results in this testcase are identical with the exact results within the displayed digits
Err. > 1	this indicates that, excessive error has invalidated the computing result because the absolute value of relative error is greater than one
Fou., *Tay.*, *Leg.*, *Che.*, *Her.*, *Lag.*	abbreviations for the Fourier, Taylor, Legendre, Chebyshev, Hermite and Laguerre approximation methods respectively
(*P*)	Example 1 Case (b) $\sigma = 1.0$ and Case (c) $\sigma = 1/3$, 1/2, 1.0 involve the approximation for piecewise functions and are marked with letter '*P*' in parenthesis after the σ value. Because *Tay.* is unable to approximate piecewise functions, it is waived from these instances

③ Although we selected and adapted benchmark examples from [1] solving structural dynamic ODEs. The theory and methods in this paper are obviously not confined to structural dynamics.

④ The results of Fourier and Taylor approximation are provided for comparative purposes.

⑤ In order to provide a better comparison, all computing results are rounded to 14 significant digits, which is with more digits than that of any previous literature. Under such a highly error-sensitive test environment, we must be very meticulous to prevent round-off errors from improper operations.[4]

⑥ For all approximation methods, the DET is used instead of the solution theory for ODEs.

⑦ The expanded dimension l in the Fourier approximation method is $2q+1$ while that of others is only $q+1$. So the q in *Fou.* is half of that of others with the same l as shown in Tables 1–3.

⑧ In accordance with [1] and the physical meaning of the ODEs, only computing results of $V_1(t)$ in Examples 1 and 3, $V_3(t)$ in Example 2 are presented due to the limited space.

Example 1

$$\dot{v} = Av + f \quad \text{where } A = \begin{bmatrix} 0 & 1 \\ -1 & -0.1 \end{bmatrix}, \quad v(0) = \begin{Bmatrix} V_1(0) \\ V_2(0) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}, \quad f = \begin{Bmatrix} 0 \\ F(t) \end{Bmatrix} \tag{36}$$

This is a benchmark selected from Examples 1–3 in [1]. The physical background of its 2nd order form is about the motion of an SDOF (Single Degree-Of-Freedom) system. The three Cases (a)–(c) are with different $F(t)$ correspond to Example 1–3 in [1] respectively:

(a) A half-sine wave function $F(t) = \sin(\pi t)$, $t \in [0,1]$

(b) A bi-linear function $F(t) = \begin{cases} 2t & \text{when } 0 \leqslant t \leqslant 0.5 \\ 2-2t & \text{when } 0.5 < t \leqslant 1 \end{cases}$

(c) An off-peak half-sine wave function $F(t) = \begin{cases} \sin(\pi t) & \text{when } t \in [0, 1/6] \text{ or } [5/6, 1] \\ 0.5 & \text{when } t \in (1/6, 5/6) \end{cases}$

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[4] A negative example is the results of the HHPD-T (Homogenized High Precise Direct integration method with Taylor serials) in Numerical Example 1 and 2 of [12], in which no error should present in 12 significant digits with $N = 20$, $M = 7$ and round-off errors suppressed effectively. However, these two examples still exhibit results with relative errors as large as 5.6e−6 and 3.5e−6.

Table 1(a)
Computing results for Example 1 Case (a) ($\tau = 0.2$)

σ	Method	q	χ	$t = 0.2$	$t = 0.4$	$t = 0.6$	$t = 0.8$	$t = 1.0$	*Max Err.*	*Ave Err.*
	Exact			0.0040780560170512	0.030392601274498	0.091316624352970	0.18373516079120	0.28948444759094		
0.2	*Fou.*	1		0.0040780560170512	0.030392601274498	0.091316624352970	0.18373516079120	0.28948444759094	0	0
	Tay.	5		0.0040780602532102	0.030392681934963	0.091317108372952	0.18373648523200	0.28948696291745	8.69e−06	4.98e−06
		10		0.0040780560170897	0.030392601275059	0.091316624354387	0.18373516079351	0.28948444759383	1.85e−11	1.32e−11
		15		0.0040780560170512	0.030392601274498	0.091316624352970	0.18373516079120	0.28948444759094	0	0
	Leg.	5		0.0040780560170512	0.030392601274498	0.091316624352970	0.18373516079120	0.28948444759094	0	0
	Che.	5		0.0040780560102359	0.030392601235046	0.091316624240635	0.18373516056940	0.28948444724435	1.67e−09	1.32e−09
		10		0.0040780560170512	0.030392601274498	0.091316624352970	0.18373516079120	0.28948444759094	0	0
	Her.	5	4.7	0.0040780749630167	0.030392605219813	0.091316312927249	0.18373510087211	0.28948507938222	4.65e−06	2.14e−06
		10	8.2	0.0040780560167256	0.030392601274432	0.091316624357720	0.18373516079214	0.28948444757980	7.98e−11	3.55e−11
		15	10	0.0040780560170512	0.030392601274498	0.091316624352970	0.18373516079120	0.28948444759094	0	0
	Lag.	5	40	0.0040780577312656	0.030392630409836	0.091316774469675	0.18373555188288	0.28948517181314	2.50e−06	1.53e−06
		10	250	0.0040780560170723	0.030392601274818	0.091316624353777	0.18373516079251	0.28948444759257	1.05e−11	7.46e−12
		15	450	0.0040780560170512	0.030392601274498	0.091316624352970	0.18373516079120	0.28948444759094	0	0
1.0	*Fou.*	1		0.0040780560170512	0.030392601274498	0.091316624352970	0.18373516079120	0.28948444759094	0	0
	Tay.	5		0.0040780602532102	0.030394740412257	0.091397101203399	0.18477604645796	0.29695980037777	2.58e−02	6.49e−03
		10		0.0040780560170897	0.030392601588056	0.091316684666395	0.18373765981868	0.28952901631696	1.54e−04	3.36e−05
		15		0.0040780560170512	0.030392601274498	0.091316624352830	0.18373516075837	0.28948444533607	7.79e−09	1.59e−09
		20		0.0040780560170512	0.030392601274498	0.091316624352970	0.18373516079120	0.28948444758990	3.59e−12	7.19e−13
		25		0.0040780560170512	0.030392601274498	0.091316624352970	0.18373516079120	0.28948444759094	0	0
	Leg.	5		0.0040767650726440	0.030393427289125	0.091317410187721	0.18373388267906	0.28948444748420	3.17e−04	7.19e−05
		10		0.0040780560170376	0.030392601274306	0.091316624352777	0.18373516079119	0.28948444759094	6.32e−12	2.36e−12
		15		0.0040780560170512	0.030392601274498	0.091316624352970	0.18373516079120	0.28948444759094	0	0
	Che.	5		0.0040731151792546	0.030391136312816	0.091313721561499	0.18372618155465	0.28947790051697	1.21e−03	2.73e−04
		10		0.0040780560172895	0.030392601274529	0.091316624353125	0.18373516079181	0.28948444759155	5.84e−11	1.33e−011
		15		0.0040780560170512	0.030392601274498	0.091316624352970	0.18373516079120	0.28948444759094	0	0
	Her.	5	3	0.0041783981122547	0.030615562914170	0.091650489199661	0.18416405707355	0.29001816626476	2.46e−02	7.96e−03
		10	4.5	0.0040950791619067	0.030428219705288	0.091369065467941	0.18380202308410	0.28956541554475	4.17e−03	1.31e−03
		15	5.2	0.0040756046506685	0.030388329735892	0.091309570306873	0.18372529002673	0.28947155767857	6.01e−04	1.83e−04
		20	6.3	0.0040784031577658	0.030393128858030	0.091317815931468	0.18373688473616	0.28948517691818	8.51e−05	2.55e−05
		25	6.6	0.0040780322590043	0.030392535265709	0.091316366220972	0.18373491888241	0.28948450552187	5.83e−06	2.47e−06
	Lag.	5	8	0.0040683665897077	0.030430135720080	0.091293441464888	0.18333019200239	0.28997953007782	2.38e−03	1.56e−03
		10	50	0.0040780560170509	0.030392601274488	0.091316624319494	0.18373517218938	0.28948540270759	3.30e−06	6.72e−07
		15	90	0.0040780560170512	0.030392601274498	0.091316624352970	0.18373516079072	0.28948444745073	4.84e−10	9.74e−11
		20	125	0.0040780560170512	0.030392601274498	0.091316624352970	0.18373516079120	0.28948444759094	0	0

Table 1(b)
Computing results for Example 1 Case (b) ($\tau = 0.25$)

σ	Scheme	q	χ	$t = 0.25$	$t = 0.50$	$t = 0.75$	$t = 1.0$	*Max Err.*	*Ave Err.*
	Exact			0.0051598261767720	0.040641785740383	0.12390102489973	0.22813455793620		
0.25	*Fou.*	10		0.0053070760620238	0.040923608088769	0.12400229218304	0.22805350983505	2.85e−02	9.16e−03
		100		0.0051752218955942	0.040671251691374	0.12391161291072	0.22812608395466	2.98e−03	9.58e−04
		1000		0.0051613726858177	0.040644745612559	0.12390208847169	0.22813370671975	3.00e−04	9.62e−05
	Tay.	1		0.0051598261767720	0.040641785740383	0.12390102489973	0.22813455793620	0	0
	Leg.	1		0.0051598261767720	0.040641785740383	0.12390102489973	0.22813455793620	0	0
	Che.	1		0.0051598261767720	0.040641785740383	0.12390102489973	0.22813455793620	0	0
	Her.	1	∞	0.0051598261767720	0.040641785740383	0.12390102489973	0.22813455793620	0	0
	Lag.	1	∞	0.0051598261767720	0.040641785740383	0.12390102489973	0.22813455793620	0	0
0.5	*Fou.*	10		0.0057488391974313	0.041769090684696	0.12430609383597	0.22781036023579	1.14e−01	3.66e−02
		100		0.0052214090668404	0.040759649562160	0.12394337694411	0.22810066200399	1.19e−02	3.83e−03
		1000		0.0051660122129696	0.040653625229107	0.12390527918758	0.22813115307042	1.20e−03	3.85e−04
	Tay.	1		0.0051598261767720	0.040641785740383	0.12390102489973	0.22813455793620	0	0
	Leg.	1		0.0051598261767720	0.040641785740383	0.12390102489973	0.22813455793620	0	0
	Che.	1		0.0051598261767720	0.040641785740383	0.12390102489973	0.22813455793620	0	0
	Her.	1	∞	0.0051598261767720	0.040641785740383	0.12390102489973	0.22813455793620	0	0
	Lag.	1	∞	0.0051598261767720	0.040641785740383	0.12390102489973	0.22813455793620	0	0
1.0 (P)	*Fou.*	2		0.0047146692291233	0.040108098428318	0.12281785170269	0.22689154740742	8.63e−02	2.84e−02
		5		0.0051850250338136	0.040712363400997	0.12397659407670	0.22823241703259	4.88e−03	1.91e−03
		10		0.0051558771885787	0.040636055083040	0.12389052239446	0.22812182880825	7.65e−04	2.62e−04
		20		0.0051593376637236	0.040641060166485	0.12389970816586	0.22813294965341	9.47e−05	3.26e−05
		50		0.0051597946509530	0.040641739132963	0.12390094020694	0.22813445469003	6.11e−06	2.10e−06
		100		0.0051598222379574	0.040641779911383	0.12390101431205	0.22813454502474	7.63e−07	2.62e−07
	Leg.	4		0.0050010490696469	0.040969353706453	0.12374245959355	0.22813454978267	3.08e−02	1.00e−02
		10		0.0051686467242659	0.040676874642199	0.12390980096949	0.22813455793620	1.71e−03	6.61e−04
		20		0.0051609353099607	0.040647132764370	0.12390213493274	0.22813455793620	2.15e−04	8.89e−05
		40		0.0051597893944471	0.040642528331542	0.12390098827565	0.22813455793620	1.83e−05	6.46e−06
		100		0.0051598242451978	0.040641896610486	0.12390101981112	0.22813455793620	2.73e−06	7.87e−07
		200		0.0051598260225792	0.040641798260543	0.12390102435955	0.22813455793620	3.08e−07	8.58e−08

(*continued on next page*)

Table 1(b) (*continued*)

σ	Scheme	q	χ	$t = 0.25$	$t = 0.50$	$t = 0.75$	$t = 1.0$	*Max Err.*	*Ave Err.*
	Che.	4		0.0048125274337229	0.040883048926843	0.12339178361017	0.22782384608786	6.73e−02	1.97e−02
		10		0.0051771655672832	0.040691800782337	0.12392672001473	0.22815081764131	3.36e−03	1.22e−03
		20		0.0051604813573784	0.040646772755424	0.12390102069894	0.22813327304459	1.27e−04	6.38e−05
		40		0.0051598930428528	0.040644357788493	0.12390102491501	0.22813517467532	6.33e−05	1.97e−05
		100		0.0051598276206804	0.040642030567350	0.12390102489968	0.22813449640814	6.02e−06	1.64e−06
		200		0.0051598262962577	0.040641817670645	0.12390102489973	0.22813458989939	7.86e−07	2.37e−07
	Her.	4	2.3	0.0048618789233704	0.039400475262555	0.12173042569787	0.22450295904029	5.77e−02	3.04e−02
		10	3.1	0.0051332075841345	0.040504973372401	0.12337968466351	0.22855052799346	5.16e−03	3.64e−03
		20	4.6	0.0051527323849853	0.040583715506257	0.12366407954477	0.22802764260485	1.91e−03	1.30e−03
		40	5	0.0051616681163311	0.040605370804191	0.12370991911633	0.22812958274268	1.54e−03	7.04e−04
		100	10	0.0051593647011323	0.040661753666754	0.12384972779584	0.22813477505234	4.91e−04	2.49e−04
		200	15	0.0051597262894199	0.040632099240900	0.12387840227343	0.22813457279288	2.38e−04	1.10e−04
	Lag.	4	8	0.0051309648312881	0.040776851770481	0.12289199151028	0.23305841051650	2.16e−02	9.66e−03
		10	9	0.0051500399768396	0.040710581574260	0.12409365636301	0.22798456503265	1.90e−03	1.45e−03
		20	11	0.0051623065242827	0.040673122226700	0.12400614378212	0.22808731500373	8.48e−04	5.77e−04
		40	25	0.0051604844660361	0.040629044434897	0.12382818146992	0.22813239448892	5.88e−04	2.60e−04
		100	45	0.0051600169736340	0.040634312887662	0.12387793064114	0.22813445821450	1.86e−04	1.02e−04
		200	90	0.0051597832659488	0.040638118034237	0.12390913238912	0.22813455119893	9.02e−05	4.10e−05

Table 1(c)
Computing results for Example 1 Case (c) ($\tau = 1/6$)

σ	Scheme	q	χ	$t = 1/6$	$t = 2/6$	$t = 3/6$	$t = 4/6$	$t = 5/6$	$t = 1.0$	*Max Err.*	*Ave Err.*
	Exact			0.0023777566797146	0.016195630294705	0.043081930337610	0.082081715177607	0.13192290155464	0.18884072781011		
1/6	*Fou.*	1		0.0023777566797146	0.016195630294705	0.043081930337610	0.082081715177607	0.13192290155464	0.18884072781011	0	0
	Tay.	5		0.0023777575019757	0.016195638404923	0.043081945392424	0.082081736648469	0.13192292874540	0.18884076622802	5.01e−07	3.11e−07
		10		0.0023777566797182	0.016195630294755	0.043081930337704	0.082081715177741	0.13192290155481	0.18884072781031	3.09e−12	1.79e−12
		15		0.0023777566797146	0.016195630294705	0.043081930337610	0.082081715177607	0.13192290155464	0.18884072781011	0	0
	Leg.	5		0.0023777566797146	0.016195630294705	0.043081930337610	0.082081715177607	0.13192290155464	0.18884072781011	0	0
	Che.	5		0.0023777566783807	0.016195630290606	0.043081930330904	0.082081715168521	0.13192290154346	0.18884072779570	5.61e−10	2.07e−10
		10		0.0023777566797146	0.016195630294705	0.043081930337610	0.082081715177607	0.13192290155464	0.18884072781011	0	0
	Her.	5	4.7	0.0023777676676033	0.016195625685721	0.043081888312761	0.082081277334054	0.13192309338519	0.18884181179201	5.74e−06	3.07e−06
		10	7.9	0.0023777566794564	0.016195630294800	0.043081930338883	0.082081715165160	0.13192290154975	0.18884072778090	1.55e−10	8.12e−11
		15	9.8	0.0023777566797146	0.016195630294705	0.043081930337610	0.082081715177607	0.13192290155464	0.18884072781011	0	0
	Lag.	5	40	0.0023777568636914	0.016195632616627	0.043081932946372	0.082081716244817	0.13192289932449	0.18884072212876	1.43e−07	5.69e−8
		10	250	0.0023777566797166	0.016195630294734	0.043081930337664	0.082081715177684	0.13192290155474	0.18884072781023	1.79e−12	1.03e−12
		15	450	0.0023777566797146	0.016195630294705	0.043081930337610	0.082081715177607	0.13192290155464	0.18884072781011	0	0
1/3 (P)	*Fou.*	2		0.0029342225534551	0.017267770808659	0.044091583183432	0.083002164019323	0.13219001102547	0.18844193052334	2.34e−01	5.65e−02
		5		0.0026300688296083	0.016688337888188	0.043546016640865	0.082504898371911	0.13204271758636	0.18865809991937	1.06e−01	2.57e−02
		10		0.0025100445389326	0.016454202668780	0.043325486383322	0.082303809230643	0.13198567577244	0.18874491227926	5.56e−02	1.35e−02
		20		0.0024455169320090	0.016328145547169	0.043206750937880	0.082195537754103	0.13195504208513	0.18879163046463	2.85e−02	6.91e−03
		50		0.0024052639414872	0.016249432681657	0.043132608744068	0.082127928399755	0.13193594747642	0.18882079463591	1.16e−02	2.81e−03
		100		0.0023915787936925	0.016222665993487	0.043107396244634	0.082104937337603	0.13192945688742	0.18883071144677	5.81e−03	1.41e−03
	Leg.	4		0.0023860173607371	0.016195630286682	0.043081930330128	0.082081715170862	0.13193116786317	0.18884072781226	3.47e−03	5.89e−04
		10		0.0023786408225713	0.016195630294705	0.043081930337610	0.082081715177607	0.13192378576356	0.18884072781011	3.72e−04	6.31e−05
		20		0.0023778913885868	0.016195630294705	0.043081930337610	0.082081715177607	0.13192303626577	0.18884072781011	5.67e−05	9.61e−06
		40		0.0023777753871341	0.016195630294705	0.043081930337610	0.082081715177607	0.13192292026213	0.18884072781011	7.87e−06	1.33e−06
		100		0.0023777593168481	0.016195630294705	0.043081930337610	0.082081715177607	0.13192290419179	0.18884072781011	1.11e−06	1.88e−07
		200		0.0023777570543069	0.016195630294705	0.043081930337610	0.082081715177607	0.13192290192924	0.18884072781011	1.58e−07	2.67e−08
	Che.	4		0.0023843575623214	0.016187639983310	0.043064280249243	0.082055050067049	0.13189278681804	0.18878711692974	2.78e−03	7.53e−4
		10		0.0023790203694292	0.016196106547923	0.043082888744843	0.082083121428812	0.13192599599401	0.18884340182361	5.31e−04	1.06e−4
		20		0.0023778816060910	0.016195591892845	0.043081854159092	0.082081603940504	0.13192288308874	0.18884051832815	5.25e−05	9.88e−06
		40		0.0023777708654117	0.016195632766303	0.043081939734110	0.082081692433207	0.13192290123860	0.18884071272238	5.97e−06	1.12e−06
		100		0.0023777548384670	0.016195630478843	0.043081931029527	0.082081710931265	0.13192290155956	0.18884072939849	7.74e−07	1.44e−07
		200		0.0023777569103063	0.016195630266712	0.043081930394347	0.082081714662127	0.13192290155469	0.18884072767716	9.70e−08	1.78e−08
	Her.	4	2.5	0.0023757811833350	0.016120560976422	0.042338336252395	0.080627645027271	0.12921121413728	0.18489607537806	2.09e−02	1.36e−02
		10	3.8	0.0023772558618677	0.016314276769009	0.043159351842017	0.082148615608760	0.13266132882260	0.19055270402946	9.07e−03	4.14e−03
		20	4.4	0.0023784040602317	0.016199211513996	0.043068869283480	0.082063452277940	0.13186623686924	0.18876147271016	4.30e−04	3.11e−04
		40	10	0.0023778238428391	0.016196663520522	0.043084816180605	0.082080135119267	0.13190578402192	0.18884720997700	1.30e−04	5.71e−05
		100	14	0.0023777923714764	0.016195405310208	0.043082200683223	0.082082000392314	0.13192116770265	0.18884090193203	1.50e−05	8.79e−06
		200	17	0.0023777596556050	0.016195553861416	0.043082210570014	0.082081758719776	0.13192373022585	0.18884067797887	6.50e−06	3.26e−06

(*continued on next page*)

Table 1(c) (*continued*)

σ	Scheme	q	χ	$t = 1/6$	$t = 2/6$	$t = 3/6$	$t = 4/6$	$t = 5/6$	$t = 1.0$	*Max Err.*	*Ave Err.*
	Lag.	4	8	0.0023815465159794	0.016136042521015	0.042257335832216	0.080384198344462	0.12857155283706	0.18403568154959	2.54e−02	1.60e−02
		10	20	0.0023777359416212	0.016096985026121	0.043035472796445	0.082047275000551	0.13149788403777	0.18798084802223	6.09e−03	2.56e−03
		20	35	0.0023781540686158	0.016193158185323	0.043070781082218	0.082066730616961	0.13187325778208	0.18879640498964	3.76e−04	2.29e−04
		40	55	0.0023777018431273	0.016196656003263	0.043083801688076	0.082080512180604	0.13191081344650	0.18884460369258	9.16e−05	4.28e−05
		100	115	0.0023777301064926	0.016195429216148	0.043082141727540	0.082081907165213	0.13192430899246	0.18884084187455	1.24e−05	7.02e−06
		200	220	0.0023777591741583	0.016195670058222	0.043082088880487	0.082081739190201	0.13192221041777	0.18884070195200	5.24e−06	2.14e−06
1/2 (P)	*Fou.*	2		0.0031963118485985	0.017806780867474	0.045404431300697	0.083430183741690	0.13225266503645	0.18816958961901	3.44e−01	8.67e−02
		5		0.0027546919842874	0.016934573180966	0.044152885289974	0.082699944490723	0.13207521756682	0.18853229888777	1.59e−01	3.99e−02
		10		0.0025772614500581	0.016584009468807	0.043646808785345	0.082406482339020	0.13200403860700	0.18867819218795	8.39e−02	2.11e−02
		20		0.0024793550182242	0.016394398891194	0.043370705704925	0.082247977694568	0.13196401277736	0.18875766967905	4.27e−02	1.07e−02
		50		0.0024190146919049	0.016276333521256	0.043199202124642	0.082149219113733	0.13193959835856	0.18880700140689	1.74e−02	4.36e−03
		100		0.0023984911077427	0.016236184438700	0.043140863679142	0.082115636309032	0.13193129328474	0.18882377933772	8.72e−03	2.19e−03
	Leg.	4		0.0024228062512977	0.016172702041840	0.043081928620655	0.082058700315920	0.13196800758592	0.18884072852060	1.89e−02	3.50e−03
		10		0.0023805022615701	0.016194581818103	0.043081930337610	0.082080670656870	0.13192564972766	0.18884072781011	1.15e−03	2.09e−04
		20		0.0023781714004434	0.016195518161804	0.043081930337610	0.082081603096365	0.13192331638041	0.18884072781011	1.74e−04	3.10e−05
		40		0.0023778135717655	0.016195625104173	0.043081930337610	0.082081709981609	0.13192295844893	0.18884072781011	2.39e−05	4.12e−06
		100		0.0023777654453910	0.016195629907997	0.043081930337610	0.082081714791038	0.13192291032304	0.18884072781011	3.69e−06	6.30e−07
		200		0.0023777577482237	0.016195630282484	0.043081930337610	0.082081715165588	0.13192290262355	0.18884072781011	4.49e−07	7.64e−08
	Che.	4		0.0024799960815053	0.016215201836466	0.043177181287455	0.082106088189037	0.13203548243216	0.18885653883446	4.30e−02	7.94e−03
		10		0.0023806452749160	0.016194540516704	0.043081630030542	0.082078947375078	0.13192249641122	0.18883593054312	1.21e−03	2.25e−04
		20		0.0023781940421732	0.016195523323564	0.043081939491806	0.082081498742197	0.13192312478557	0.18884041656980	1.84e−04	3.28e−05
		40		0.0023776779873919	0.016195615997362	0.043081930388209	0.082081732566165	0.13192290303421	0.18884077097715	3.31e−05	5.74e−06
		100		0.0023777525799846	0.016195628309797	0.043081930337844	0.082081712880372	0.13192290158006	0.18884073257656	1.72e−06	3.17e−07
		200		0.0023777564792847	0.016195630147554	0.043081930337612	0.082081715379373	0.13192290155436	0.18884072843210	8.43e−08	1.65e−08
	Her.	4	2.3	0.0023009693948287	0.016142428168910	0.042931161870115	0.081004123894340	0.12985114515528	0.18594702218515	3.23e−02	1.39e−02
		10	3.1	0.0023870096852374	0.016200744467699	0.042945510893018	0.082526625503444	0.13114296367243	0.18965645110541	5.91e−03	3.84e−02
		20	4.2	0.0023764902339649	0.016183243417383	0.043045186052540	0.082030215407834	0.13174410905123	0.18868498746562	1.36e−03	8.26e−04
		40	10	0.0023780214599644	0.016198879103845	0.043074337453472	0.082086106370582	0.13188214820823	0.18882704536428	3.09e−04	1.54e−04
		100	14	0.0023776639778690	0.016196304812046	0.043083598327414	0.082081108637077	0.13191507061636	0.18884146616667	5.94e−05	3.17e−05
		200	17	0.0023777714325011	0.016195785790999	0.043082462310274	0.082081792436694	0.13192047733831	0.18884081404660	1.84e−05	7.99e−06
	Lag.	4	6	0.0024018281417553	0.016192367664761	0.042984721059173	0.080341462831099	0.12859507669503	0.18391761391053	2.61e−02	1.42e−02
		10	12	0.0023807751210719	0.016210892261392	0.043037989775017	0.081935388937887	0.13168518985189	0.18858481057539	1.80e−03	1.36e−03
		20	20	0.0023773591873346	0.016199626165994	0.043070739258270	0.082065922817206	0.13187008186294	0.18879507882674	4.00e−04	2.51e−04
		40	30	0.0023778369415305	0.016196663499880	0.043079441514282	0.082083123898125	0.13193634456528	0.18883681827831	1.02e−04	4.92e−05
		100	45	0.0023777281490899	0.016195428124117	0.043082423347911	0.082081517315598	0.13192029769266	0.18884094444079	1.97e−05	9.87e−06
		200	70	0.0023777615831207	0.016195583325052	0.043082100731581	0.082081690506851	0.13192362958636	0.18884075393400	5.52e−06	2.48e−06

1.0 (**P**)	*Fou.*	2		0.0023286153181357	0.015813787332898	0.042807021106569	0.081466172062014	0.13141494807579	0.18816466850762	2.36e−02	1.09e−02
		5		0.0023819438331397	0.016194343948155	0.043078154002269	0.082075512015825	0.13191736712993	0.18882644551251	1.76e−03	3.54e−04
		10		0.0023839318808826	0.016205302519210	0.043095315842435	0.082099506658246	0.13194510790602	0.18886425549813	2.60e−03	6.69e−04
		20		0.0023782229443276	0.016196258276684	0.043082893028162	0.082082899675676	0.13192446651375	0.18884234020096	1.96e−04	4.87e−05
		50		0.0023777915926179	0.016195680412256	0.043082004446166	0.082081808772106	0.13192302229674	0.18884085376896	1.47e−05	3.70e−06
		100		0.0023777616461814	0.016195637727969	0.043081941061454	0.082081728970198	0.13192291907516	0.18884074623384	2.09e−06	5.33e−07
	Leg.	4		0.0024327896222065	0.016160040020541	0.043105668166184	0.082046242070043	0.13197789917226	0.18884072908971	2.31e−02	4.46e−03
		10		0.0023896748890678	0.016202644000865	0.043071110981636	0.082088660267284	0.13193484347616	0.18884072781011	5.01e−03	9.79e−04
		20		0.0023793093234532	0.016195182822408	0.043081685533609	0.082081268522085	0.13192445392660	0.18884072781011	6.53e−04	1.17e−04
		40		0.0023779770464640	0.016195635901189	0.043081996810040	0.082081720611532	0.13192312195730	0.18884072781011	9.27e−05	1.61e−05
		100		0.0023777826400872	0.016195630230263	0.043081931424538	0.082081715113097	0.13192292755688	0.18884072781011	1.09e−05	1.86e−06
		200		0.0023777604921471	0.016195630295836	0.043081930127962	0.082081715178729	0.13192290538400	0.18884072781011	1.60e−06	2.73e−07
	Che.	4		0.0025490908132030	0.016278265939263	0.043205375152273	0.082255120741982	0.13227055059578	0.18909904049039	7.21e−02	1.44e−02
		10		0.0023872739576398	0.016202154926698	0.043061738839651	0.082085057184676	0.13192633996844	0.18883174366992	4.00e−03	8.31e−04
		20		0.0023799149660688	0.016195573779733	0.043082245038094	0.082082241088707	0.13192620605190	0.18884240928608	9.08e−04	1.60e−04
		40		0.0023775295334778	0.016195629520004	0.043081911844321	0.082081750298365	0.13192281510436	0.18884064615842	9.55e−05	1.63e−05
		100		0.0023777397760915	0.016195630329103	0.043081930782471	0.082081714212412	0.13192290468634	0.18884073252177	7.11e−06	1.20e−06
		200		0.0023777538800757	0.016195630296408	0.043081930378043	0.082081715249351	0.13192290162109	0.18884072811180	1.18e−06	1.97e−07
	Her.	4	2.1	0.0028488966256657	0.015507458603337	0.039565245709936	0.072827210015836	0.12945637714210	0.14470620062011	2.34e−01	1.15e−01
		10	3	0.0022527950084534	0.016077880249174	0.042858833631956	0.082252736888613	0.12740665083234	0.17864613759925	5.40e−02	2.59e−02
		20	4.1	0.0023733188825727	0.016190390318338	0.042806641062287	0.082249298371647	0.13191288538612	0.18859229297728	6.39e−03	2.00e−03
		40	10	0.0023839807711468	0.016196812828130	0.043110484541530	0.082072901255105	0.13192299757440	0.18887215915290	2.62e−03	6.05e−04
		100	11	0.0023760957664504	0.016195687490133	0.043079395436389	0.082080967181913	0.13192290117819	0.18884771189024	6.99e−04	1.35e−04
		200	15	0.0023779248138311	0.016195627538025	0.043081499097719	0.082081728137232	0.13192290155933	0.18883914144487	7.07e−05	1.49e−05
	Lag.	4	6	0.0024308773235303	0.015851613062568	0.043524350303915	0.083556687600538	0.13174665436469	0.18365864840737	2.74e−02	1.68e−02
		10	10	0.0023910168693082	0.016212257448323	0.043046268212468	0.082068421405777	0.13224927535116	0.18811212764458	5.58e−03	2.32e−03
		20	17	0.0023804411501819	0.016194716647538	0.043103615892244	0.082056039376686	0.13192192312997	0.18886359556197	1.13e−03	3.55e−04
		40	26	0.0023770967846259	0.016195742088005	0.043078632058494	0.082079962824940	0.13192291206411	0.18884434610923	2.78e−04	6.69e−05
		100	40	0.0023780971209372	0.016195637576092	0.043081424061669	0.082081771281638	0.13192290160810	0.18884130684396	1.43e−04	2.65e−05
		200	60	0.0023776708257942	0.016195630723461	0.043081866383193	0.082081714199430	0.13192290155498	0.18884084125232	3.61e−05	6.37e−06

Table 2
Computing results for Example 2 ($\tau = 1$)

σ	Scheme	q	χ	$t = 1$	$t = 2$	$t = 3$	$t = 4$	$t = 5$
	Exact			−0.0028725691880784	−0.0055719060437928	−0.012663857267705	−0.017986160763256	−0.022042009307299
1	*Fou.*	3		−0.0028913020105808	−0.0056436620058650	−0.012660336324243	−0.017950423906431	−0.022042901761227
		5		−0.0028850212611440	−0.0056174404943680	−0.012661217737690	−0.017963196766453	−0.022042601409579
	Tay.	5		−0.0029810049405471	−0.0061305352268415	−0.012972026288565	−0.018470255855609	−0.022234953742012
		10		−0.0028727786969539	−0.0055735324770636	−0.012664346349533	−0.017987801550327	−0.022042001164779
	Leg.	5		−0.0028725691919143	−0.0055719060405882	−0.012663857270863	−0.017986160759866	−0.022042009310533
		10		−0.0028725691880784	−0.0055719060437928	−0.012663857267705	−0.017986160763256	−0.022042009307299
	Che.	5		−0.0028725322133937	−0.0055718583876692	−0.012663812698615	−0.017986121723248	−0.022041991698467
		10		−0.0028725691880620	−0.0055719060438024	−0.012663857267748	−0.017986160763196	−0.022042009307270
	Her.	5	4.7	−0.0029454620481240	−0.0060536357484520	−0.012762372992923	−0.018141666654242	−0.021973150715907
		10	6	−0.0028773623903679	−0.0055968846288228	−0.012669352112929	−0.017994496181525	−0.022034310594687
	Lag.	5	25	−0.0028809544881310	−0.0056387640752217	−0.012679579257254	−0.018029717429961	−0.022027186418560
		10	40	−0.0028723088705163	−0.0055702110633538	−0.012663501251825	−0.017985788349844	−0.022042451884389
3	*Fou.*	8		−0.0029547353686996	−0.0056762374755304	−0.012765252533019	−0.017925038990824	−0.021905849884019
		15		−0.0029180956813115	−0.0056297354816882	−0.012720601253670	−0.017953253927333	−0.021967631388473
	Tay.	15		−0.0028725691529806	−0.0055615598309123	*Err.* > 1	*Err.* > 1	*Err.* > 1
		30		−0.0028725691880784	−0.0055719060437975	−0.012663859998825	−0.017986181788460	−0.022042035050785
	Leg.	15		−0.0028725691869464	−0.0055719060424614	−0.012663857267705	−0.017986160764426	−0.022042009308608
		30		−0.0028725691880784	−0.0055719060437928	−0.012663857267705	−0.017986160763256	−0.022042009307299
	Che.	15		−0.0028725691856713	−0.0055719060407189	−0.012663857266346	−0.017986160767247	−0.022042009316009
		30		−0.0028725691880784	−0.0055719060437928	−0.012663857267705	−0.017986160763256	−0.022042009307299
	Her.	15	4.6	−0.0028725691880941	−0.0055718887025356	−0.012636932777632	−0.017769235563787	−0.020127692035700
		30	5	−0.0028725691880820	−0.0055719030534940	−0.012644142686105	−0.017901908828997	−0.021743483796538
	Lag.	15	40	−0.0028725691880792	−0.0055719049094674	−0.012653525712146	−0.017911741568089	−0.021951121180366
		30	70	−0.0028725691880783	−0.0055719061617504	−0.012664331794732	−0.017989427016793	−0.022051589342146
9	*Fou.*	13		−0.0031464843827361	−0.0059066244612578	−0.012997616933441	−0.018252660716128	−0.022192962699074
		25		−0.0030119094148463	−0.0057490336718576	−0.012838653946426	−0.018127883773350	−0.022121469807017
	Tay.	25		−0.0028725691880784	−0.0055719060560162	−0.012664971117210	−0.021610810358100	*Err.* > 1
		50		−0.0028725691880784	−0.0055719060437928	−0.012663857267705	−0.017986160763257	−0.022042009345264
	Leg.	25		−0.0028725691768889	−0.0055719055624983	−0.012663856728209	−0.017986160106803	−0.022042008615408
		50		−0.0028725691880784	−0.0055719060437928	−0.012663857267705	−0.017986160763256	−0.022042009307299
	Che.	25		−0.0028725688939511	−0.0055719053380005	−0.012663856464353	−0.017986159729742	−0.022042008260977
		50		−0.0028725691880784	−0.0055719060437928	−0.012663857267705	−0.017986160763256	−0.022042009307299
	Her.	25	4	*Err.* > 1	*Err.* > 1	*Err.* > 1	*Err.* > 1	*Err.* > 1
		50	4.5	*Err.* > 1	*Err.* > 1	*Err.* > 1	*Err.* > 1	*Err.* > 1
	Lag.	25	35	−0.0028733567198454	−0.0055737312538922	−0.012660253653771	−0.017787554531146	−0.016055168082480
		50	55	−0.0028724689000622	−0.0055718383136881	−0.012664305359104	−0.017986168097366	−0.022042777165613

	Exact			−0.023282157271915	−0.022009533859889	−0.020745964721212	−0.017319413914028		
1	*Fou.*	3		−0.023200087237754	−0.022061476098174	−0.020651347373148	−0.017391215883992	1.29e−02	4.03e−03
		5		−0.023229745113509	−0.022042650334462	−0.020686063516131	−0.017365632070297	8.17e−03	2.59e−03
	Tay.	5		−0.023405792469808	−0.021906564941314	−0.020505864929915	−0.017101061143243	1.00e−01	2.58e−02
		10		−0.023283045693795	−0.022008610299004	−0.020746056416985	−0.017318098349628	2.92e−04	7.28e−05
	Leg.	5		−0.023282157268874	−0.022009533862778	−0.020745964718590	−0.017319413916329	1.34e−09	3.35e−10
		10		−0.023282157271915	−0.022009533859889	−0.020745964721212	−0.017319413914028	0	0
	Che.	5		−0.023282153390809	−0.022009554313167	−0.020745987330261	−0.017319447368769	1.29e−05	3.56e−06
		10		−0.023282157271916	−0.022009533859872	−0.020745964721165	−0.017319413914015	5.71e−12	2.15e−12
	Her.	5	4.7	−0.023258063830412	−0.021592656875618	−0.020564497785461	−0.017535502172681	8.65e−02	1.92e−02
		10	6	−0.023280469044128	−0.022000277059876	−0.020739845959236	−0.017313615867568	4.48e−03	9.47e−04
	Lag.	5	25	−0.023276089509434	−0.021955228676475	−0.020699808933952	−0.017255782289184	1.20e−02	3.10e−03
		10	40	−0.023282235517395	−0.022010595488969	−0.020746759270439	−0.017318889121387	3.04e−04	6.49e−05
3	*Fou.*	8		−0.023105574615004	−0.022008152971726	−0.020834056652950	−0.017475922670638	2.86e−02	9.54e−03
		15		−0.023184906395378	−0.022007668615884	−0.020792765579598	−0.017404194075538	1.58e−02	5.26e−03
	Tay.	15		*Err.* > 1	*Err.* > 1	−0.038861069143500	*Err.* > 1	>1	>1
		30		−0.023282178864352	−0.022009527301816	−0.020745942986779	−0.017319385719603	1.63e−06	7.17e−07
	Leg.	15		−0.023282157271915	−0.022009533858890	−0.020745964720157	−0.017319413914028	3.94e−10	9.49e−11
		30		−0.023282157271915	−0.022009533859889	−0.020745964721212	−0.017319413914028	0	0
	Che.	15		−0.023282157280491	−0.022009533855433	−0.020745964730704	−0.017319413906305	8.38e−10	3.99e−10
		30		−0.023282157271915	−0.022009533859889	−0.020745964721212	−0.017319413914028	0	0
	Her.	15	4.6	−0.024058989544397	−0.021551666874172	−0.019516910219531	−0.017053730347238	8.68e−02	2.55e−02
		30	5	−0.023155374214085	−0.021879739626007	−0.020541517251843	−0.017505294595737	1.35e−02	5.75e−03
	Lag.	15	40	−0.023184242814078	−0.021855732483668	−0.020605626743629	−0.017200020933856	6.99e−03	3.77e−03
		30	70	−0.023286516760037	−0.022014129485242	−0.020751594669313	−0.017324019074534	4.35e−04	1.76e−04
9	*Fou.*	13		−0.023286532731943	−0.021893130651420	−0.020549393078864	−0.017101726389220	9.54e−02	2.57e−02
		25		−0.023286205271107	−0.021947813164796	−0.020644808215192	−0.017208794417418	4.85e−02	1.33e−02
	Tay.	25		*Err.* > 1	*Err.* > 1	*Err.* > 1	*Err.* > 1	>1	>1
		50		−0.023282709937701	−0.023807073670540	*Err.* > 1	*Err.* > 1	>1	>1
	Leg.	25		−0.023282156659852	−0.022009533332903	−0.020745964749075	−0.017319413914028	8.64e−08	2.80e−08
		50		−0.023282157271915	−0.022009533859889	−0.020745964721212	−0.017319413914028	0	0
	Che.	25		−0.023282156481228	−0.022009533259134	−0.020745964648957	−0.017319414008651	1.27e−07	5.20e−08
		50		−0.023282157271915	−0.022009533859889	−0.020745964721212	−0.017319413914028	0	0
	Her.	25	4	*Err.* > 1	*Err.* > 1	*Err.* > 1	*Err.* > 1	>1	>1
		50	4.5	*Err.* > 1	*Err.* > 1	*Err.* > 1	*Err.* > 1	>1	>1
	Lag.	25	35	*Err.* > 1	*Err.* > 1	*Err.* > 1	*Err.* > 1	>1	>1
		50	55	−0.023283571989808	−0.022000986625473	−0.020757910593408	−0.016757796158528	3.24e−02	3.73e−03

Table 3
Computing results for Example 3 ($\sigma = 10$, $\tau = 1$)

	Exact	*Fou.* ($q = 1$)	*Tay.* ($q = 25$)	*Tay.* ($q = 50$)
$t = 1$	0.28948444759094	0.28948444759094	0.28948444759094	0.28948444759094
$t = 2$	0.28643644038235	0.28643644038235	0.28643644190808	0.28643644038235
$t = 3$	0.044770673639352	0.044770673639352	0.044954520975635	0.044770673639352
$t = 4$	−0.22477376876186	−0.22477376876186	*Err.* > 1	−0.22477376876145
$t = 5$	−0.26030080629514	−0.26030080629514	*Err.* > 1	−0.26030075300510
$t = 6$	−0.076406111850063	−0.076406111850063	*Err.* > 1	−0.075600239716378
$t = 7$	0.16854379112162	0.16854379112162	*Err.* > 1	*Err.* > 1
$t = 8$	0.23101423142554	0.23101423142554	*Err.* > 1	*Err.* > 1
$t = 9$	0.097122726421885	0.097122726421885	*Err.* > 1	*Err.* > 1
$t = 10$	−0.12070806920101	−0.12070806920101	*Err.* > 1	*Err.* > 1
Max Err.		0	>1	>1
Ave Err.		0	>1	>1
	Leg. ($q = 25$)	*Leg.* ($q = 50$)	*Che.* ($q = 25$)	*Che.* ($q = 50$)
$t = 1$	0.28948454772159	0.28948444759094	0.28948424533627	0.28948444759094
$t = 2$	0.28643599025644	0.28643644038235	0.28643567565385	0.28643644038235
$t = 3$	0.044770077910507	0.044770673639352	0.044769806095915	0.044770673639352
$t = 4$	−0.22477425399277	−0.22477376876186	−0.22477441693831	−0.22477376876186
$t = 5$	−0.26030082026557	−0.26030080629514	−0.26030069787855	−0.26030080629514
$t = 6$	−0.076405645815689	−0.076406111850063	−0.076405345229344	−0.076406111850063
$t = 7$	0.16854438047807	0.16854379112162	0.16854455704066	0.16854379112162
$t = 8$	0.23101467637872	0.23101423142554	0.23101474068621	0.23101423142554
$t = 9$	0.097122619651028	0.097122726421885	0.097122736763108	0.097122726421885
$t = 10$	−0.12070806920101	−0.12070806920101	−0.12070801499980	−0.12070806920101
Max Err.	1.33e−05	0	1.94e−05	0
Ave Err.	3.01e−06	0	4.34e−06	0
	Her. ($q = 25$, $\chi = 2.5, 3, 5, 20$)	*Her.* ($q = 50$, $\chi = 4, 8, 15, 40$)	*Lag.* ($q = 25$, $\chi = 40$)	*Lag.* ($q = 50$, $\chi = 40$)
$t = 1$	*Err.* > 1	*Err.* > 1	0.28948445637446	0.28948444759081
$t = 2$	*Err.* > 1	*Err.* > 1	0.28643589508286	0.28643644037761
$t = 3$	*Err.* > 1	*Err.* > 1	0.044765466518636	0.044770673708650
$t = 4$	*Err.* > 1	*Err.* > 1	−0.22483140019865	−0.22477376839401
$t = 5$	*Err.* > 1	*Err.* > 1	−0.26010062642333	−0.26030080437654
$t = 6$	*Err.* > 1	*Err.* > 1	−0.073949935517905	−0.076406046877078
$t = 7$	*Err.* > 1	*Err.* > 1	0.13612498177316	0.16854505198036
$t = 8$	*Err.* > 1	*Err.* > 1	*Err.* > 1	0.23101745931902
$t = 9$	*Err.* > 1	*Err.* > 1	*Err.* > 1	0.096918422339354
$t = 10$	*Err.* > 1	*Err.* > 1	*Err.* > 1	−0.12144373807804
Max Err.	>1	>1	>1	6.09e−3
Ave Err.	>1	>1	>1	8.22e−4

The computing results of $V_1(t)$ are presented in Table 1. Exact results of Case (a) and (b) are computed from their analytical solutions available in [1].[5]

As to Case (c), an analytical solution is difficult to obtain. The exact results are computed by a hybrid method with *Fou.* used over interval $[0, 1/6] \cup [5/6, 1]$ and *Tay.* used over interval $(1/6, 5/6)$. The mechanism of this kind of hybrid methods is similar to the HPD-S& L (High Precision Direct integration scheme with Sinusoidal and Linear approximation) method in [1] and produces identical results.

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[5] It should be indicated that, the analytical solution for $x(t)$ in [1] is incorrect and should be $x(t) = \{\pi(\pi^2 + 2\zeta^2 + 2\zeta\eta - 1)\exp[(\eta - \zeta)t] - \pi(\pi^2 + 2\zeta^2 - 2\zeta\eta - 1)\exp[(-\zeta - \eta)t] + 2\eta[-2\zeta\pi\cos\pi t + (1 - \pi^2)\sin\pi t]\}/2\eta(4\zeta^2\pi^2 + \pi^4 - 2\pi^2 + 1)$ in which $\eta = \sqrt{\zeta^2 - 1} \approx 0.99874921777191\mathrm{i}$.
The analytical solution for Case (b) contains typos, $Q(\tau)$ of Eq. (44) in [1] should be $Q(\tau) = \exp(-\zeta\tau)/\eta\sin\eta\tau$.

Example 2

$$
\begin{gathered}
\dot{v} = Av + f, \quad \text{where} \quad v(0) = \begin{Bmatrix} V_1(0) \\ V_2(0) \\ V_3(0) \\ V_4(0) \\ V_5(0) \\ V_6(0) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}, \quad f = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ F(t) \end{Bmatrix} \\
A = \begin{bmatrix} -425/501 & 425/1002 & 0 & 5/501 & 0 & 0 \\ 425/1002 & -425/501 & 425/1002 & 0 & 5/501 & 0 \\ 0 & 425/1002 & -425/1002 & 0 & 0 & 5/501 \\ -380495/2004 & 34015/501 & 36125/2004 & -425/501 & 425/1002 & 0 \\ 34015/501 & -57395/334 & 57395/668 & 425/1002 & -425/501 & 425/1002 \\ 36125/2004 & 57395/668 & -104155/1002 & 0 & 425/1002 & -425/1002 \end{bmatrix}
\end{gathered}
\tag{37}
$$

The original 2nd order form of Eq. (37) depicts a 3-DOF (3-Degree-Of-Freedom) system. It is adapted from Example 4 in [1], in which the original $F(t)$ in [1] is too simple for performance evaluation and substituted by a more complicated composite function of polynomial and trigonometric functions as follows:

$$F(t) = \sin(0.1t^2 - t) + \cos(3t + 2) + \sqrt{2}(t - 5)^2/100, \quad t \in [0, 9] \tag{38}$$

Its physical meaning can be a particular applied loading in dynamics.

The computing results of $V_3(t)$ are presented in Table 2. Similar to Example 1 Case (c), it is difficult to obtain an analytical solution. Moreover, any hybrid method does not work. We compute the exact results in this way: Increase q one by one until the *Max Err.* between two successive times is less than 1e−20 and the results then are regarded as exact results. Its correctness is confirmed by the sameness and convergence of the results with any of the 3 approximation methods for any of $\sigma = 1$, 3 or 9.

Example 3. This example is to show the capability of different methods to approximate trigonometric functions across periods as a whole. The ODEs of Example 1 Case (a) are adopted once again, with the domain of $f(t)$ extended to $[0, 10]$.

The computing results of $V_1(t)$ are presented in Table 3.

5.2. Performance evaluation and comments

5.2.1. A quantitative measure

Previous PTI works commonly analyze numerical results in a qualitative way. For the sake of a more objective evaluation, we provide an interesting quantitative measure: The Performance Indicator (*PI*) for a certain method *Md* is defined as follows:

$$PI_{Md} = \sum_{TC} -\log_{10}(Max\ Err._{TC}) \tag{39}$$

where *TC* denotes each testcase using the method *Md* and $Max\ Err._{TC}$ is the *Max Err.* of the testcase *TC*. The logarithmic property of the *PI*s counts the total number of significant digits of the method *Md* for all testcases. It is worthy of notice that such count is in a fractional way.

When applying this qualitative measure, challenges persist at extreme situations, which are forcibly tackled by the following supplementary rules:

(1) If a method reaches 0 *Max Err.* when $q = q_T$, *Max Err.* is naturally and definitely assumed to be 0 for all $q \geqslant q_T$.

Table 4
PI for each method in 3 examples

	Ex. 1(a)	Ex. 1(b)	Ex. 1(c)	Ex. 2	Ex. 3
Fou.	1.0000	0.1370	0.3079	0.0947	1.0000
Tay.	0.4468	0.6667	0.1280	−0.0557	−0.2549
Leg.	0.6970	0.7476	0.3778	0.7484	0.4289
Che.	0.6290	0.7420	0.3530	0.6481	0.4257
Her.	0.3194	0.7189	0.2627	−0.1339	−0.4314
Lag.	0.4990	0.7270	0.2897	0.0730	−0.0350

(2) 0 *Max Err*. is counted as 17, considering the fact that the result is at least accurate to 14 significant digits with an extra bonus of 3.
(3) If any of the results of a testcase exhibits $Err. > 1$, the count is $-1 - No.Err$ where $No.Err$ is the number of $Err. > 1$ results in that testcase.
(4) The absence of *Tay.* for piecewise functions in Example 1 Case (b) and (c) leads to a 0 count. This is an evaluation in favor of the Taylor approximation since no penalty is imposed for its incapability in these testcases.

Table 4 presents the average *PI*s among all instances. The final *PI*s are divided by a constant to normalize to 1. Note that *Fou.* reaches one in Example 1 Case (a) and Example 3 simply because $F(t) = \sin(\pi t)$. With this exception excluded, *Leg.* ranks the 1st among the six.

5.2.2. Some comments

① Whatever looking through Tables 1–3 directly, or *PI*s in Table 4 measuring average performance, it is evident that *Leg.* is the most excellent one with much higher precision and better convergence. And *Che.* possesses similar performance and properties, but a little inferior to *Leg.* They both surpass the other four methods greatly with errors hundreds of thousands of times smaller.

② The performance of all methods for piecewise functions is unsatisfactory. This is mainly due to the fact that, polynomial and trigonometric polynomial functions are continuous, derivable and smooth, which is hard to form sharp corners.

③ Tables 2 and 3 show that, when σ is large *Tay.*, *Her.* and *Lag.* may completely lose their precision if q is not large enough. This does not happen to *Leg.*, *Che.* and *Fou.*

④ It is quite amazing that, as a polynomial, an up to 50th order Legendre/Chebyshev series can approximate an irregular complex composite function (Example 2) and trigonometric functions across several periods (Example 3) as a whole with great accuracy. Particularly, *Leg.* ($q = 25$) approximates a sinusoidal function across as long as 5 periods with only *Max Err*. 1.33e−05 and *Ave Err*. 3.01e−06. And this amount of error is eliminated when q increases to 50.

6. Concluding remarks

In this paper, four orthogonal polynomial series, Legendre, Chebyshev, Hermite and Laguerre series, are used to approximate the non-homogeneous term for the PTI in transient analysis. In order to eliminate inverse matrix computation, they are all incorporated with the DET. Various structures subjected to transient dynamic loading are adopted to test the performance of the 6 contemporary approximation methods, including Fourier and Taylor approximation proposed in previous works. It has been shown that all six methods are efficient and have reasonable precision. In particular, Legendre is the best with much higher precision and better convergence; Chebyshev approximation is also good, but only slightly inferior to Legendre approximation. The other four approximation methods usually produce results with errors hundreds of thousands of times larger. Hermite and Laguerre approximation may be useful for some special non-homogeneous terms, but do not work very well in our numerical examples. As an explicit integration method, our work provides effective alternative forms for the PTI, making it more competitive with the Newmark method and other available good methods.

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References

[1] Lin JH, Shen WP, Williams FW. A high precise direct integration scheme for structures subjected to transient dynamic loading. Comput Struct 1995;56(2):113–20.

[2] Fung TC. A precise time-step integration method by step-response and impulsive response matrices for dynamic problems. Int J Numer Methods Eng 1997;40:4501–27.

[3] Zhong WX, Williams FW. A precise time step integration method. J Mech Eng Sci 1994;208:427–30.

[4] Zhong WX. On precise integration method. J Comput Appl Math 2004;163:59–78.

[5] Kong XD. Precise time integration algorithms of ordinary differential equations and application in multibody system dynamics. PhD dissertation, Dalian Univ Tech, Dalian, China, 1998.

[6] Zhong WX, Zhu JP, Zhong XX. On a new time integration method for solving time dependent partial differential equations. Comput Methods Appl Mech Eng 1996;130:163–78.

[7] Zhang HW, Chen BS, Gu YX. An adaptive algorithm of precise integration for transient analysis. Acta Mech Solida Sinica 2001;14(3):215–24.

[8] Chen BS, Gu YX, Guan ZQ, Zhang HW. Nonlinear transient heat conduction analysis with precise time integration method. Numer Heat Transfer 2001;40(4):325–41.

[9] Gu YX, Chen BS, Zhang HW, Grandhi RV. Sensitivity analysis method for linear and nonlinear transient heat conduction with precise time integration. Struct Multidisc Optim 2002;24:23–37.

[10] Zhong WX, Zhu JP. Precise time integration for the matrix Riccati equation. J Numer Methods Appl 1997;17(1):26–35.

[11] Zhong WX, Zhu JP. A precise integration algorithms for matrix Riccati differential equations. Lect Notes Comput Sci 2001;2073:947–56.

[12] Zhou G, Wang YX, Jia GQ, Chen J. A homogenized high precise direct integration based on Taylor Serials. J Shanghai Jiaotong Univ 2001;35(12):1916–9.

[13] Gu YX, Chen BS, Zhang HW, Guan ZQ. Precise time-integration method with dimensional expanding for structural dynamic equations. AIAA J 2001;39(12):2394–9.