Manifold-valued Dirichlet Processes

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http://pages.cs.wisc.edu/~hwkim/projects/dp-mglm/

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Motivation

Regression

$$f : \mathbf{R}^d \to \mathcal{M}$$

$$\mathcal{M} = \mathrm{SPD}(n)$$

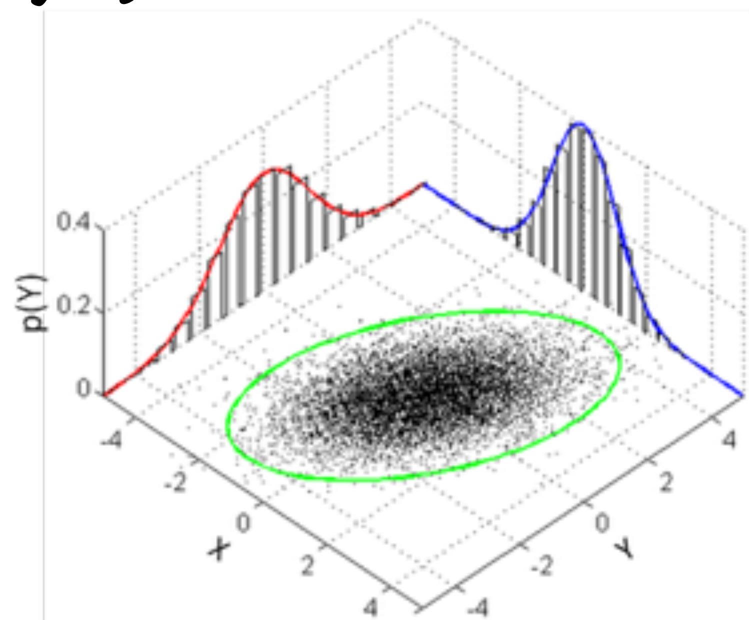
Motivation

Regression

$$f : \mathbf{R}^d \to \mathcal{M}$$

$$\mathcal{M} = \mathrm{SPD}(n)$$

- Covariance matrix

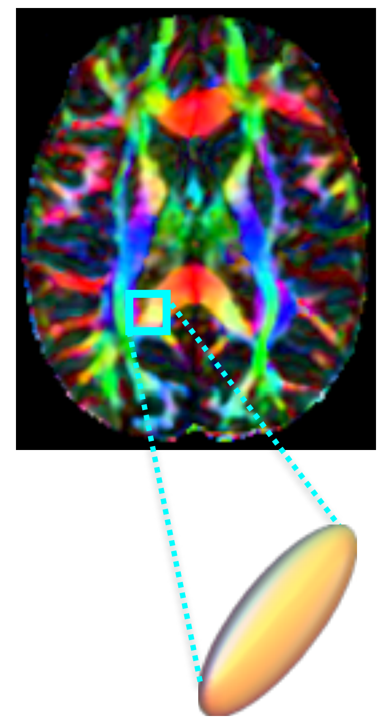


Motivation

Regression

$$f : \mathbf{R}^d \to \mathcal{M}$$

DTI



$$\mathcal{M} = \mathrm{SPD}(n)$$

- Covariance matrix
- Diffusion tensor imaging

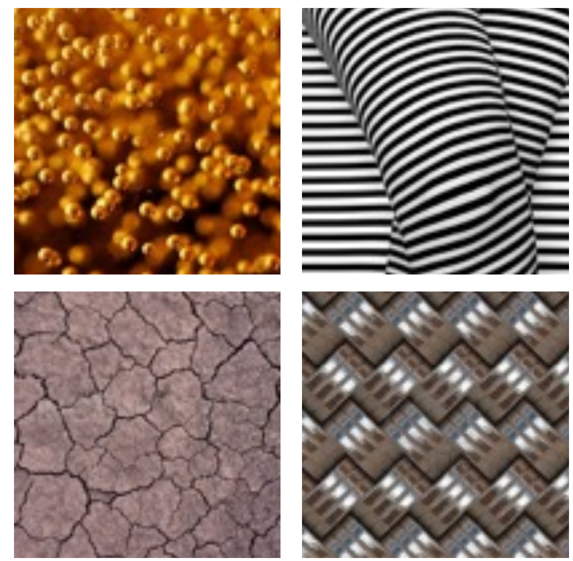
Motivation

Regression

$$f : \mathbf{R}^d \to \mathcal{M}$$

$$\mathcal{M} = \mathrm{SPD}(n)$$

- Covariance matrix
- Diffusion tensor imaging
- Region covariance
- …



M.Cimpoi et al., CVPR 2014

Linear regression on manifolds

$$f : R \to \mathcal{M} \qquad y_i = \mathrm{EXP}(\mathrm{EXP}(B, vx_i), \epsilon)$$

GR — Fletcher, IJCV 2013.

$$f : R^n \to \mathcal{M} \qquad y_i = \mathrm{EXP}(\mathrm{EXP}(B, \sum_{j=1}^{d} V^j x_i^j), \epsilon)$$

MGLM — Kim et al., CVPR 2014.

IR — Zhu et al., JASA, 2009.

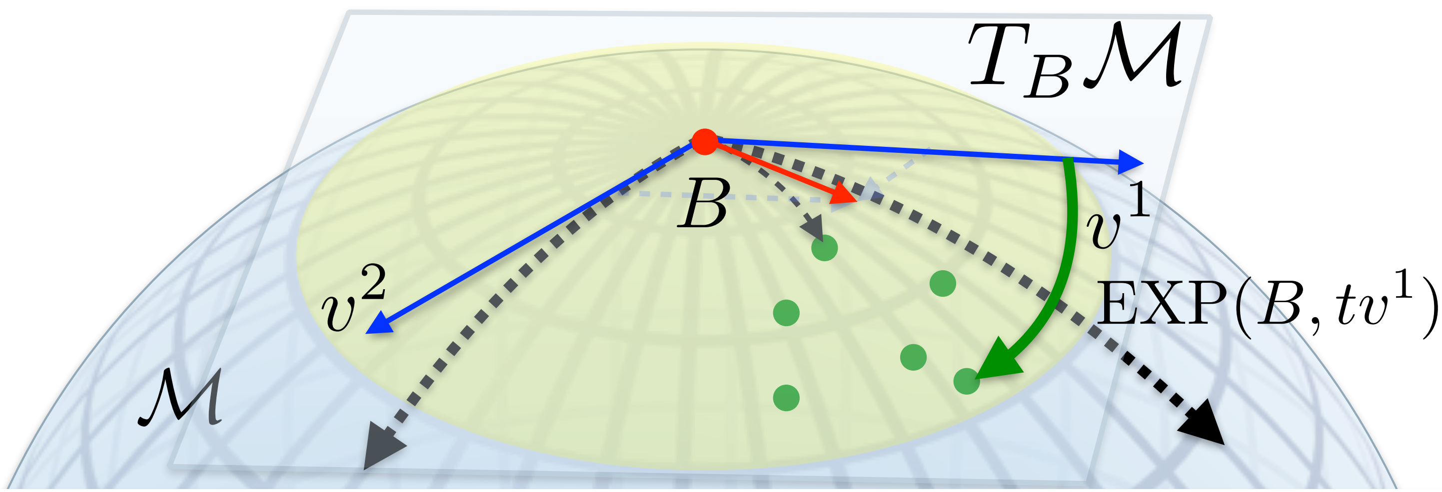
Motivations

Build models on manifolds

Locally defined parametric models

PGA (Fletcher et al., 2004), GR (Fletcher, IJCV 2013),

MGLM (Kim et al., CVPR 2014), RCCA (Kim et al., ECCV 2014)



Euclidean models

- General Linear Model

$$\boldsymbol{y}_i = \boldsymbol{\beta}^0 + \boldsymbol{\beta}^1 x_i^1 + \ldots + \boldsymbol{\beta}^d x_i^d + \epsilon$$

What if data have nonlinear correlation?

Euclidean models

- General Linear Model

$$\boldsymbol{y}_i = \boldsymbol{\beta}^0 + \boldsymbol{\beta}^1 x_i^1 + \ldots + \boldsymbol{\beta}^d x_i^d + \epsilon$$

- Generalized Linear Model (fixed link function)

$$\boldsymbol{y}_i = g^{-1}(\boldsymbol{\beta}^0 + \boldsymbol{\beta}^1 x_i^1 + \ldots + \boldsymbol{\beta}^d x_i^d) + \epsilon$$

- Single Index Model (searching link function)

$$\boldsymbol{y}_i = g^{-1}(\beta^0 + \beta^1 x_i^1 + \ldots + \beta^d x_i^d) + \epsilon$$

Euclidean models

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- Single Index Model (searching link function)

$$\boldsymbol{y}_i = g^{-1}(\beta^0 + \beta^1 x_i^1 + \ldots + \beta^d x_i^d) + \epsilon$$

Link functions for manifold-valued response?

Euclidean models

- GLM

$$\boldsymbol{y}_i = \boldsymbol{\beta}^0 + \boldsymbol{\beta}^1 x_i^1 + \ldots + \boldsymbol{\beta}^d x_i^d + \epsilon$$

- ?-GLM

$$\boldsymbol{y}_i = \underline{\boldsymbol{\beta}_{\color{red}i}}^0 + \underline{\boldsymbol{\beta}_{\color{red}i}}^1 x_i^1 + \ldots + \underline{\boldsymbol{\beta}_{\color{red}i}}^d x_i^d + \epsilon$$

Euclidean models

- GLM

$$\boldsymbol{y}_i = \boldsymbol{\beta}^0 + \boldsymbol{\beta}^1 x_i^1 + \ldots + \boldsymbol{\beta}^d x_i^d + \epsilon$$

- DP-GLM

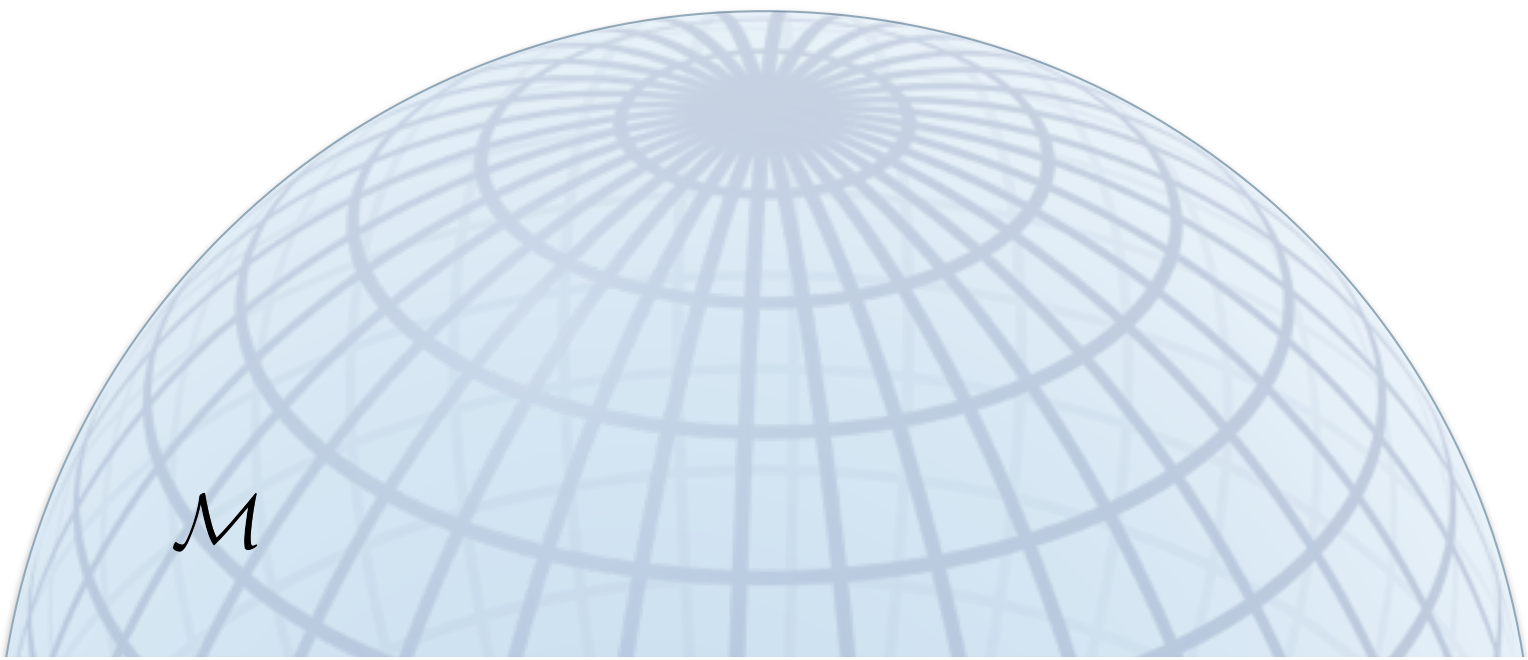
$$\boldsymbol{y}_i = \underline{\boldsymbol{\beta}_{\boldsymbol{i}}}^0 + \underline{\boldsymbol{\beta}_{\boldsymbol{i}}}^1 x_i^1 + \ldots + \underline{\boldsymbol{\beta}_{\boldsymbol{i}}}^d x_i^d + \epsilon$$

$$(x_i, y_i)|\theta_i \sim F(\theta_i), \theta_i|G \sim G, G \sim DP(G_0, \nu)$$

Hannah et al., JMLR 2011

MGLM on manifolds

$$y_i = \mathrm{EXP}(\mathrm{EXP}(B, \sum_{j=1}^{d} V^j x_i^j), \epsilon)$$



Kim. et al., CVPR 2014

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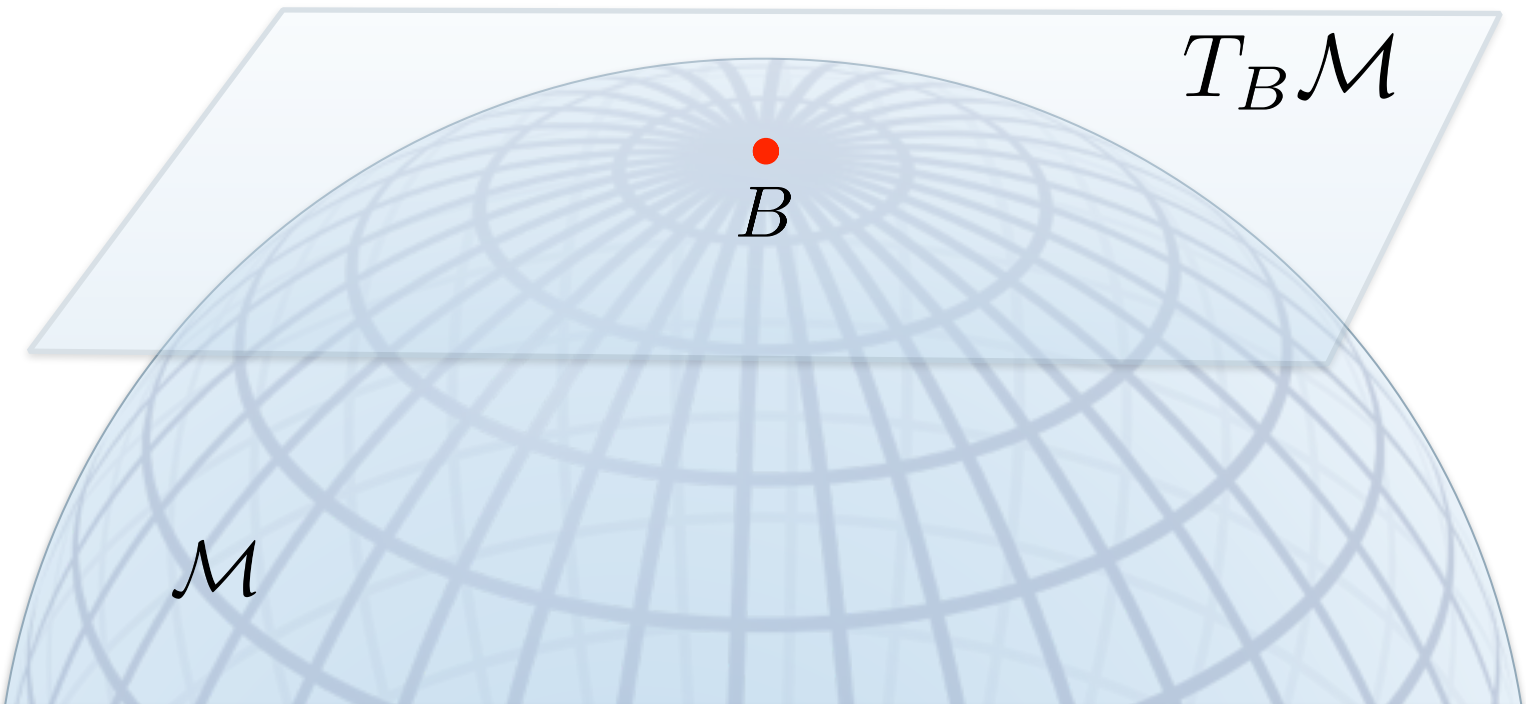

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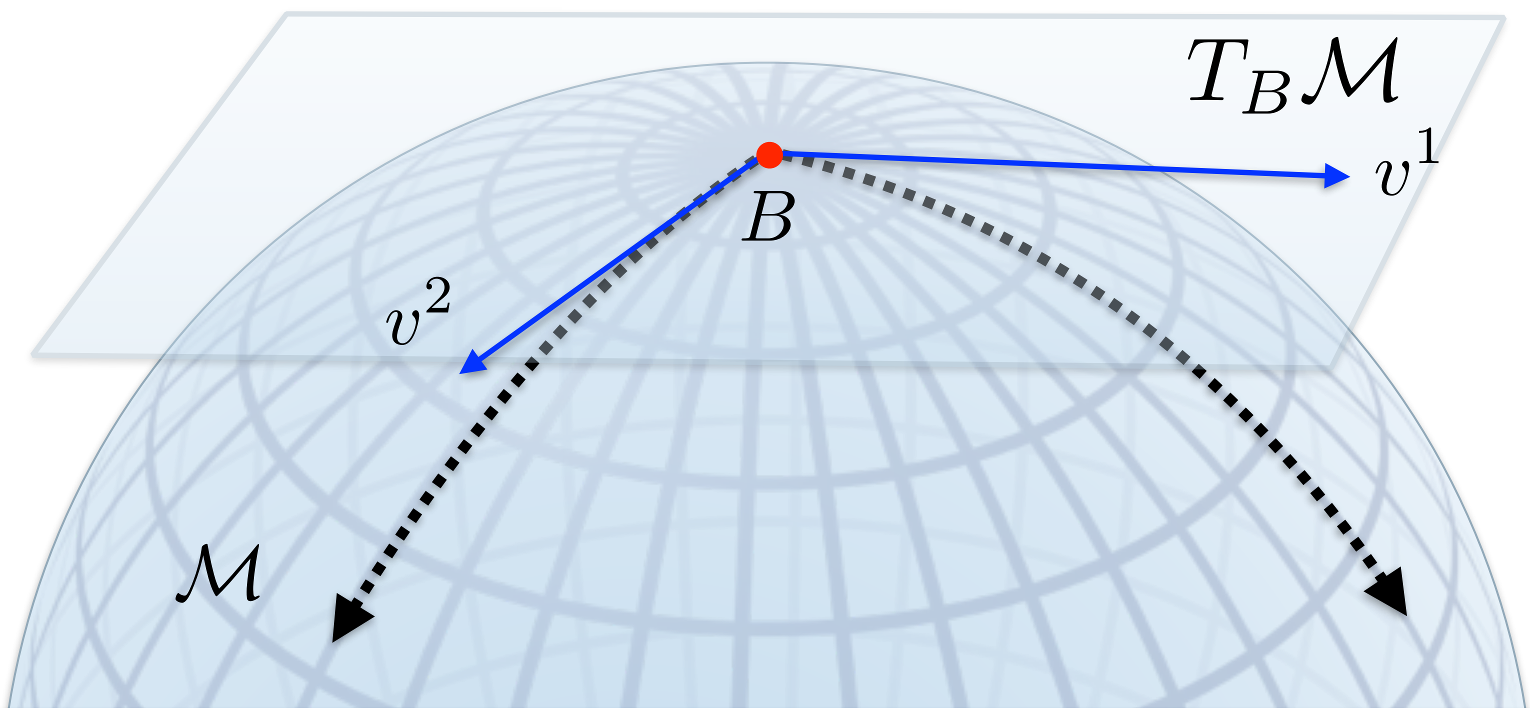

Kim. et al., CVPR 2014

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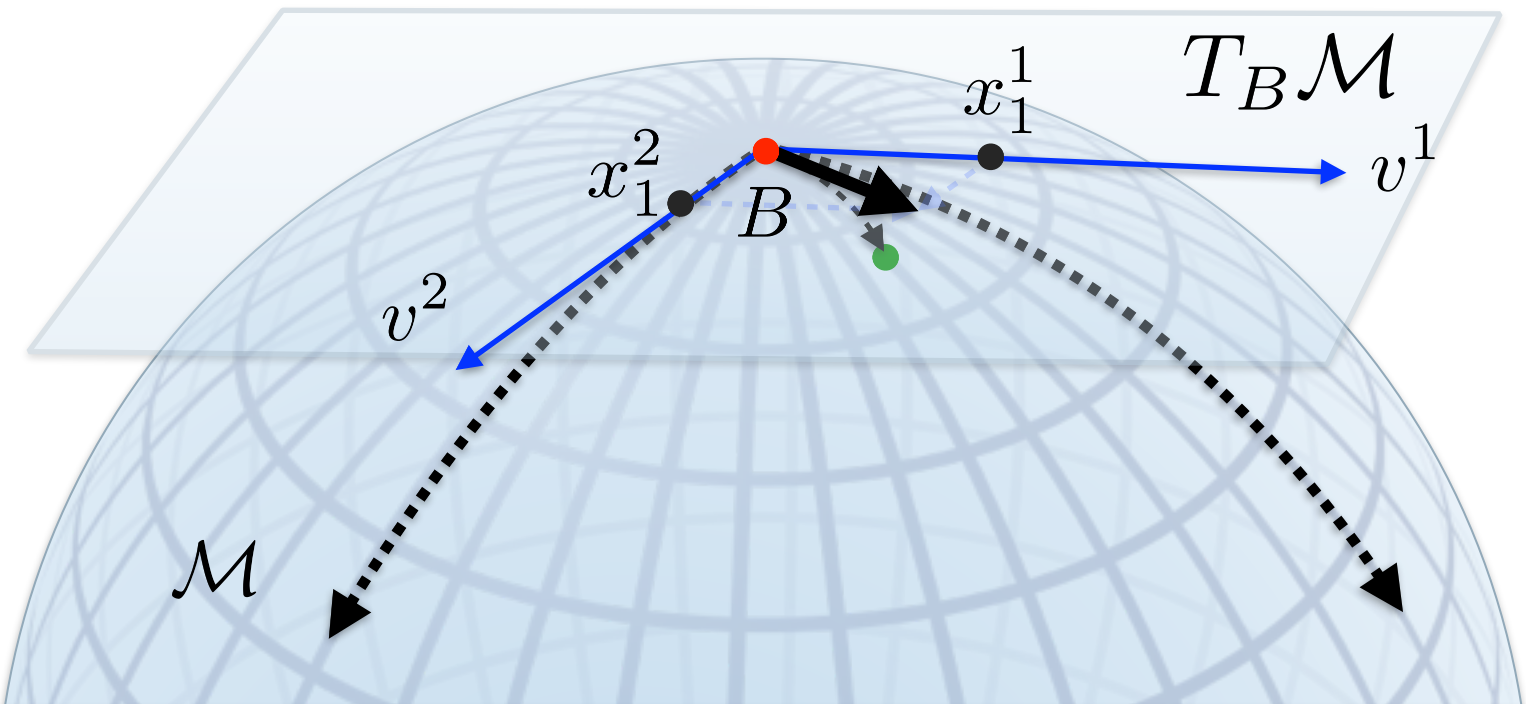

Kim. et al., CVPR 2014

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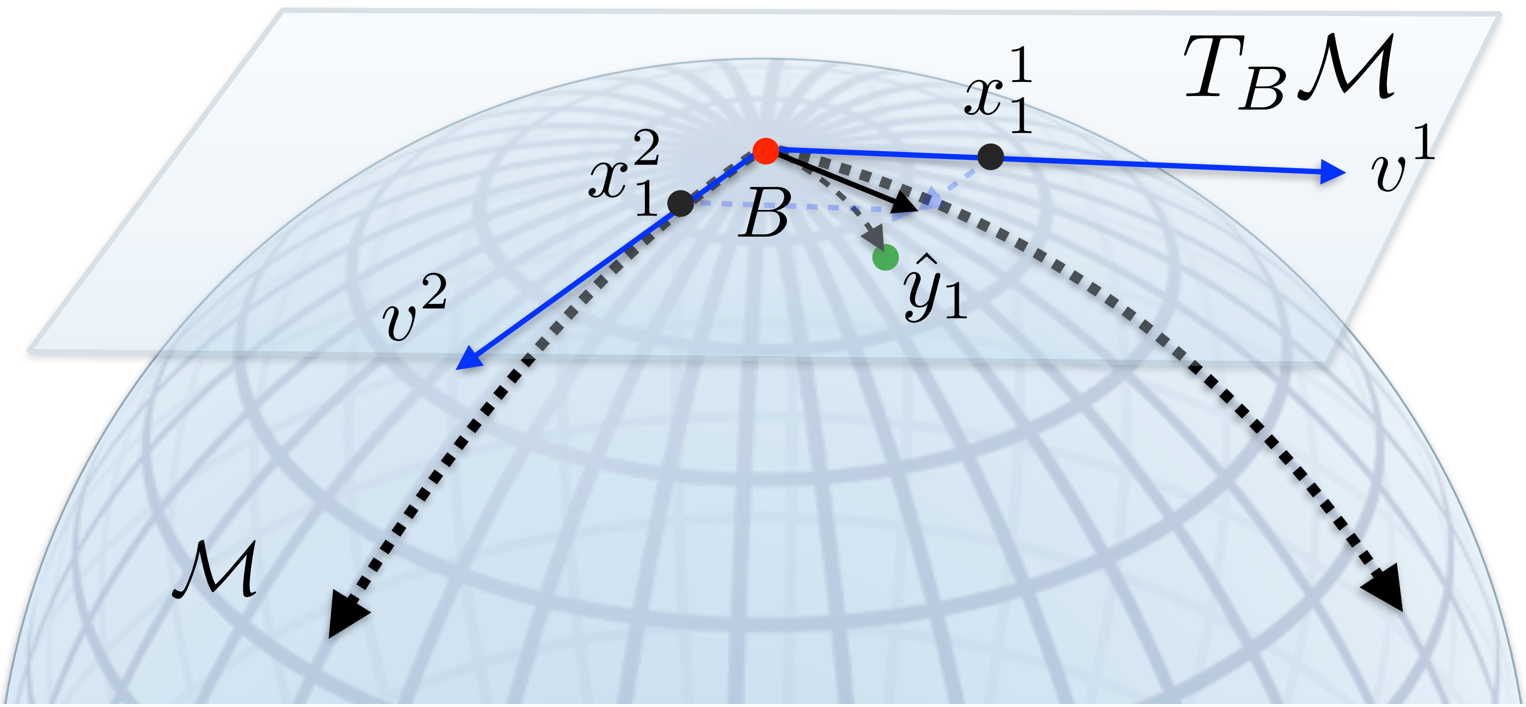
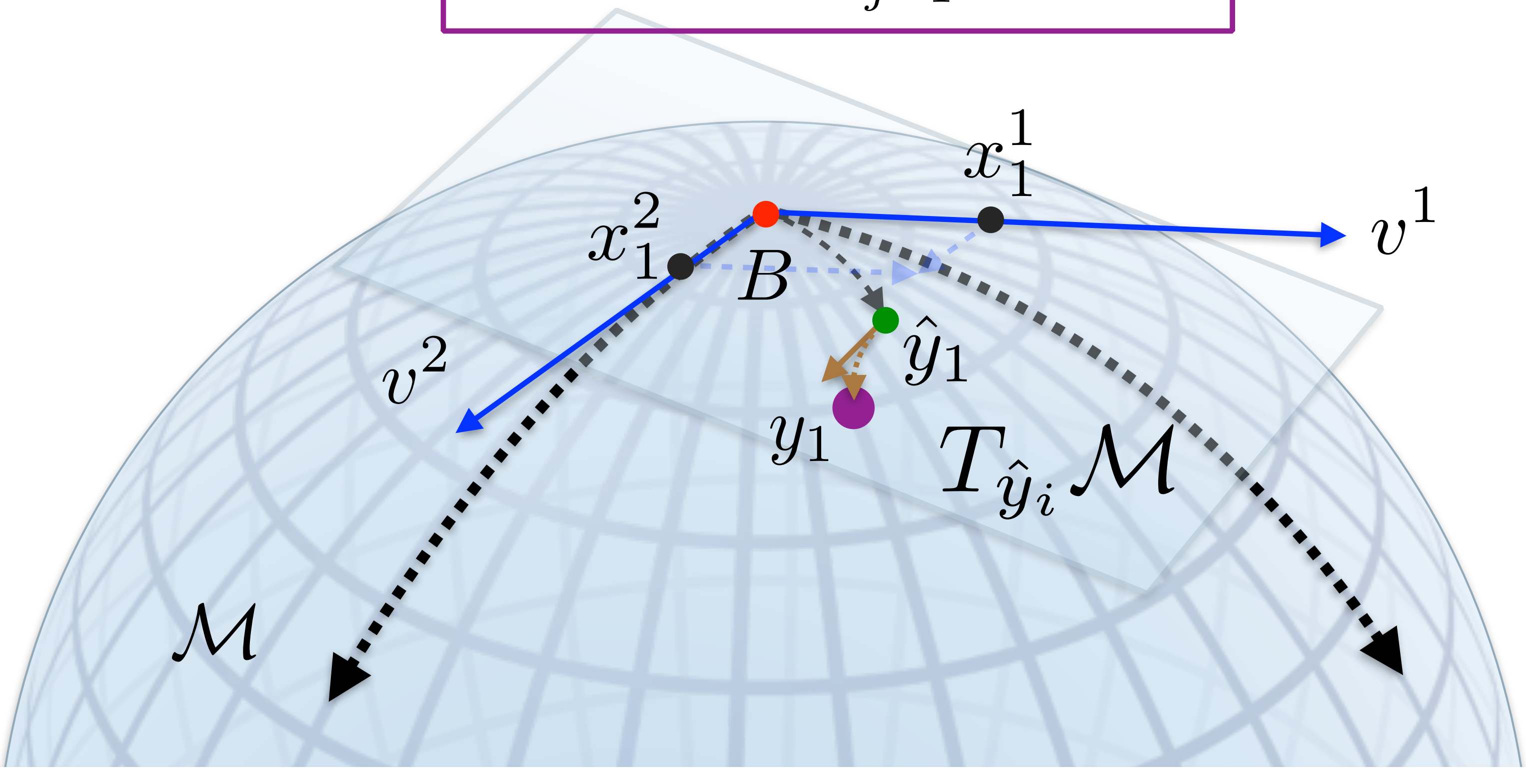


Kim. et al., CVPR 2014

MGLM on manifolds

$$y_i = \boxed{\mathrm{EXP}(\mathrm{EXP}(B, \sum_{j=1}^{d} V^j x_i^j), \epsilon)}$$

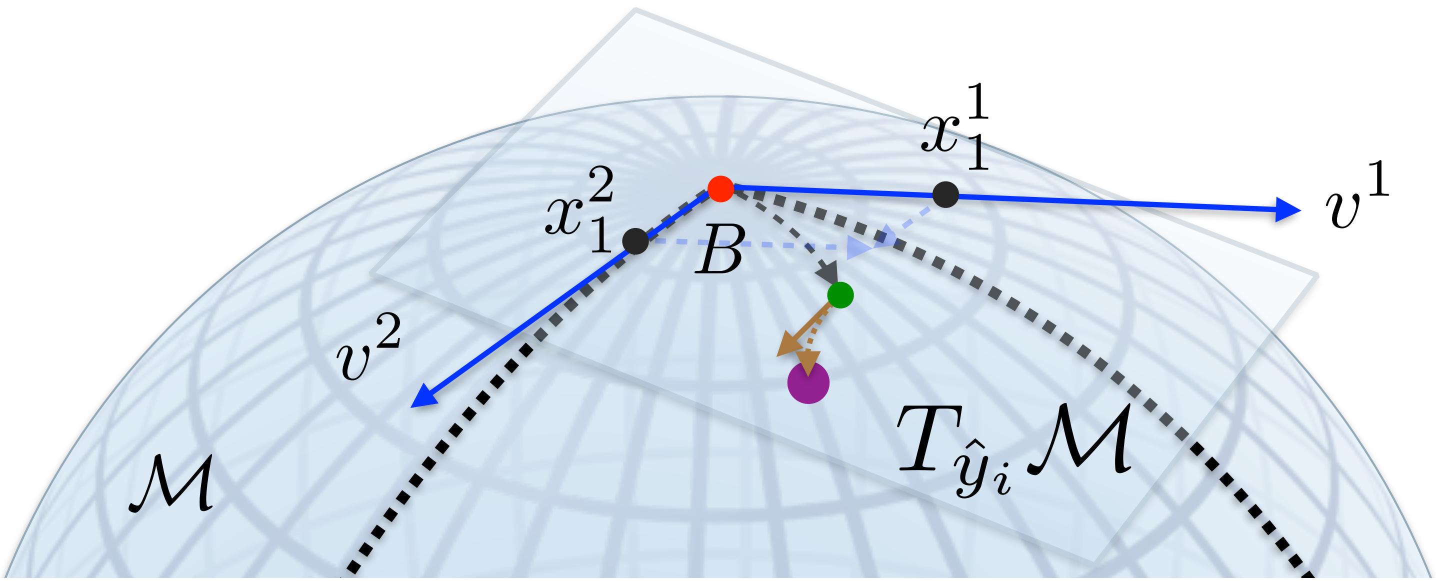


Kim. et al., CVPR 2014

MGLM on manifolds

$$\underline{y_i} = \text{EXP}(\text{EXP}(\underline{B}, \sum_{j=1}^{d} \underline{V^j} x_i^j), \underline{\epsilon})$$

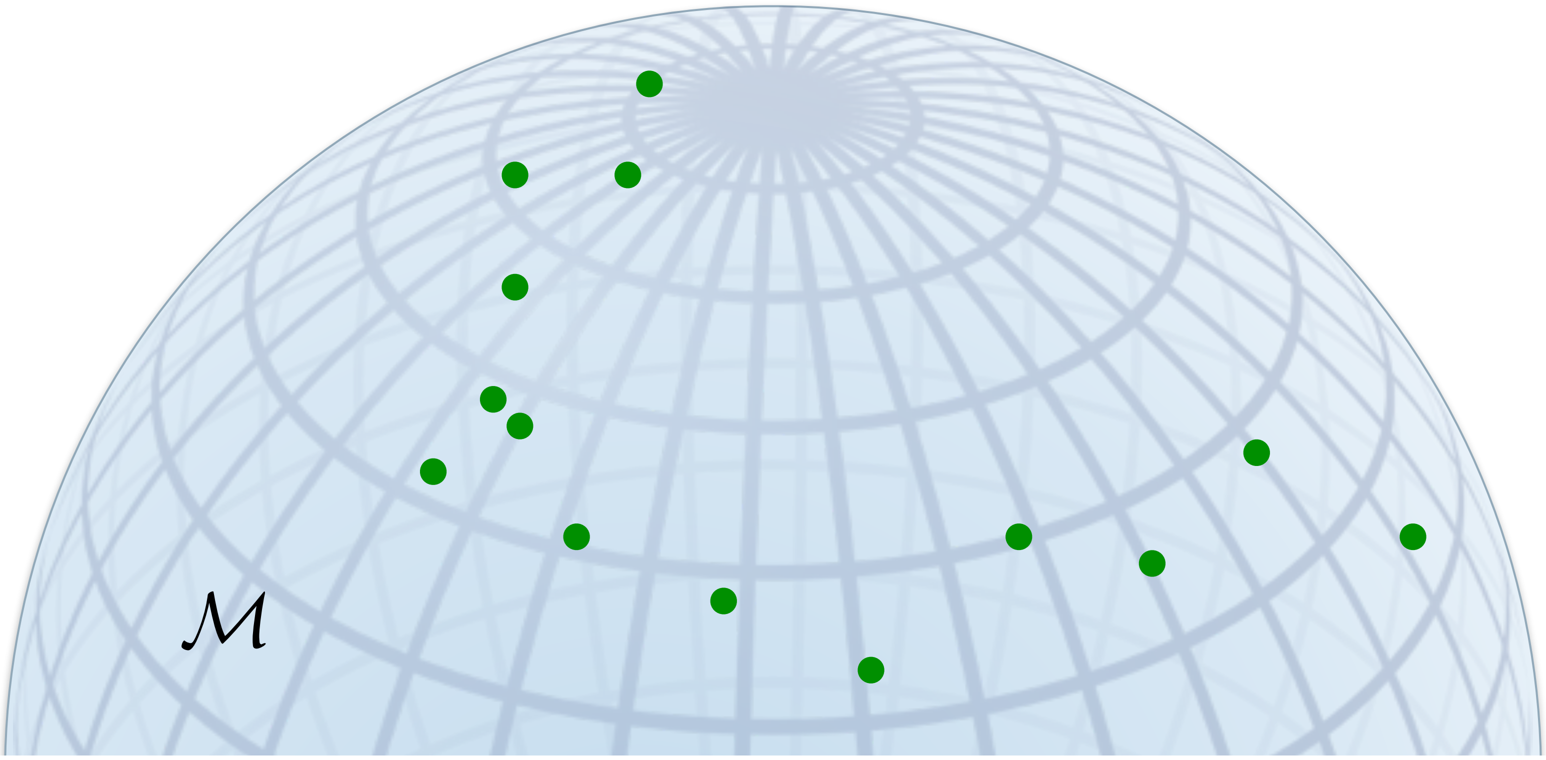
$$\underline{y_i} = \underline{\beta^0} + \underline{\beta^1} x_i^1 + \underline{\beta^2} x_i^2 + \ldots + \underline{\beta^d} x_i^d + \underline{\epsilon}$$



Kim. et al., CVPR 2014

DP-MGLM

$$y_i = \mathrm{EXP}(\mathrm{EXP}(B_i, \sum_{j=1}^{d} V_i^j x_i^j), \epsilon)$$



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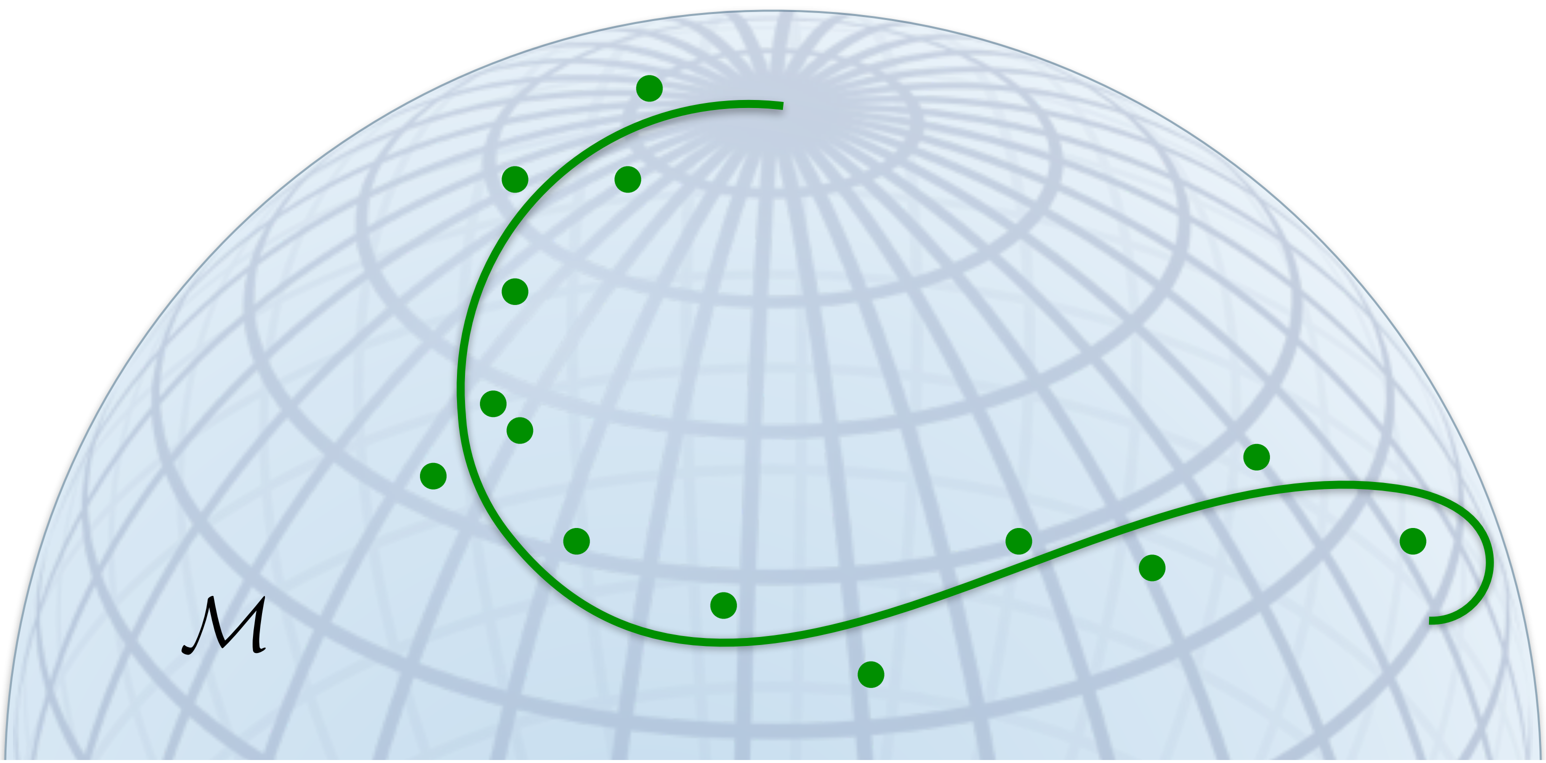

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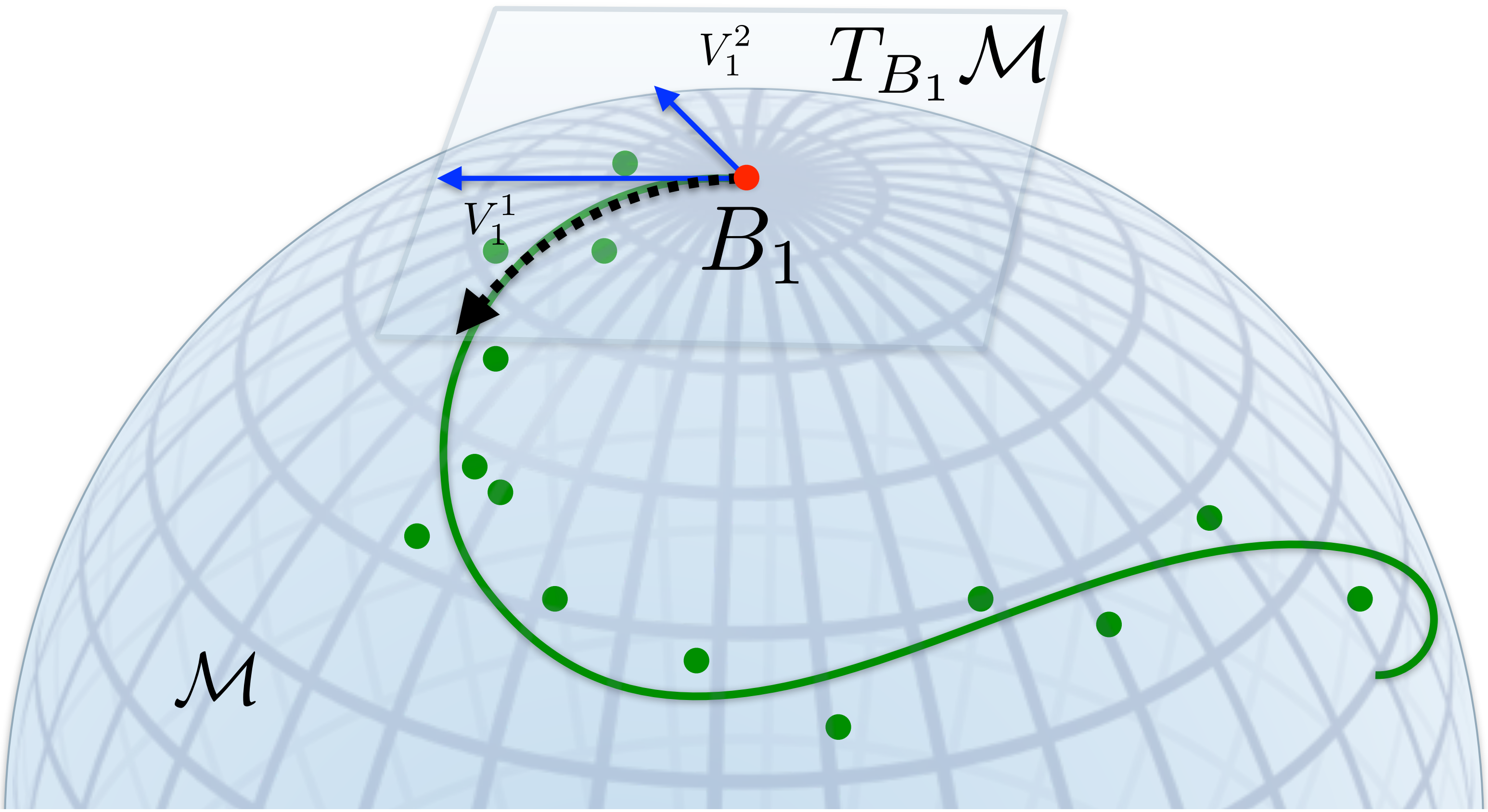

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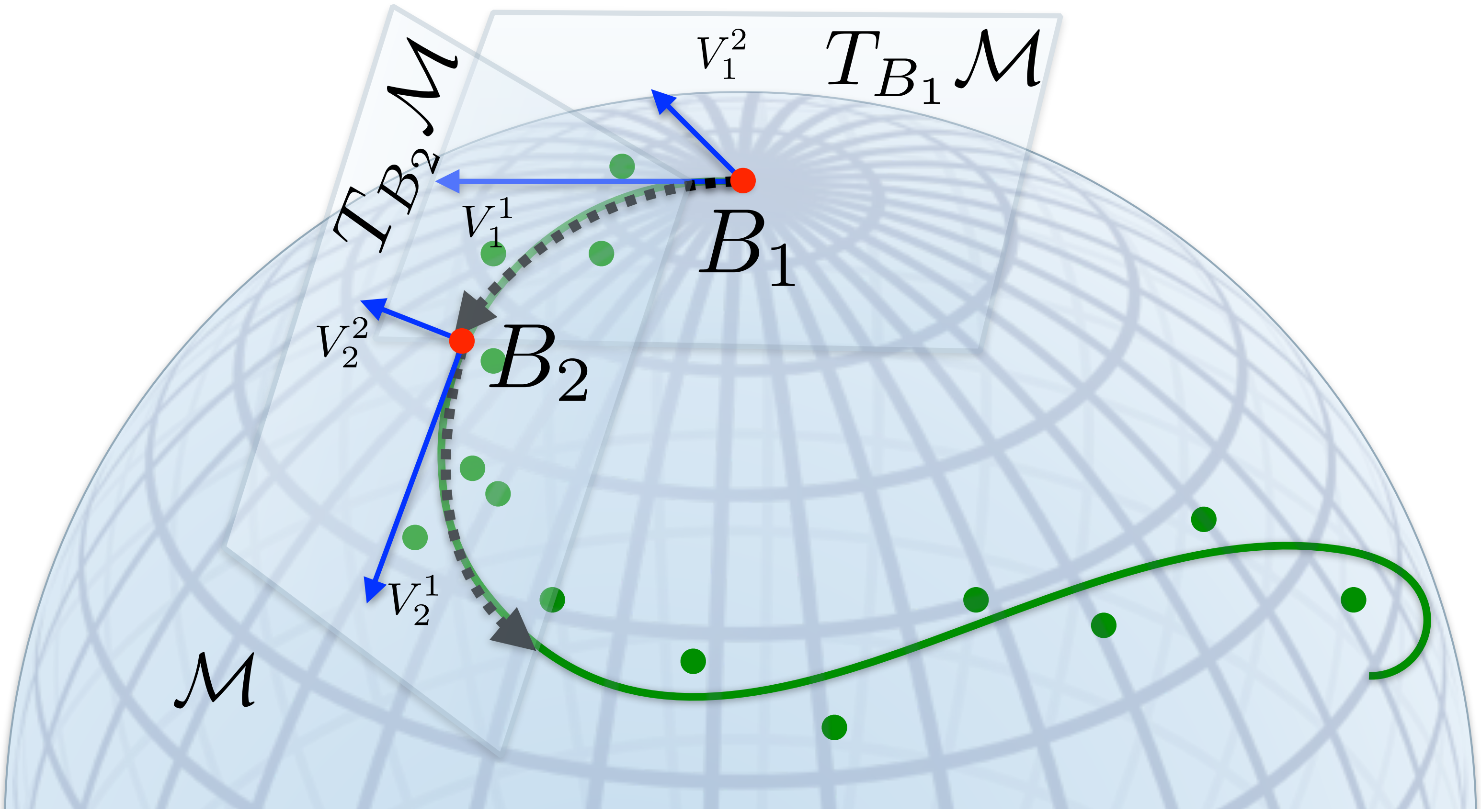

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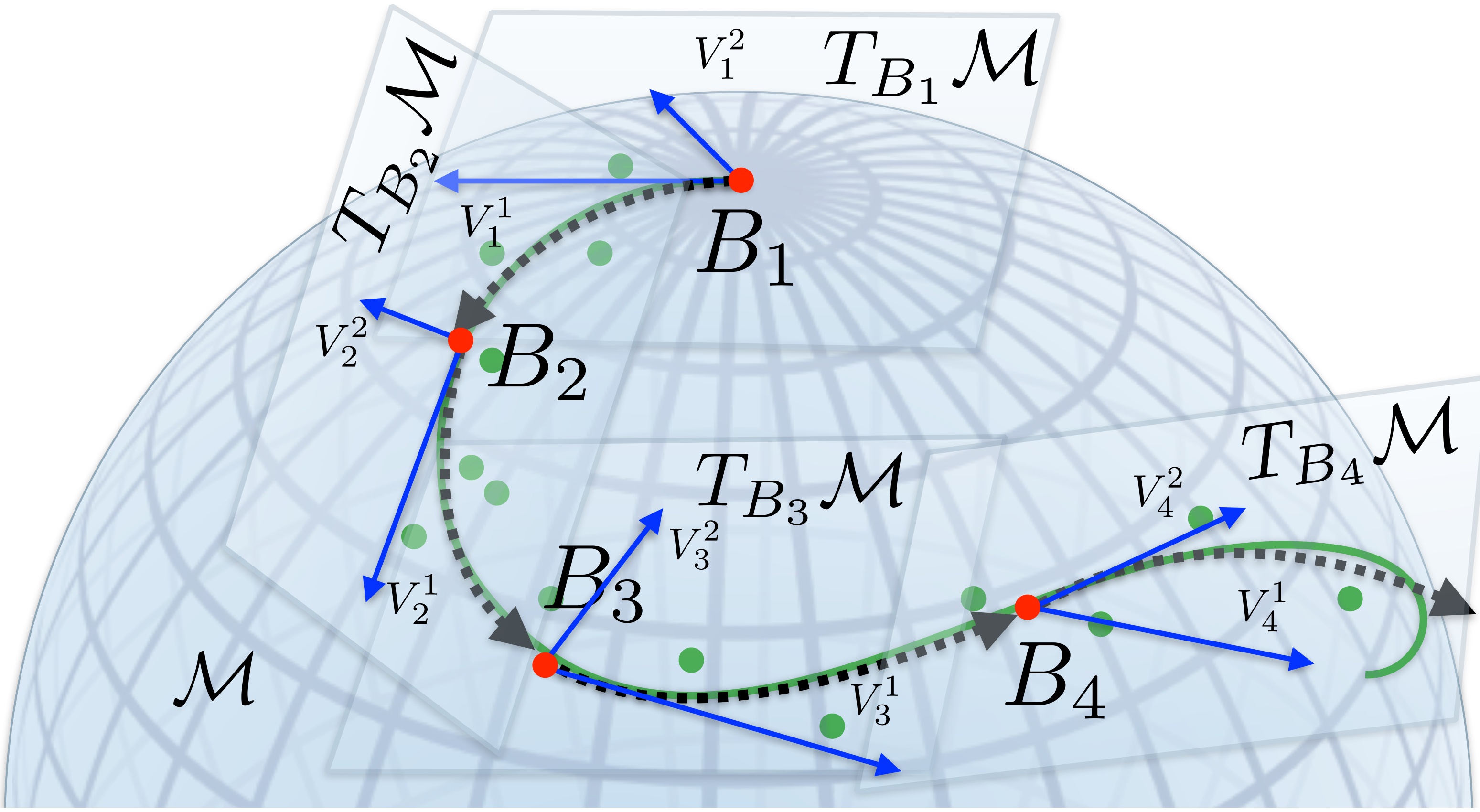

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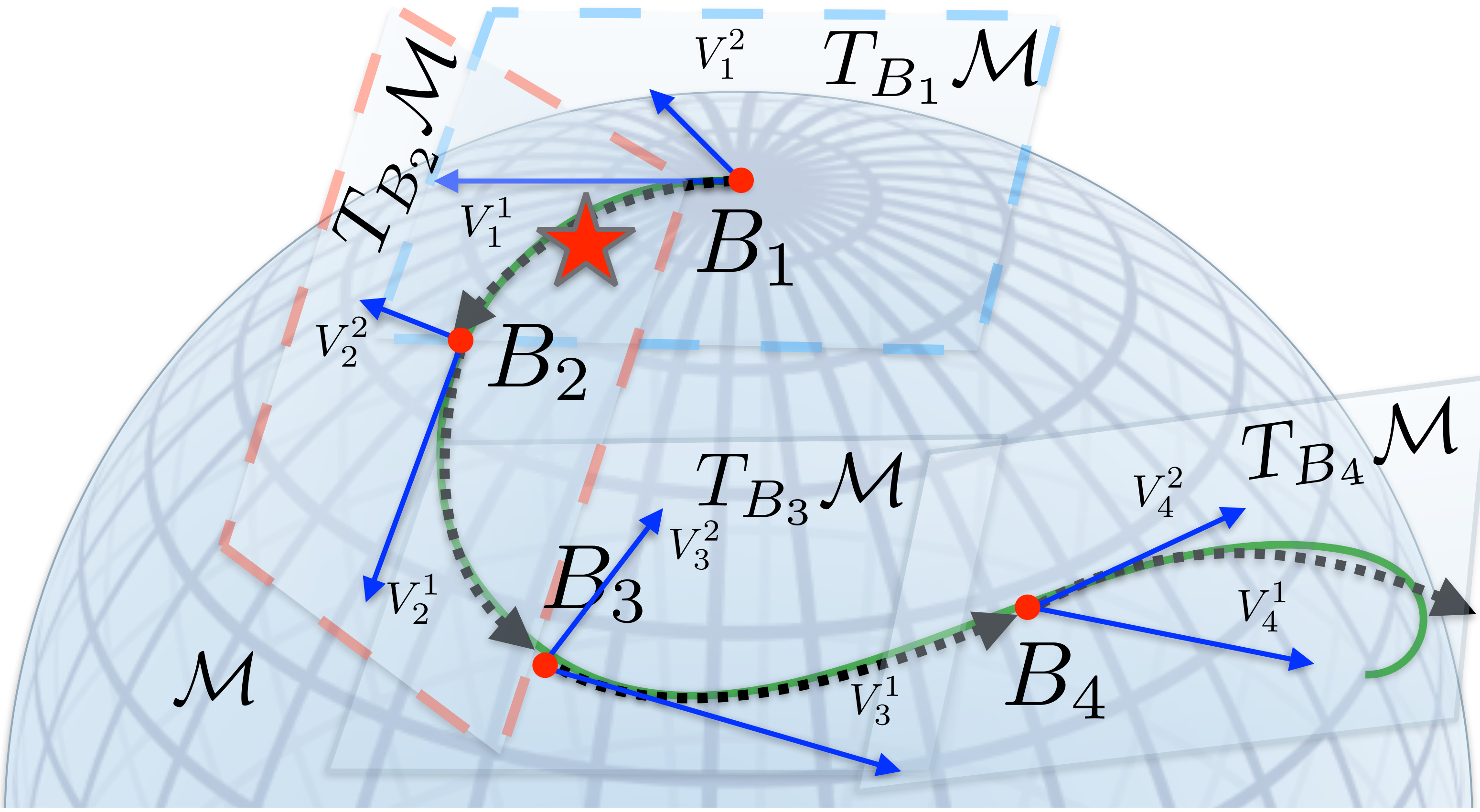

DP-GLM

$$
\begin{aligned}
G &\sim DP(G_0, \nu) \\
\theta_i = (\theta_{x_i}, \theta_{y_i})|G &\sim G, \\
x_i|\theta_{x_i} &\sim f_x(\theta_{x_i}), \\
y_i|x_i, \theta_{y_i} &\sim GLM(x_i, \theta_{y_i}),
\end{aligned}
$$

Hannah et al. JMLR 2011

DP-GLM

Manifold-valued?

$y_i \in \mathcal{M}$

$$G \sim DP(G_0, \nu)$$

$$\theta_i = (\theta_{x_i}, \theta_{y_i})|G \sim G,$$

$$x_i|\theta_{x_i} \sim f_x(\theta_{x_i}),$$

$$y_i|x_i, \theta_{y_i} \sim GLM(x_i, \theta_{y_i}),$$

Hannah et al. JMLR 2011

DP-MGLM

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$$\theta_i = (\theta_{x_i}, \theta_{y_i})|G \sim G,$$

MGLM on manifolds

$$x_i|\theta_{x_i} \sim f_x(\theta_{x_i}),$$

$$y_i|x_i, \theta_{y_i} \sim MGLM(x_i, \theta_{y_i}),$$

DP-MGLM

$$G \sim DP(G_0, \nu)$$

$$\theta_i = (\theta_{x_i}, \theta_{y_i})|G \sim G,$$

$$x_i|\theta_{x_i} \sim f_x(\theta_{x_i}),$$

$$y_i|x_i, \theta_{y_i} \sim MGLM(x_i, \theta_{y_i}),$$

manifold-valued parameters?

$$\theta_{y_i} \in \mathrm{SPD}(n) \times \mathrm{Sym}(n)^d$$

DP-MGLM

- Distribution on manifolds
- Intrinsic metric

$$G \sim DP(G_0, \nu)$$

$$\theta_i = (\theta_{x_i}, \theta_{y_i}) | G \sim G,$$

$$x_i | \theta_{x_i} \sim f_x(\theta_{x_i}),$$

$$y_i | x_i, \theta_{y_i} \sim MGLM(x_i, \theta_{y_i}),$$

$$\theta_{y_i} \in \mathrm{SPD}(n) \times \mathrm{Sym}(n)^d$$

DP-MGLM

Hamiltonian Monte Carlo (HMC) sampling

$$G \sim DP(G_0, \nu)$$

$$\theta_i = (\theta_{x_i}, \theta_{y_i})|G \sim G,$$

$$x_i|\theta_{x_i} \sim f_x(\theta_{x_i}),$$

$$y_i|x_i, \theta_{y_i} \sim MGLM(x_i, \theta_{y_i}),$$

$$\theta_{y_i} \in \mathrm{SPD}(n) \times \mathrm{Sym}(n)^d$$

HMC
Hamiltonian/Hybrid Monte Carlo

$$H(q, p) = U(q) + K(p)$$

Duane, S., et al, “Hybrid Monte Carlo”, 1987.

Neal, R. “MCMC using Hamiltonian dynamics”, 2011

HMC
Hamiltonian/Hybrid Monte Carlo

$$H(q,p) = U(q) + K(p)$$

$$q = \theta \in \boldsymbol{R}^d$$

$$p = \dot{\theta} \in \boldsymbol{R}^d$$

Duane, S., et al, “Hybrid Monte Carlo”, 1987.

Neal, R. “MCMC using Hamiltonian dynamics”, 2011

HMC
Hamiltonian/Hybrid Monte Carlo

$$H(q,p) = U(q) + K(p)$$

$$U(q) := -\log f(q)$$

$$K(p) := \frac{1}{2}p^T M^{-1} p$$

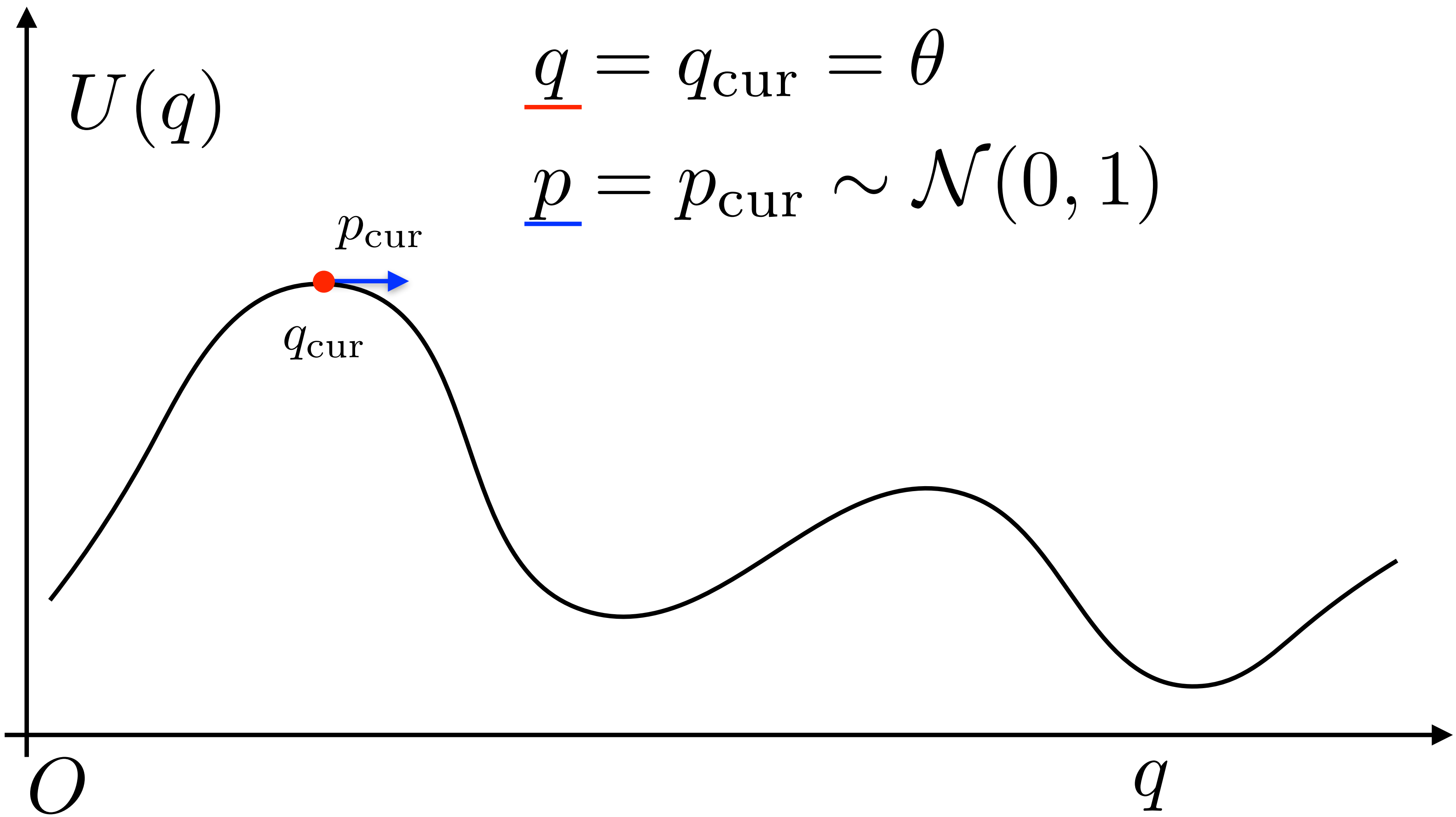
Duane, S., et al, “Hybrid Monte Carlo”, 1987.

Neal, R. “MCMC using Hamiltonian dynamics”, 2011

HMC
Hamiltonian/Hybrid Monte Carlo

$$q = q_{\rm cur} = \theta$$

$$p = p_{\rm cur} \sim \mathcal{N}(0, 1)$$



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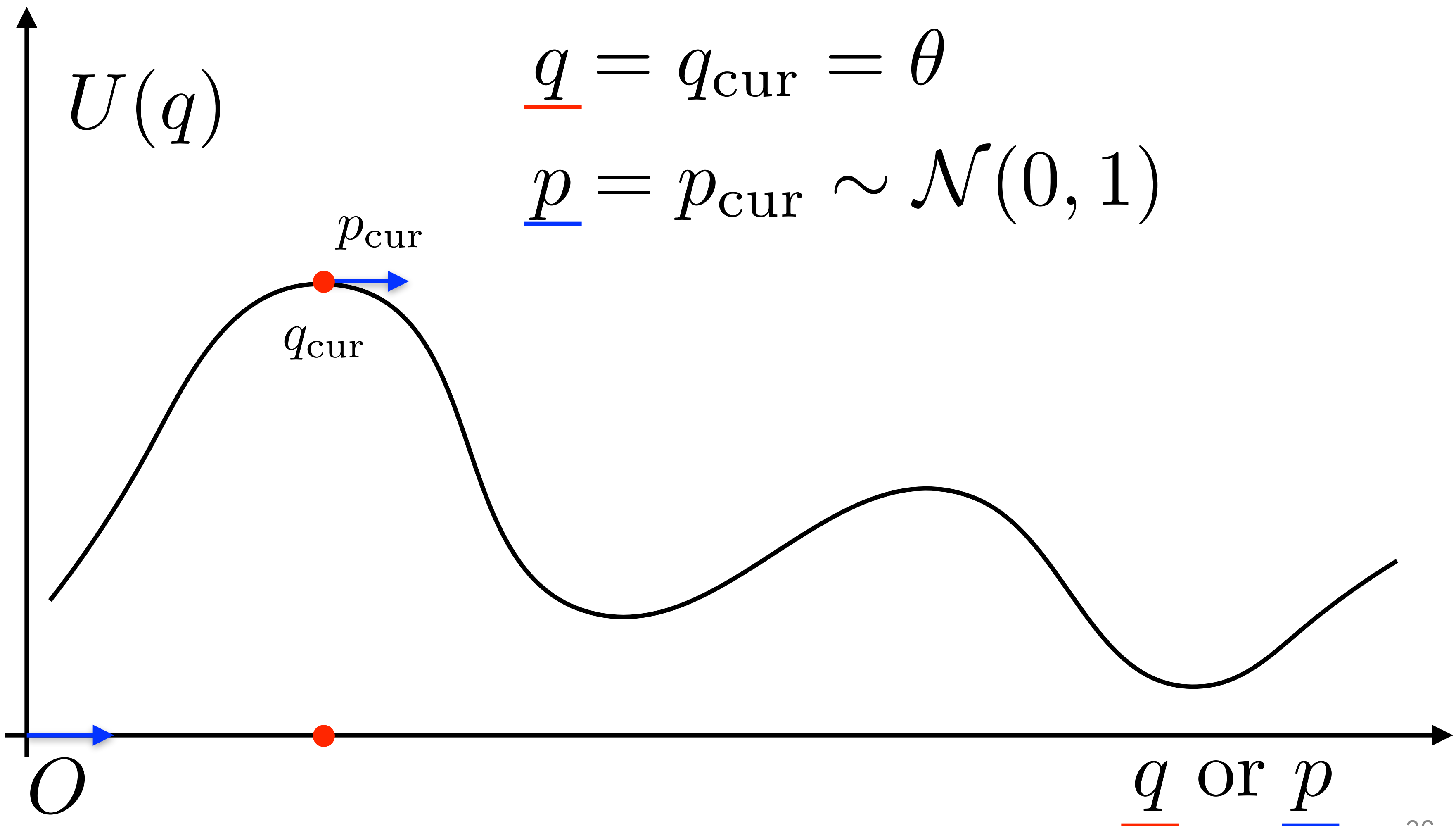
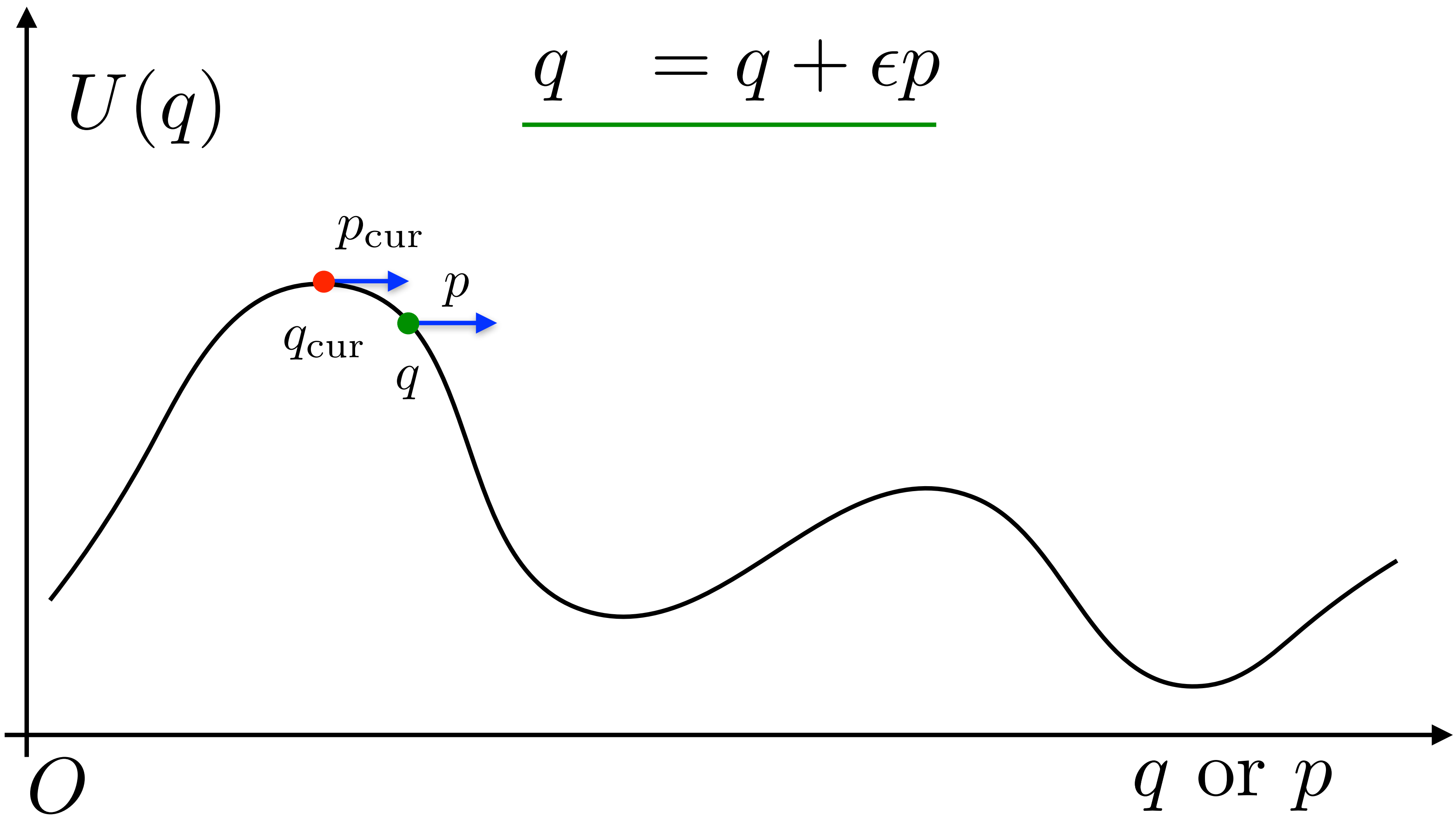


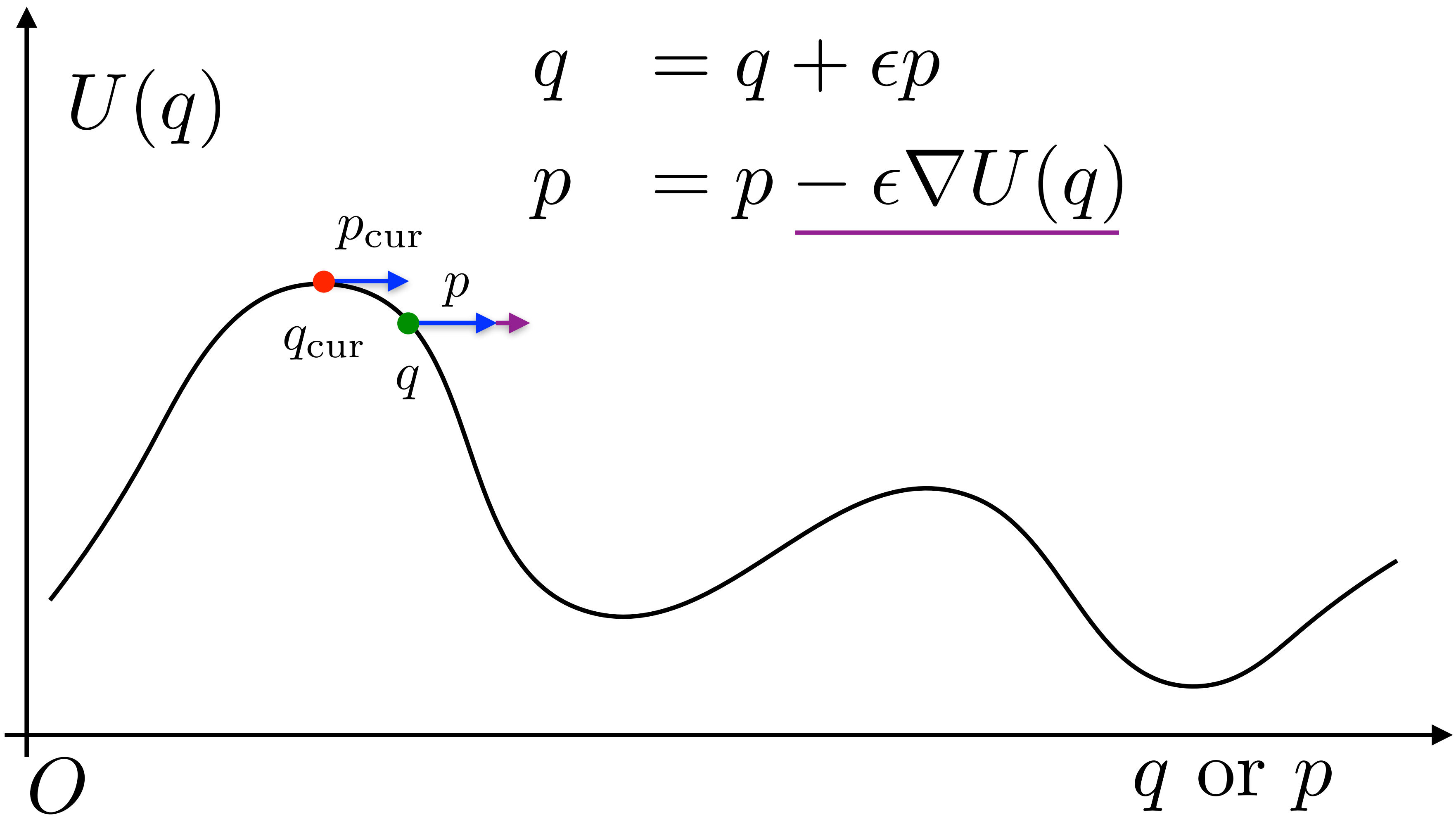
HMC
Hamiltonian/Hybrid Monte Carlo

$$q \quad = q + \epsilon p$$



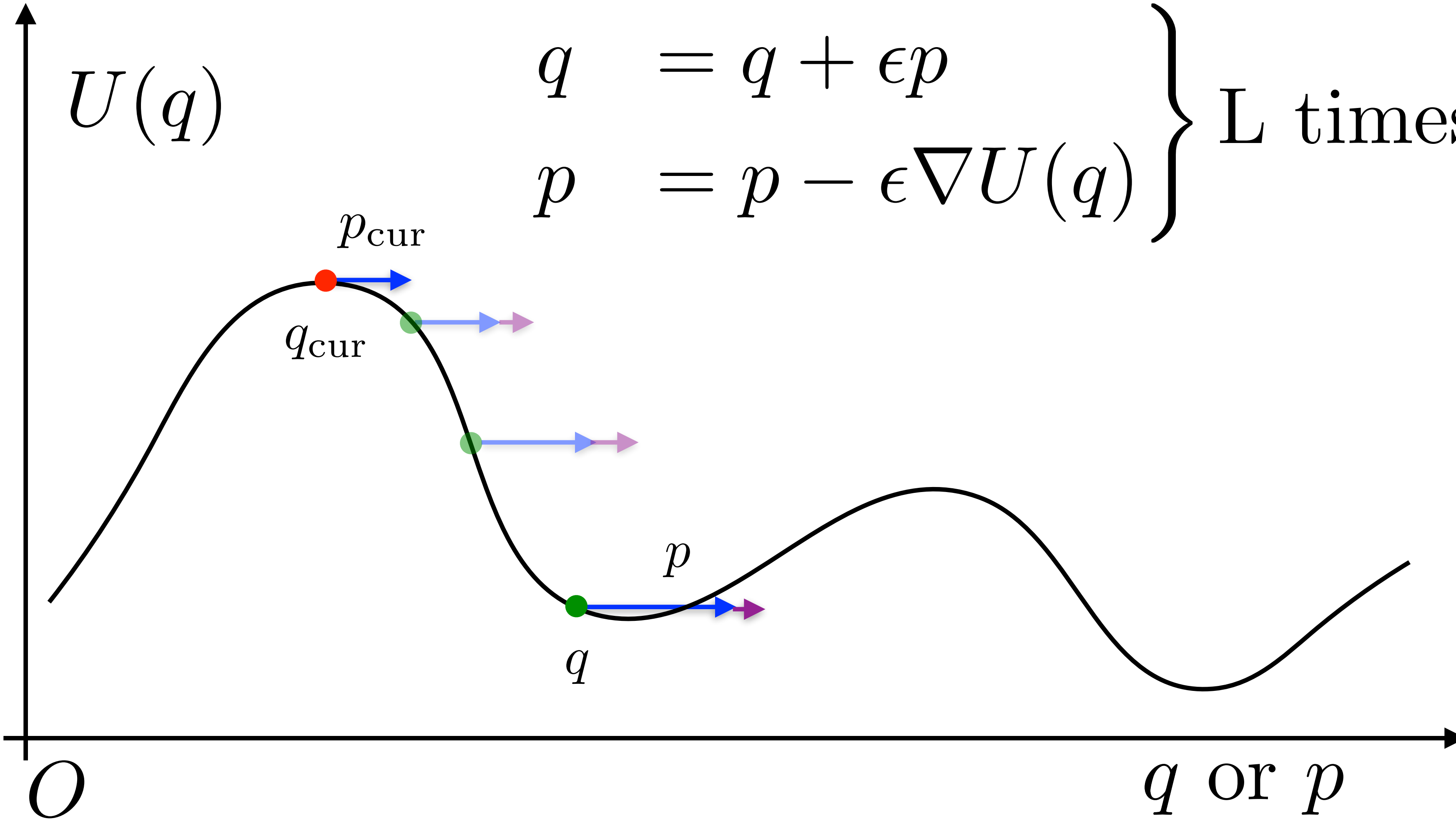
HMC
Hamiltonian/Hybrid Monte Carlo

$$q = q + \epsilon p$$
$$p = p - \epsilon \nabla U(q)$$



HMC
Hamiltonian/Hybrid Monte Carlo

$$\left.\begin{aligned} q &= q + \epsilon p \\ p &= p - \epsilon \nabla U(q) \end{aligned}\right\} \text{L times}$$



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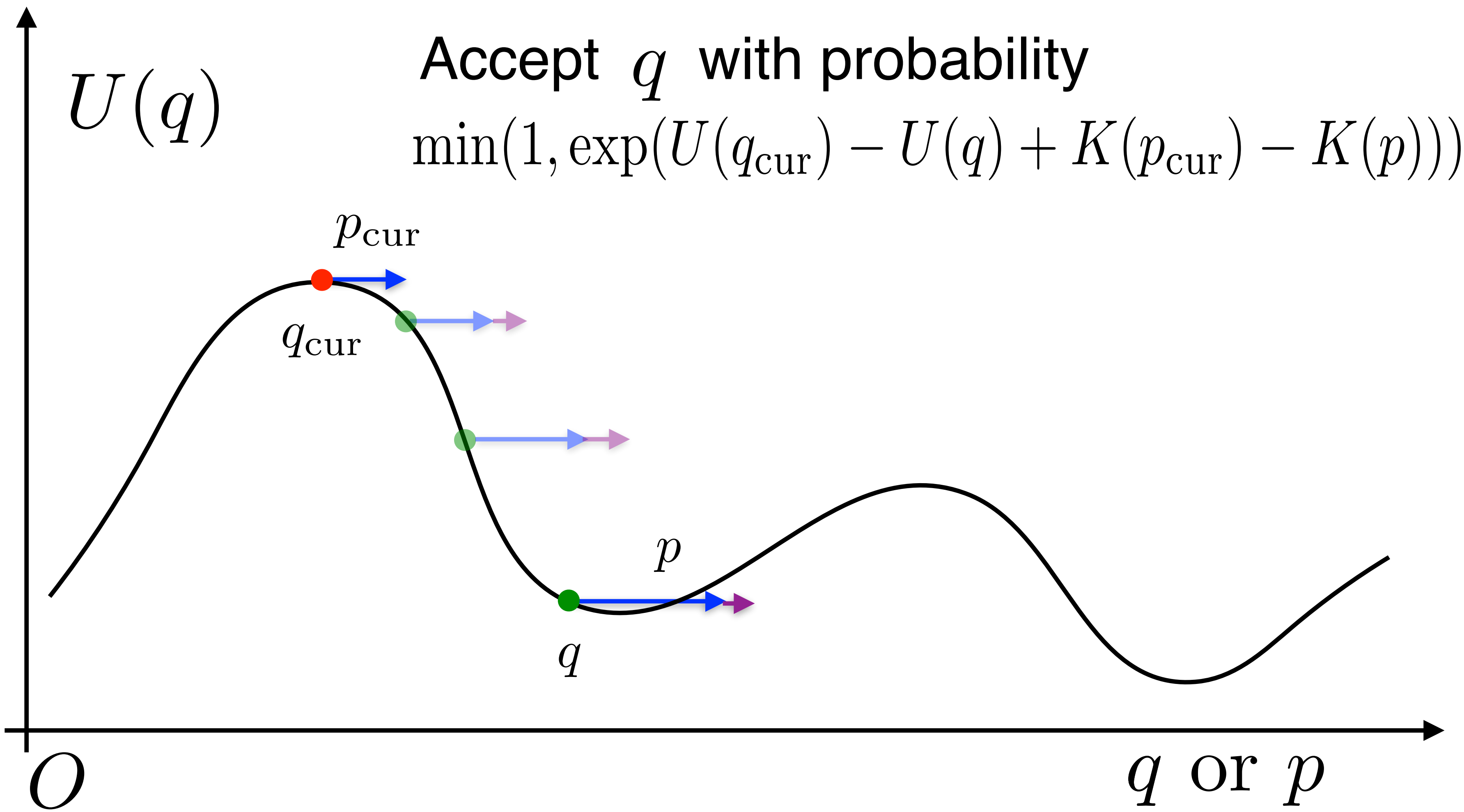
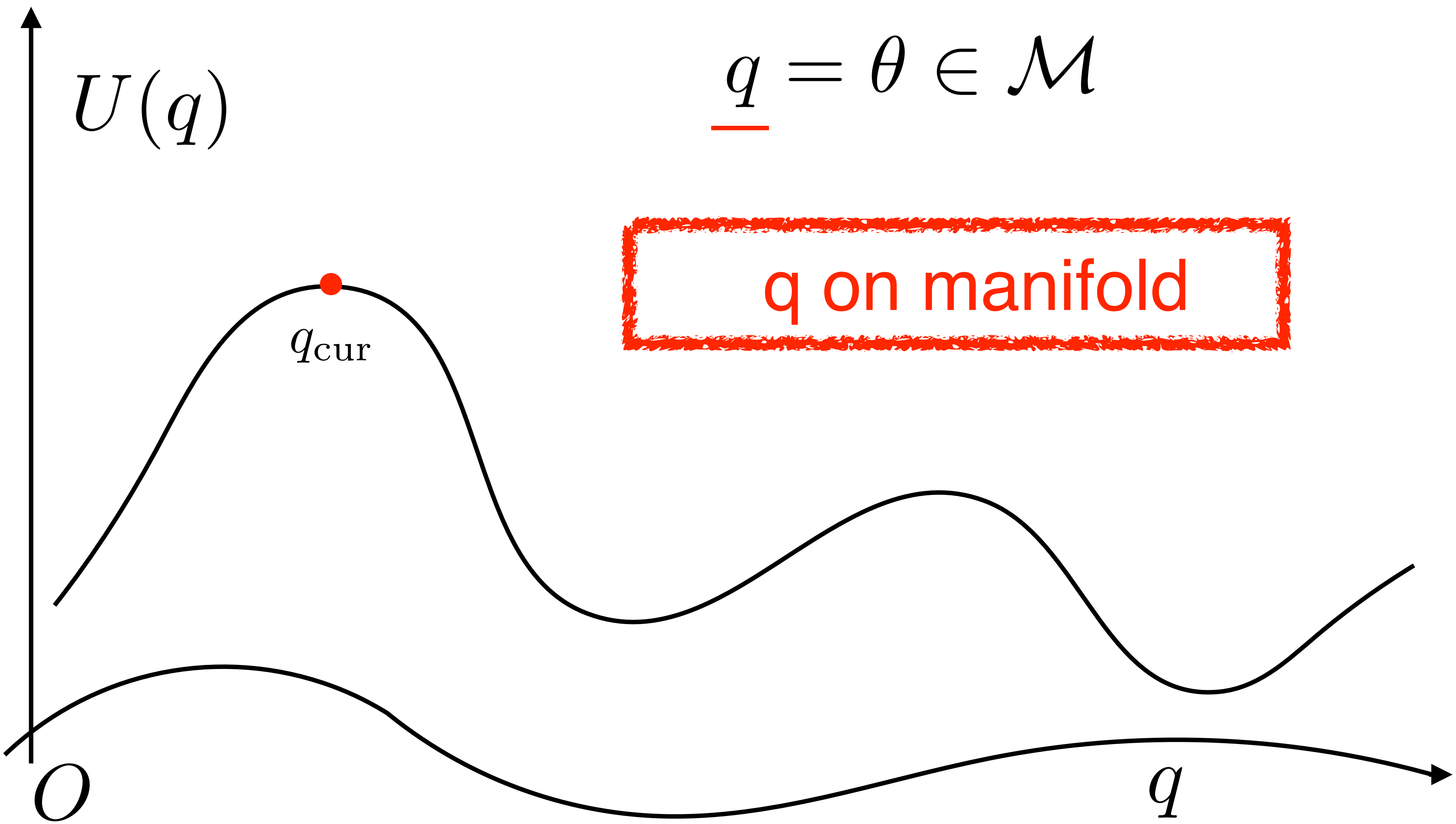


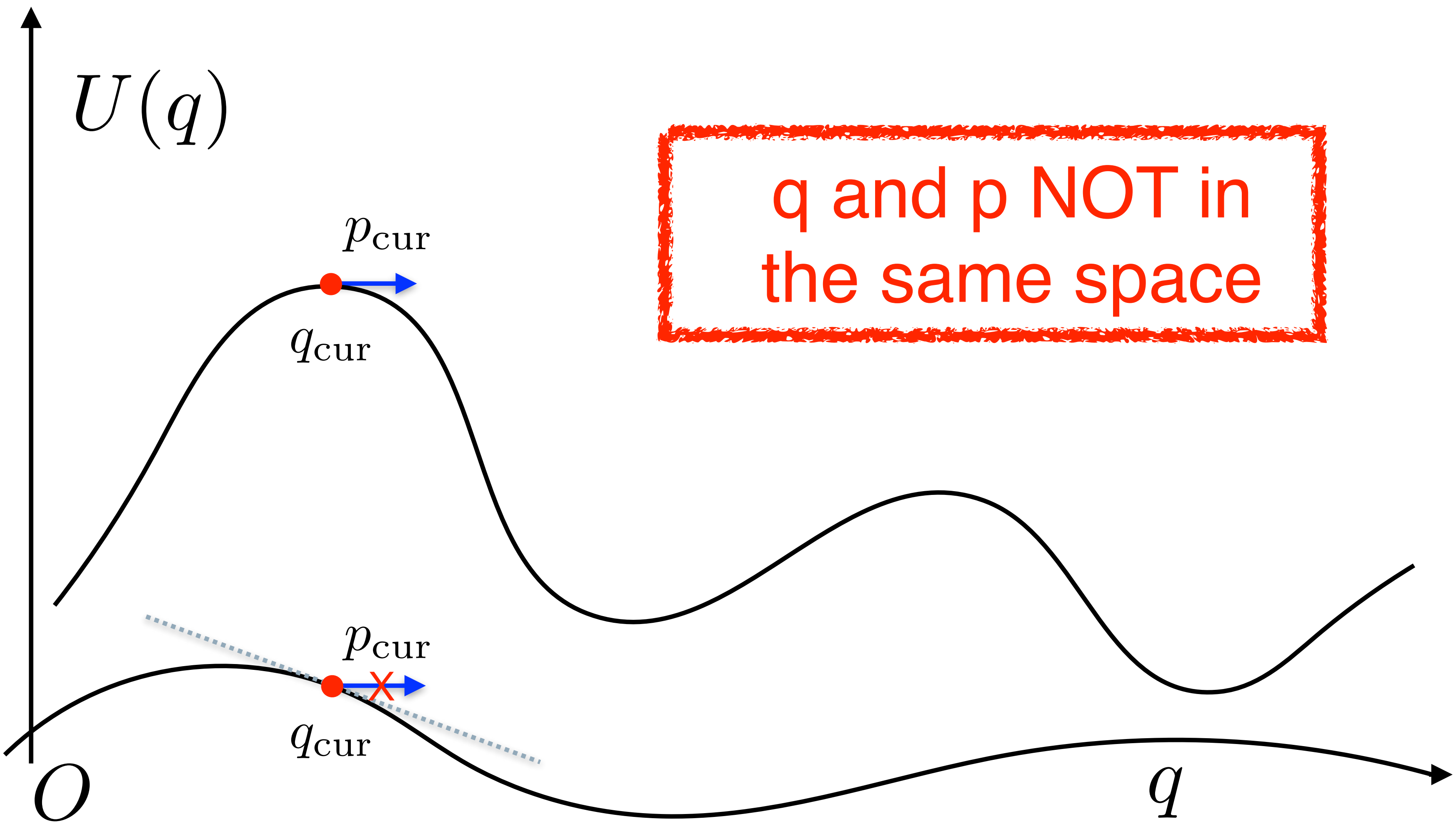
HMC for DP-MGLM

$$q = \theta \in \mathcal{M}$$



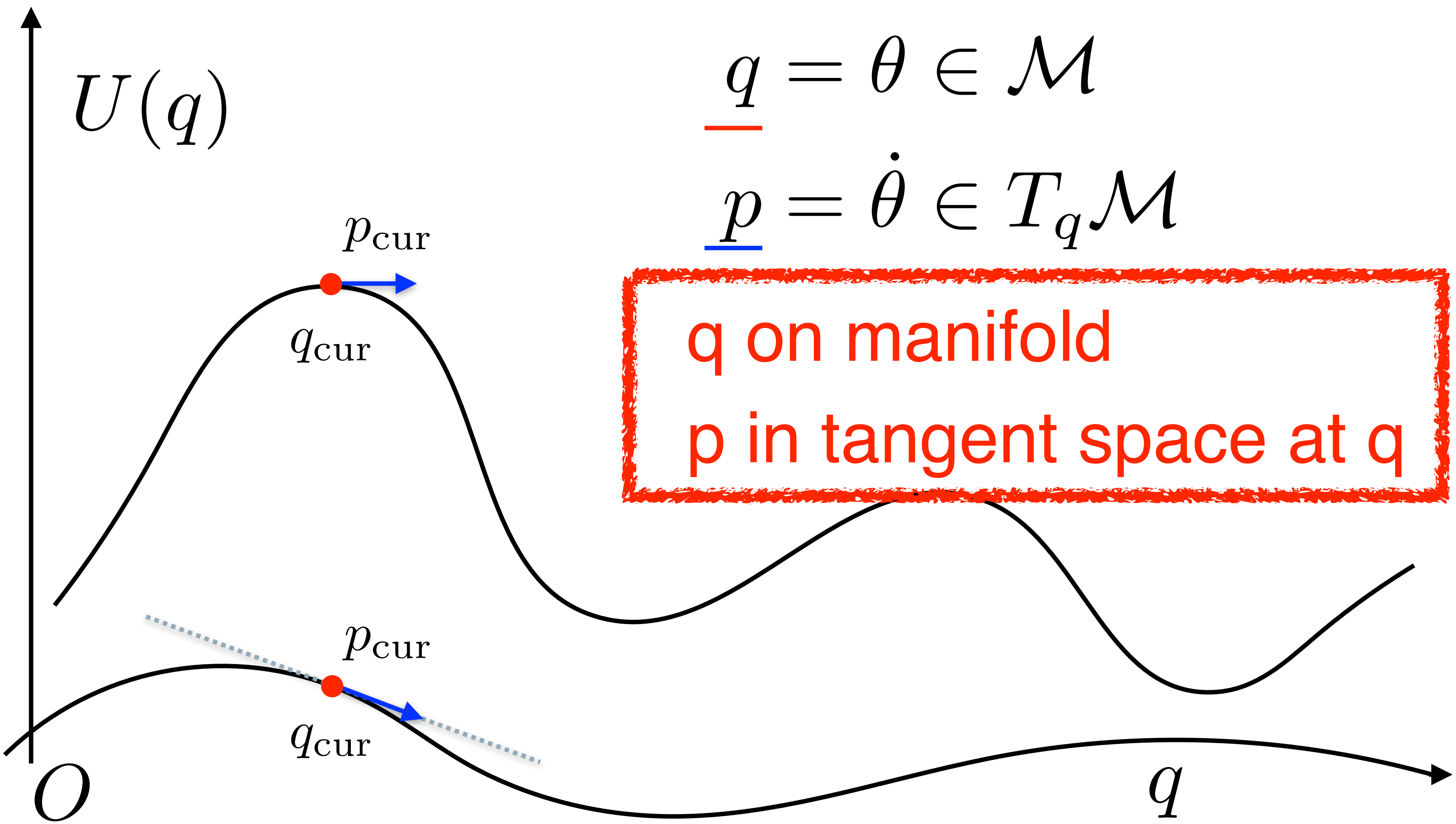
q on manifold

HMC for DP-MGLM



q and p NOT in the same space

HMC for DP-MGLM



$$q = \theta \in \mathcal{M}$$

$$p = \dot{\theta} \in T_q\mathcal{M}$$

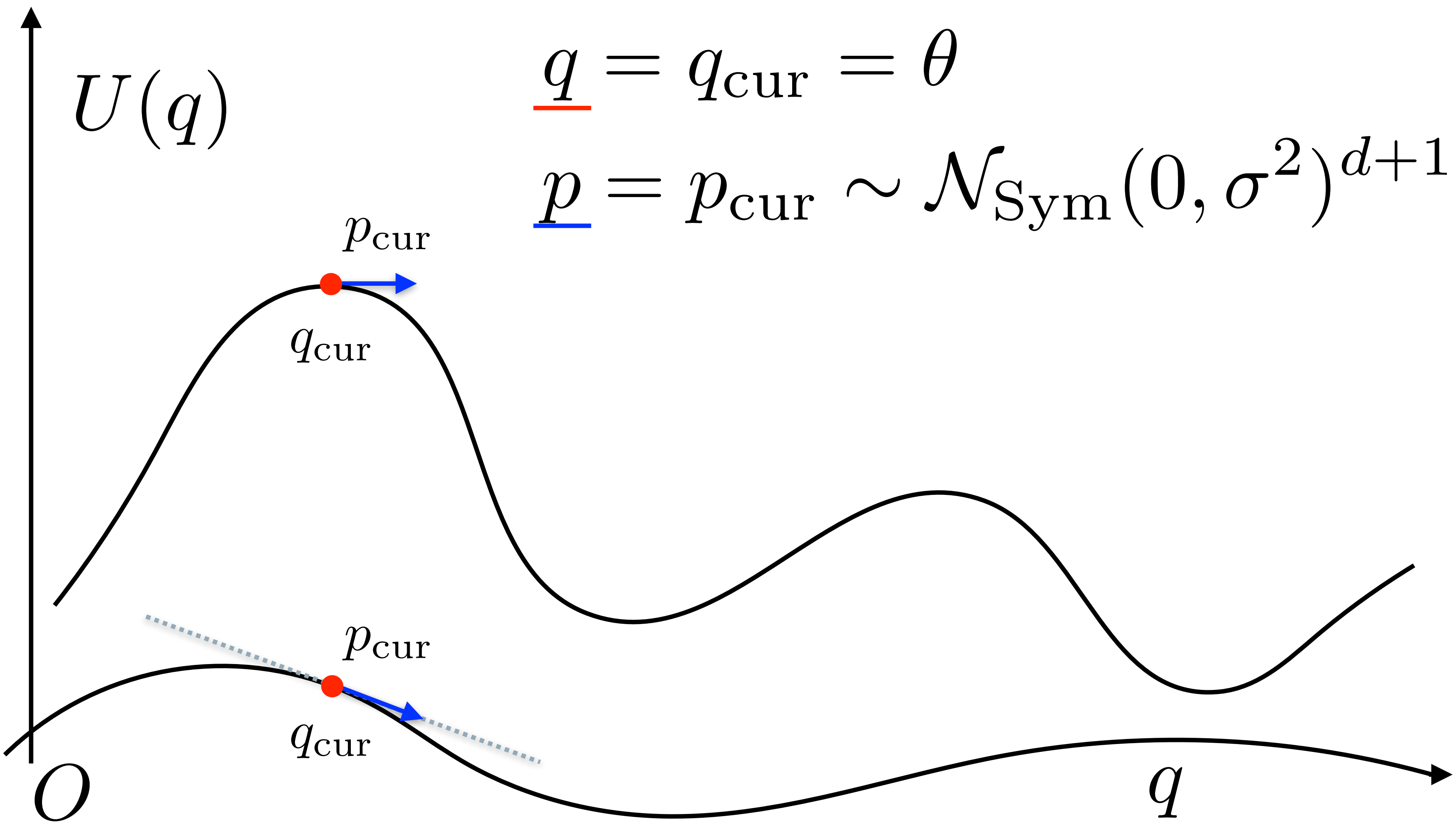
q on manifold

p in tangent space at q

HMC for DP-MGLM

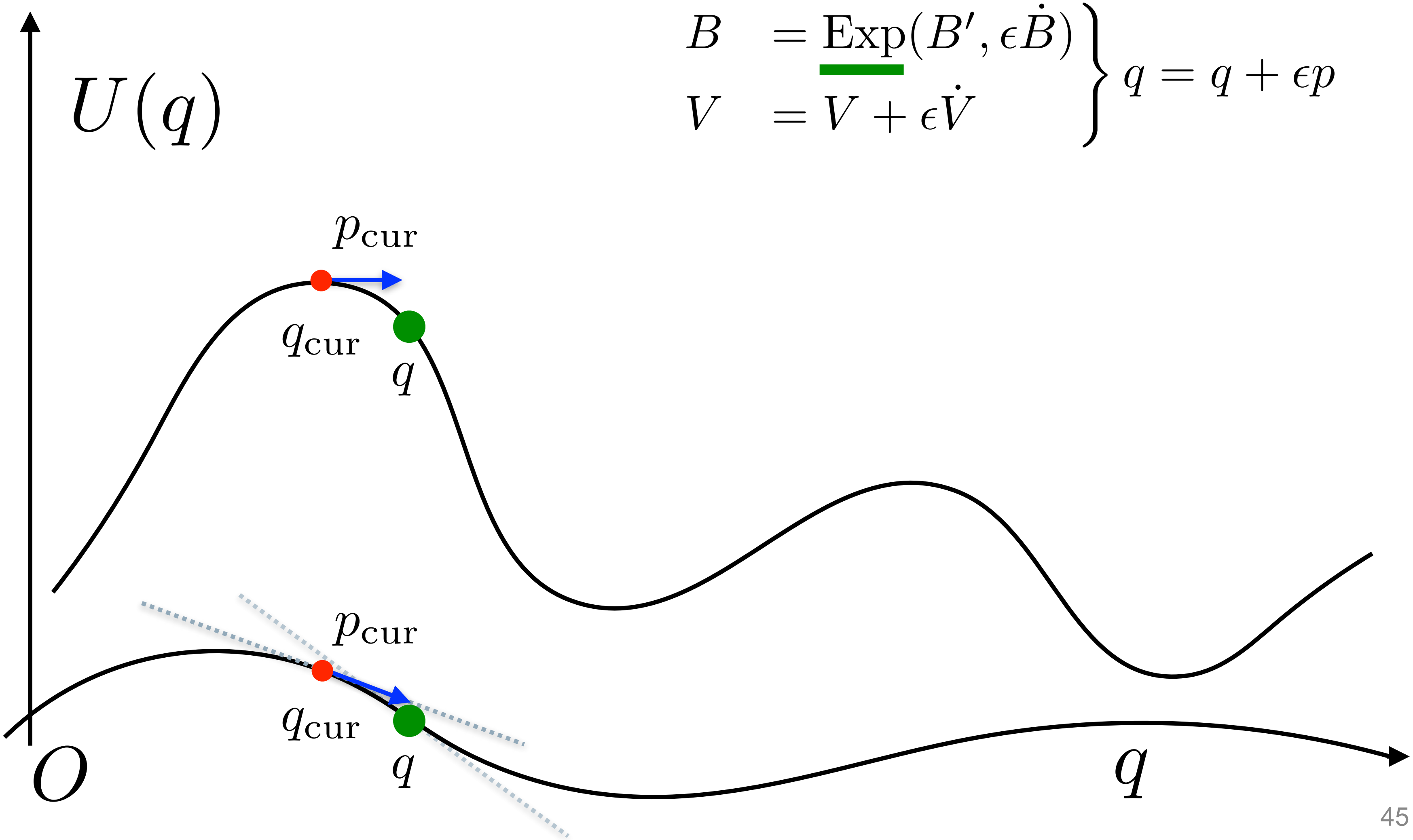
$$q = q_{\mathrm{cur}} = \theta$$

$$p = p_{\mathrm{cur}} \sim \mathcal{N}_{\mathrm{Sym}}(0, \sigma^2)^{d+1}$$



HMC for DP-MGLM

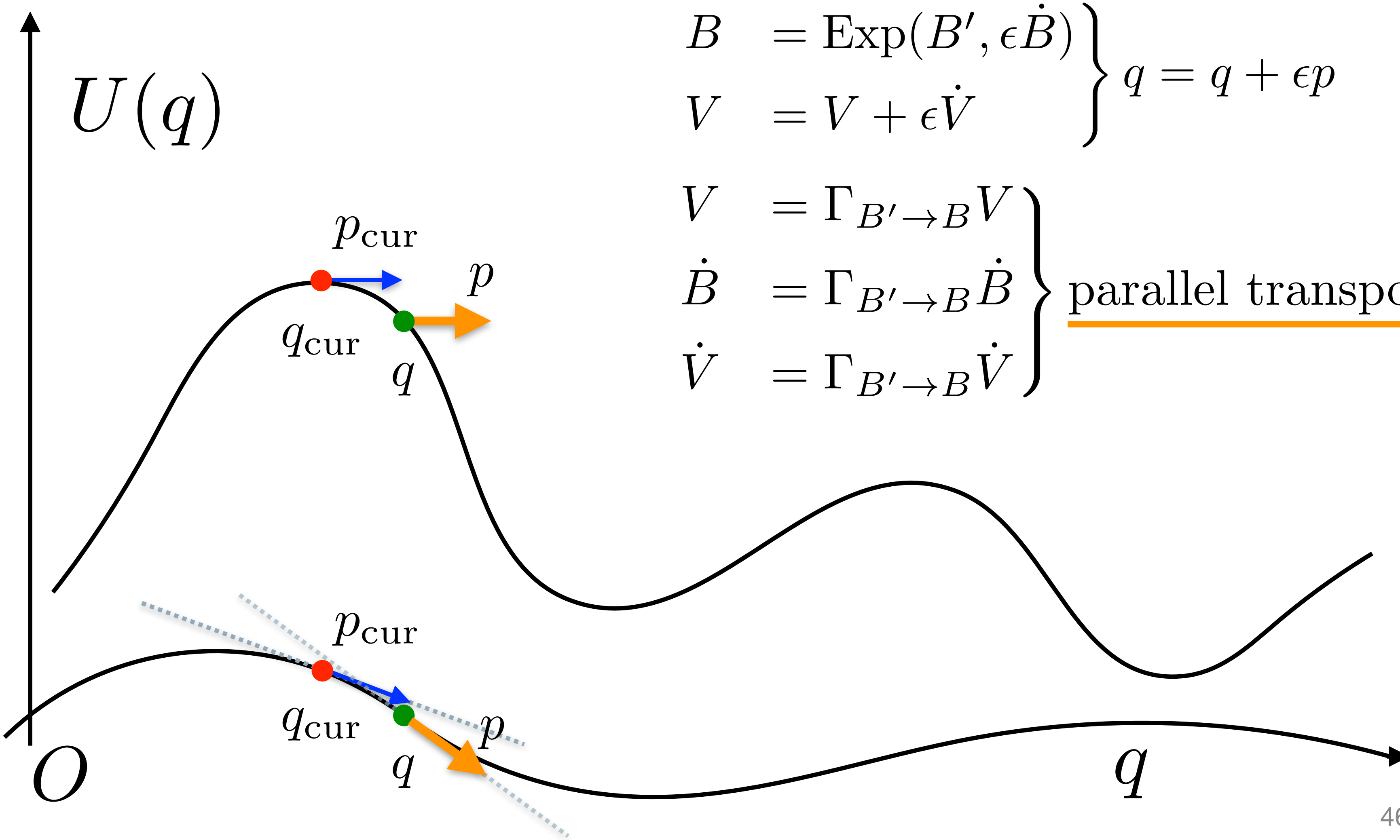
$$\left.\begin{aligned} B &= \underline{\mathrm{Exp}}(B', \epsilon \dot{B}) \\ V &= V + \epsilon \dot{V} \end{aligned}\right\} q = q + \epsilon p$$



HMC for DP-MGLM

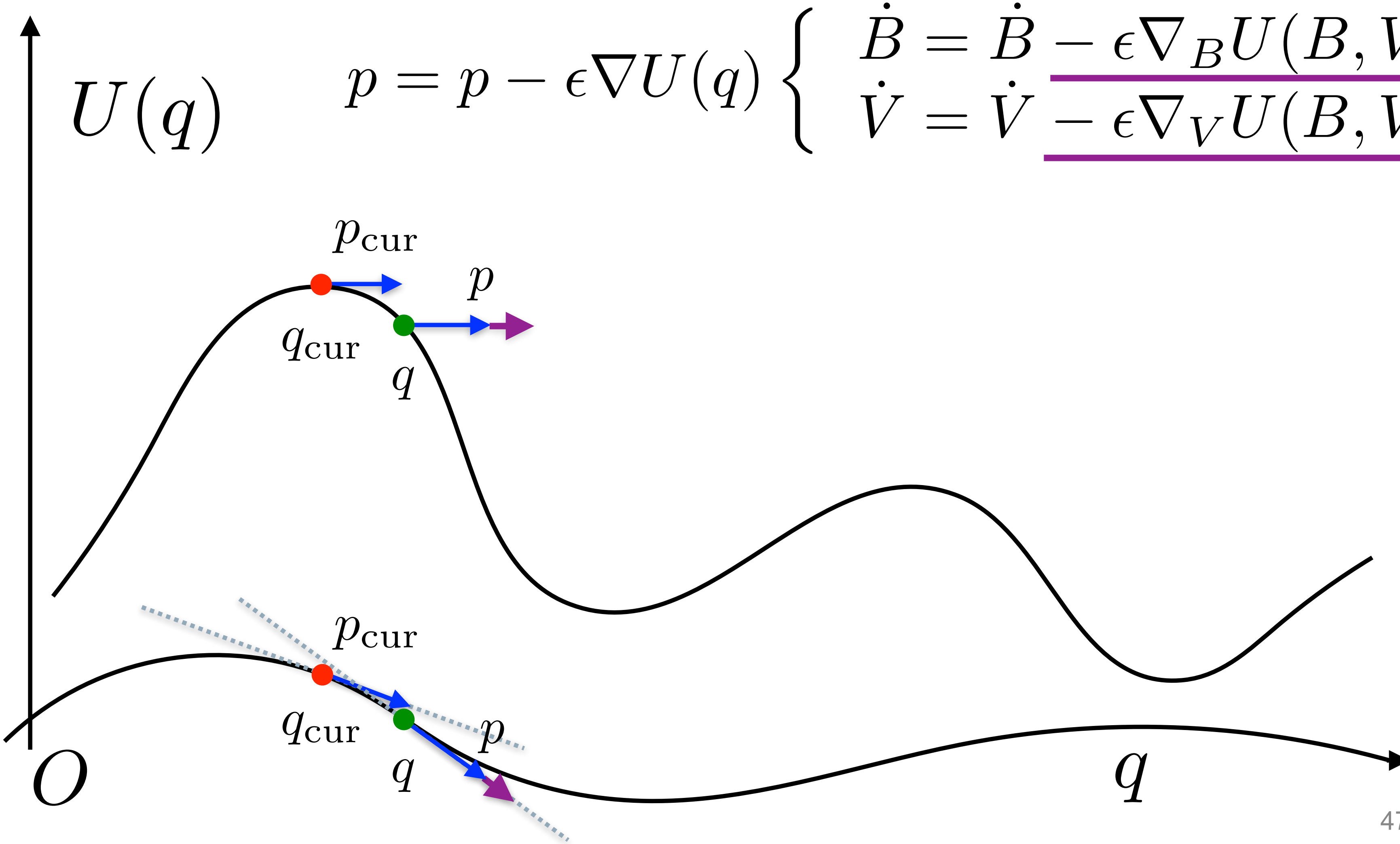
$$\left.\begin{aligned} B &= \mathrm{Exp}(B', \epsilon \dot{B}) \\ V &= V + \epsilon \dot{V} \end{aligned}\right\} q = q + \epsilon p$$

$$\left.\begin{aligned} V &= \Gamma_{B' \to B} V \\ \dot{B} &= \Gamma_{B' \to B} \dot{B} \\ \dot{V} &= \Gamma_{B' \to B} \dot{V} \end{aligned}\right\} \text{parallel transport}$$



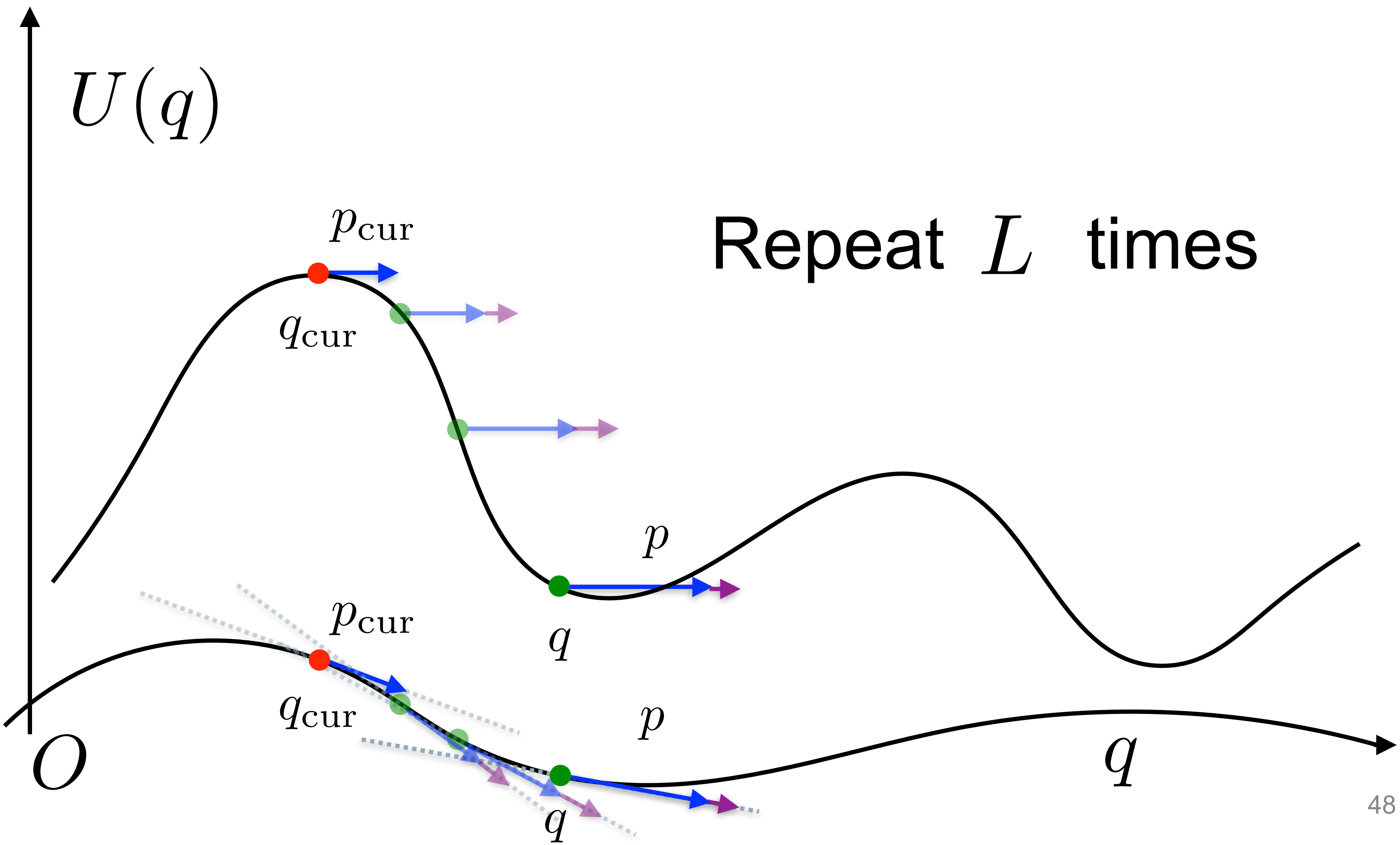
HMC for DP-MGLM

$$p = p - \epsilon \nabla U(q) \begin{cases} \dot{B} = \dot{B} - \epsilon \nabla_B U(B,V) \\ \dot{V} = \dot{V} - \epsilon \nabla_V U(B,V) \end{cases}$$



HMC for DP-MGLM

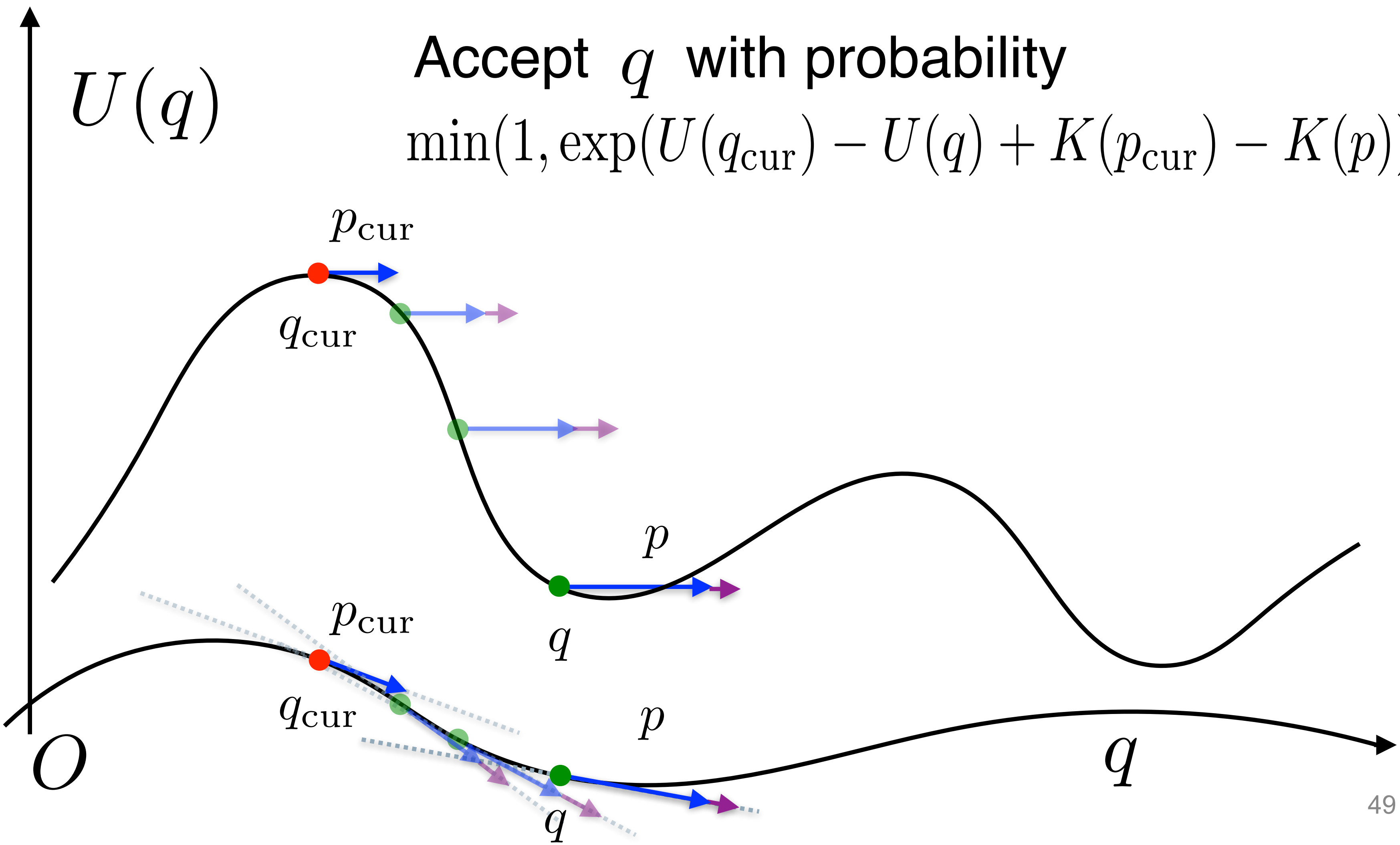
Repeat L times



HMC for DP-MGLM

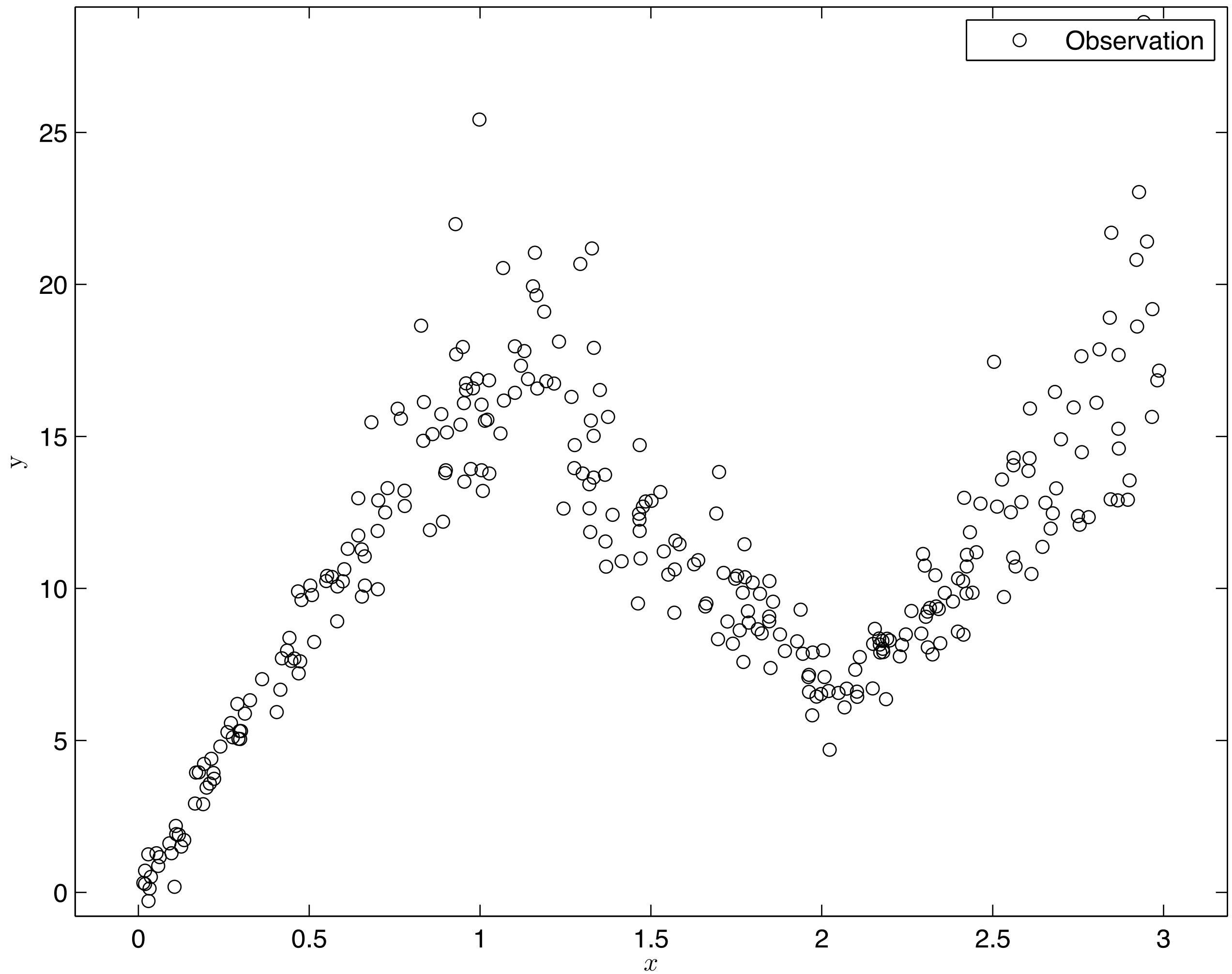
Accept q with probability

$$\min(1, \exp(U(q_{\text{cur}}) - U(q) + K(p_{\text{cur}}) - K(p)))$$

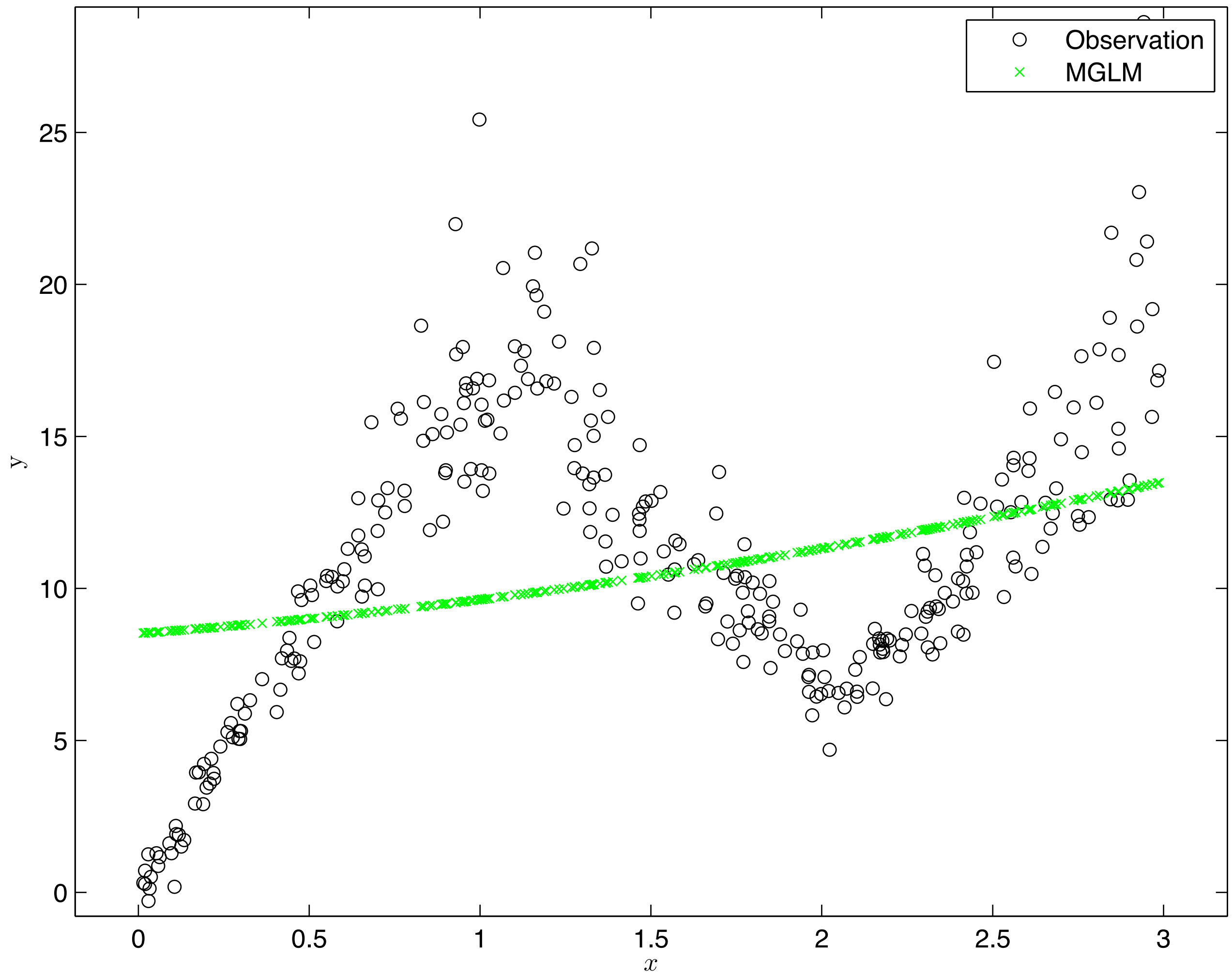


Experiments

Experiment 1
MGLM vs DP-MGLM



Experiment 1
MGLM vs DP-MGLM



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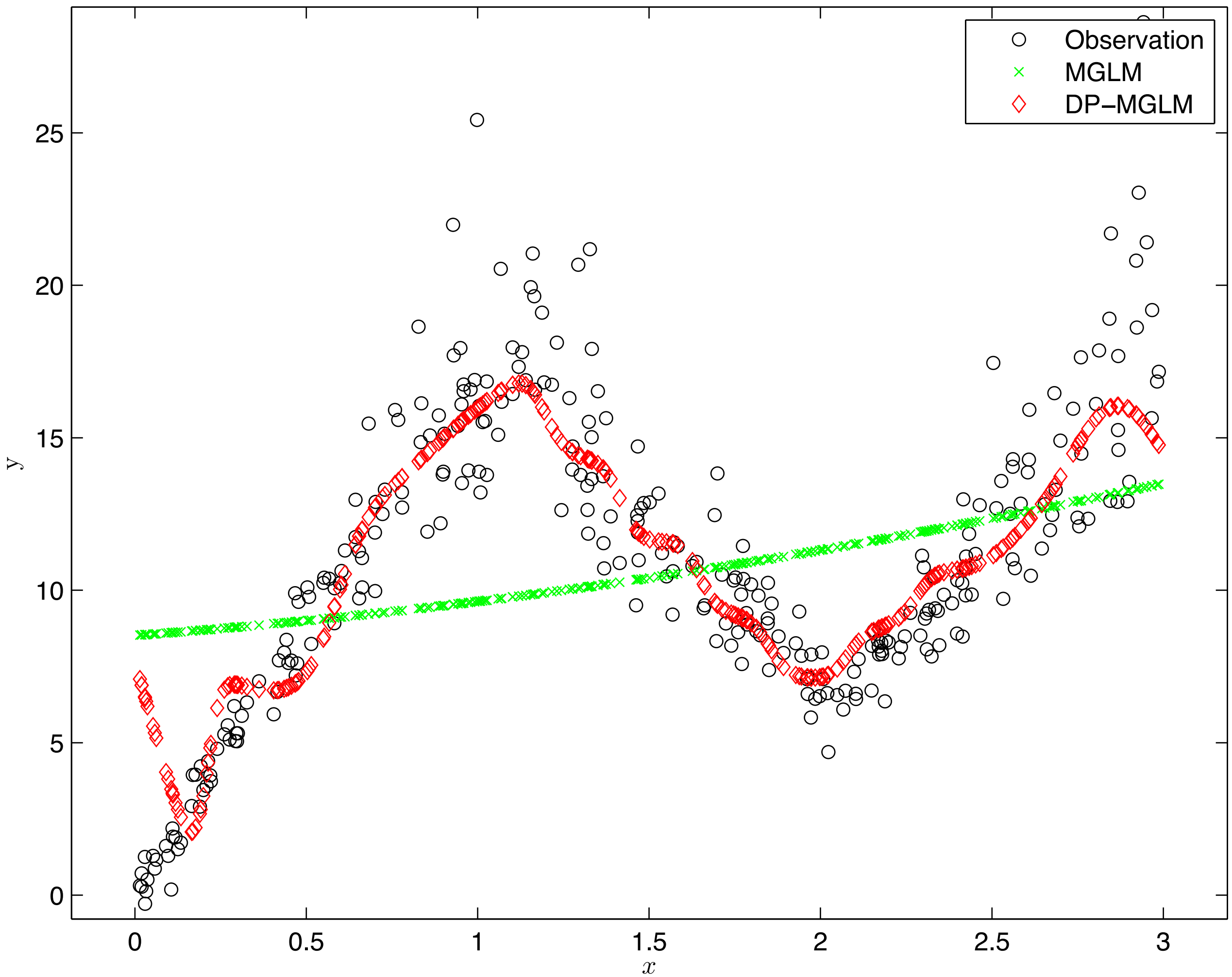

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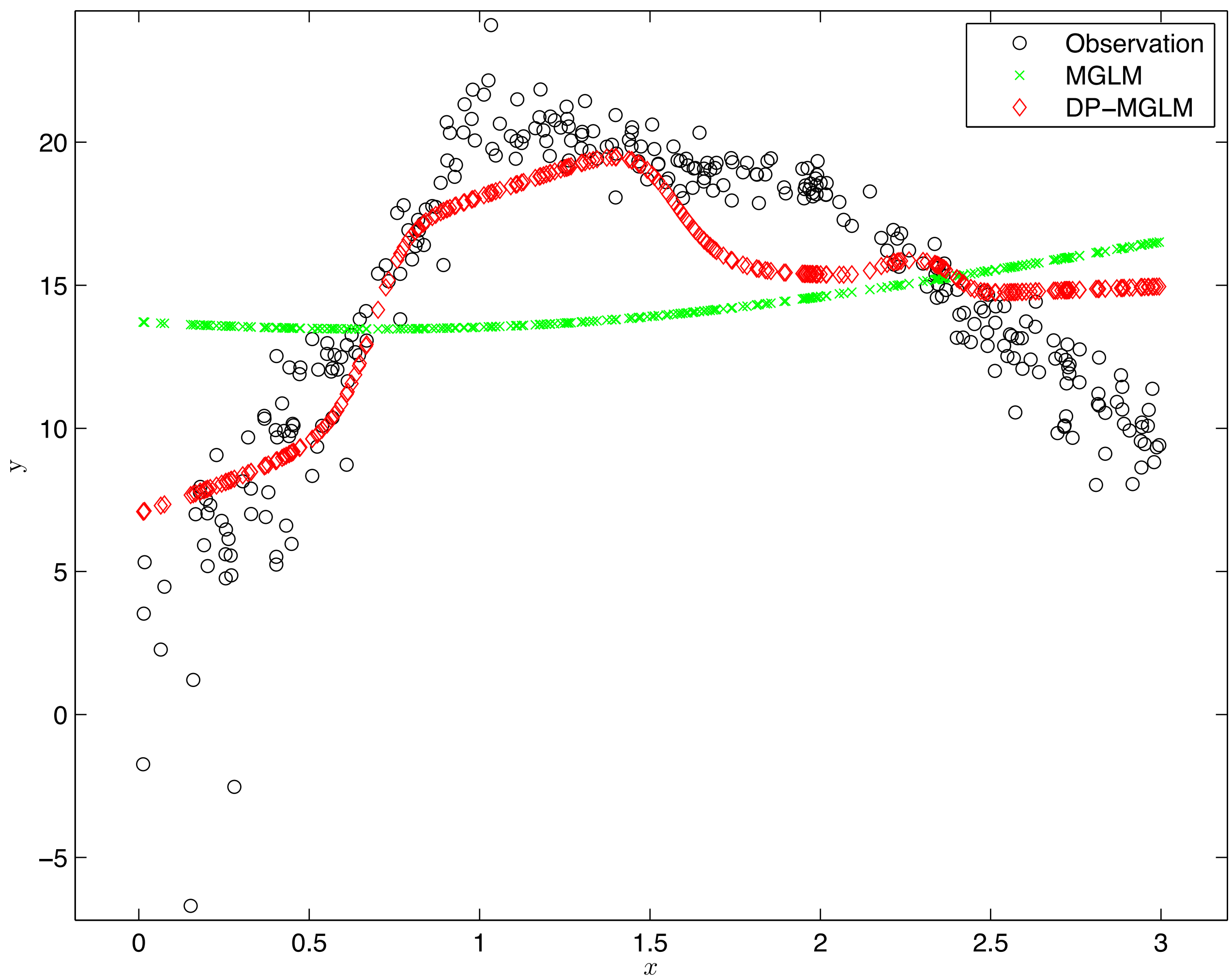
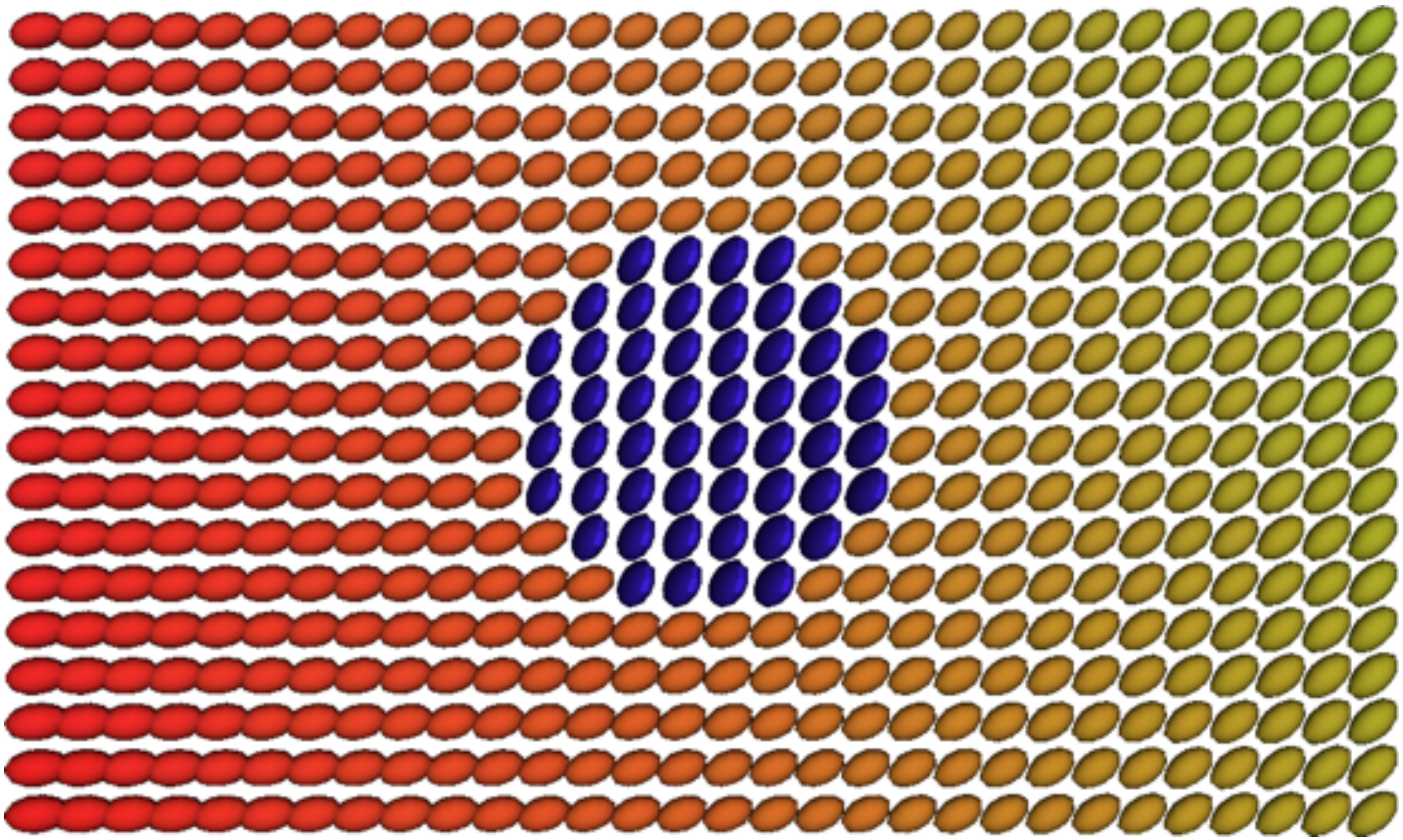
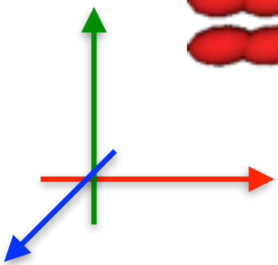


Experiment 2
Clustering on SPD(3)



3D ellipsoid patch 1



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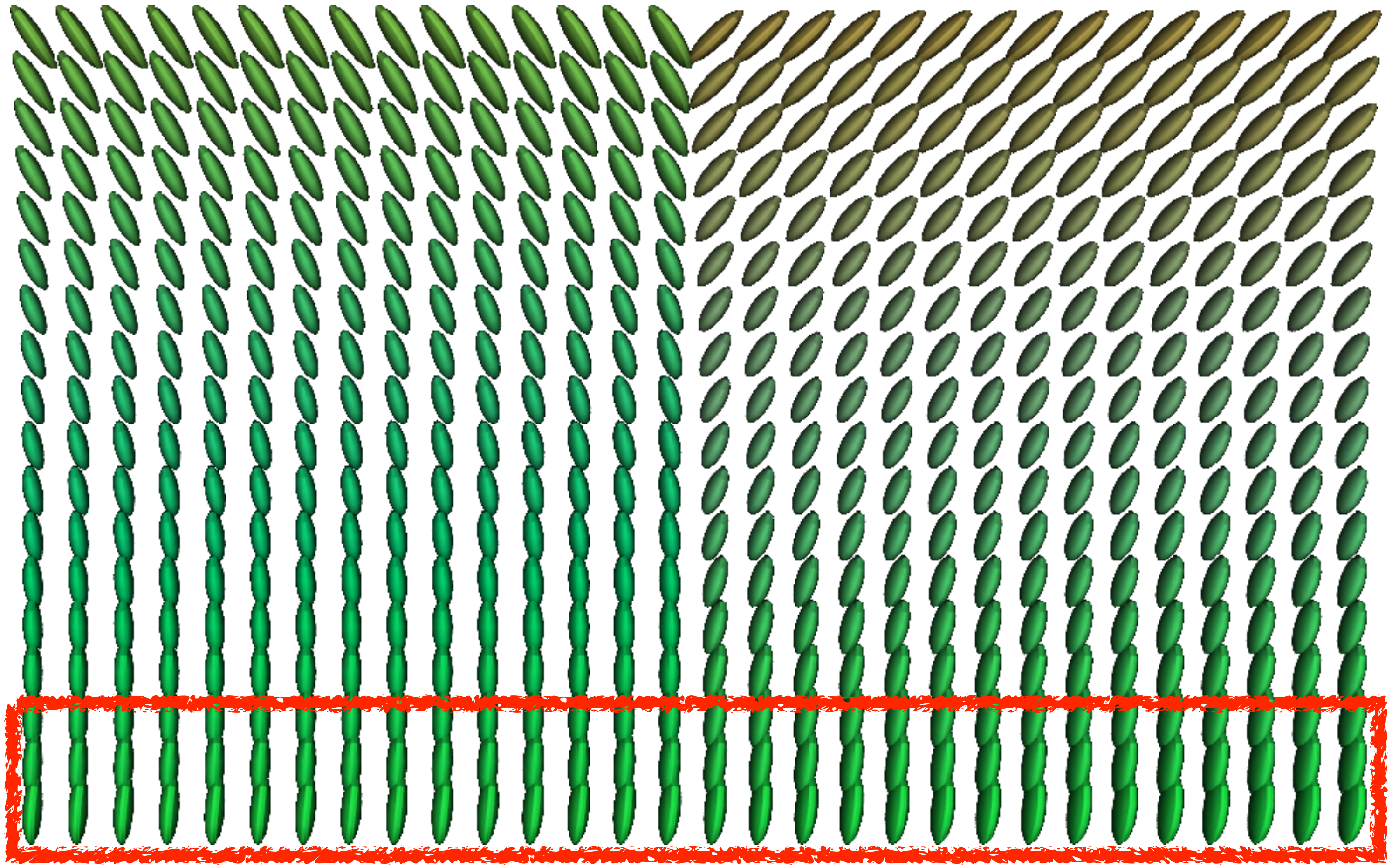
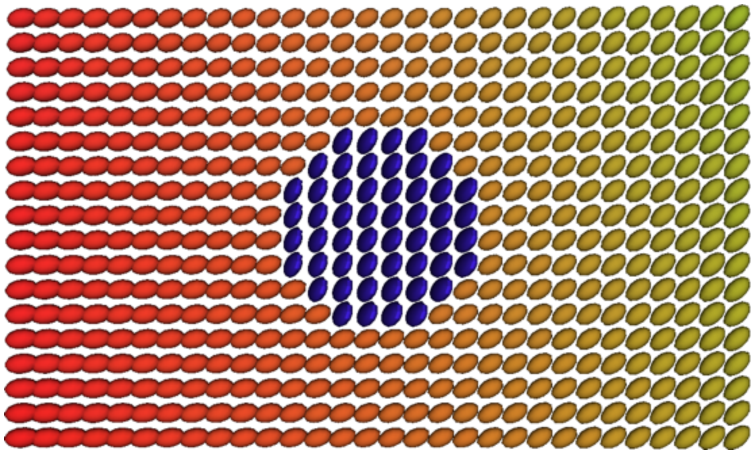
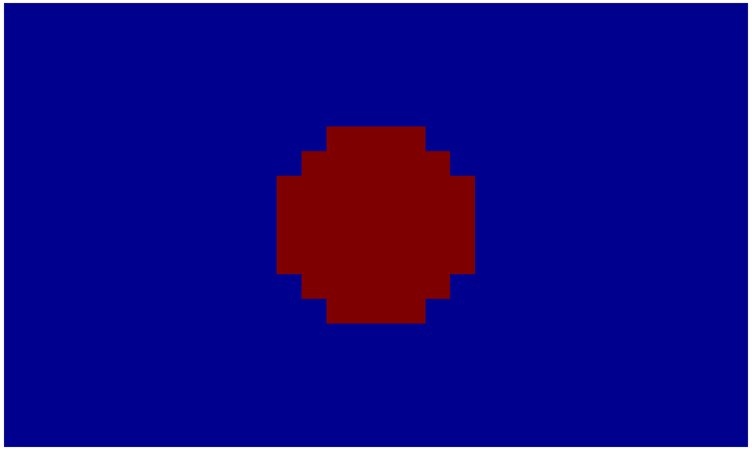


3D ellipsoid patch 2

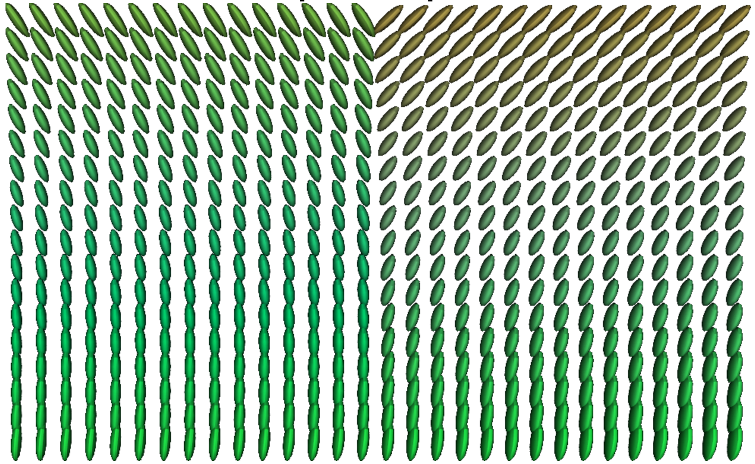
Experiment 2
Clustering on SPD(3)



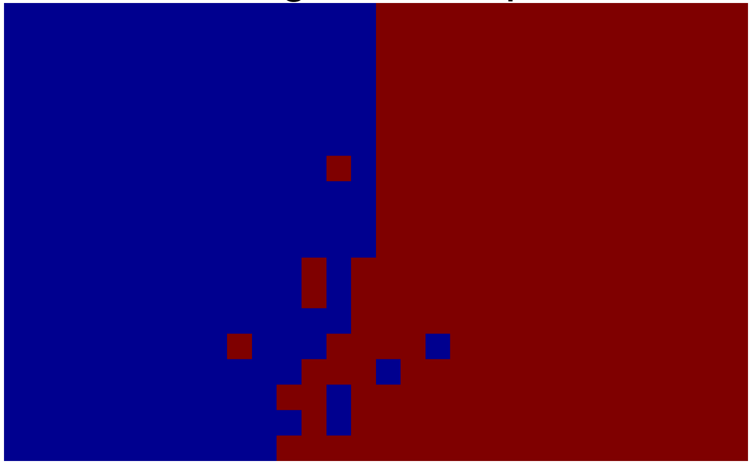
3D ellipsoid patch 1



Clustering result of patch 1

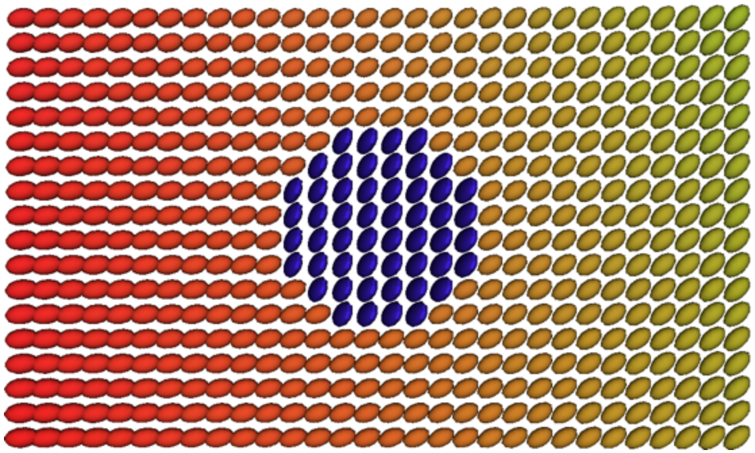


3D ellipsoid patch 2

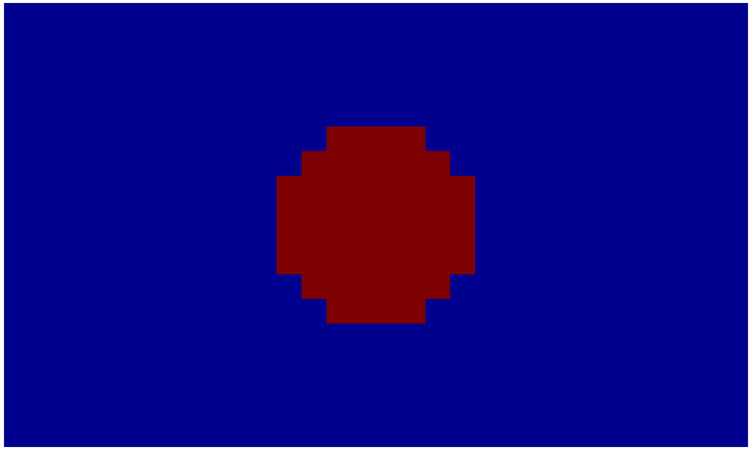


Clustering result of patch 2

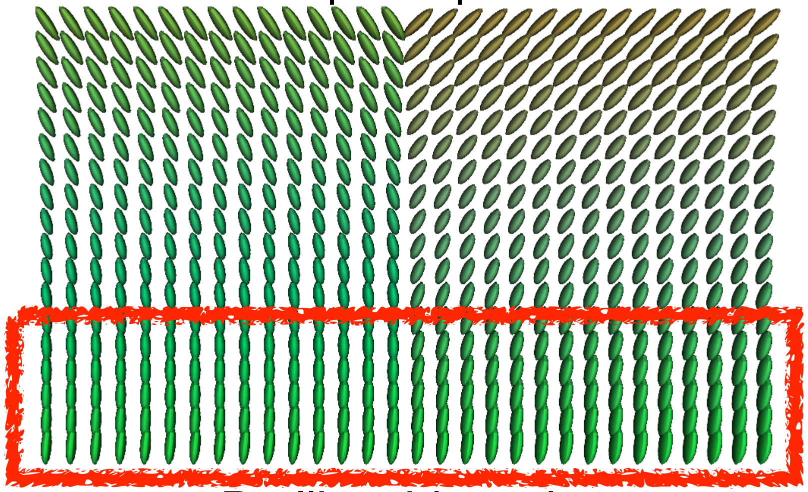
Experiment 2
Clustering on SPD(3)



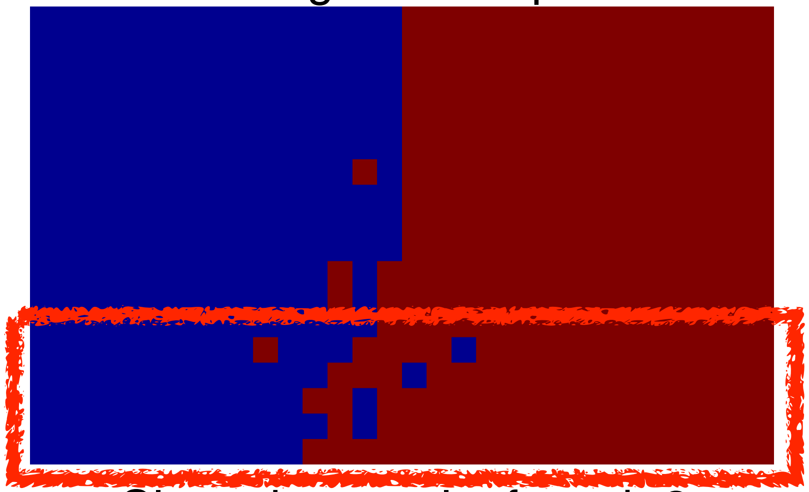
3D ellipsoid patch 1



Clustering result of patch 1

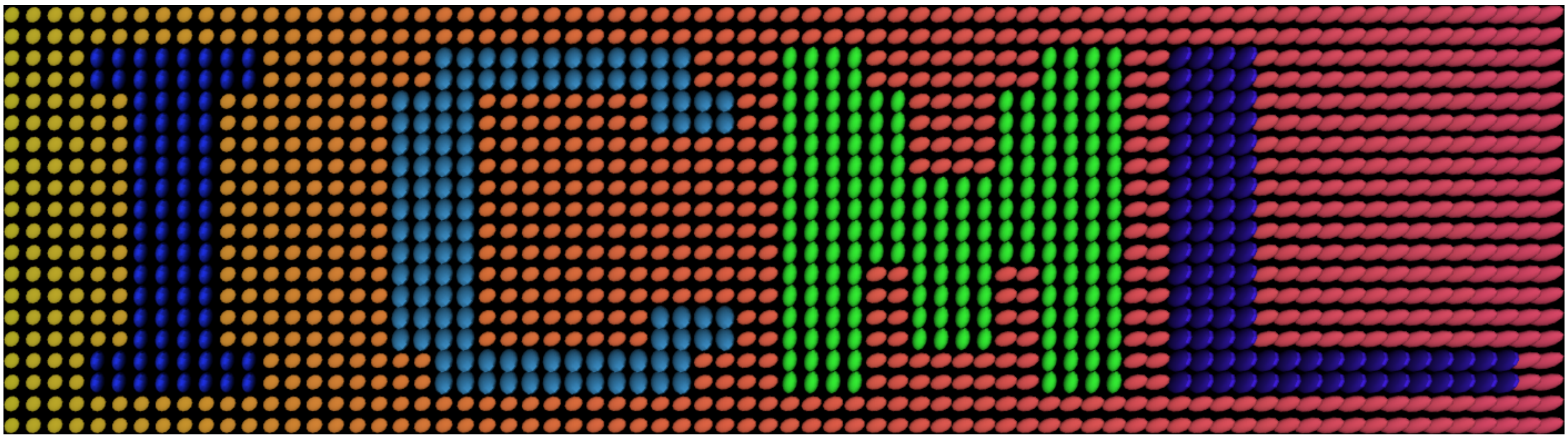


3D ellipsoid patch 2



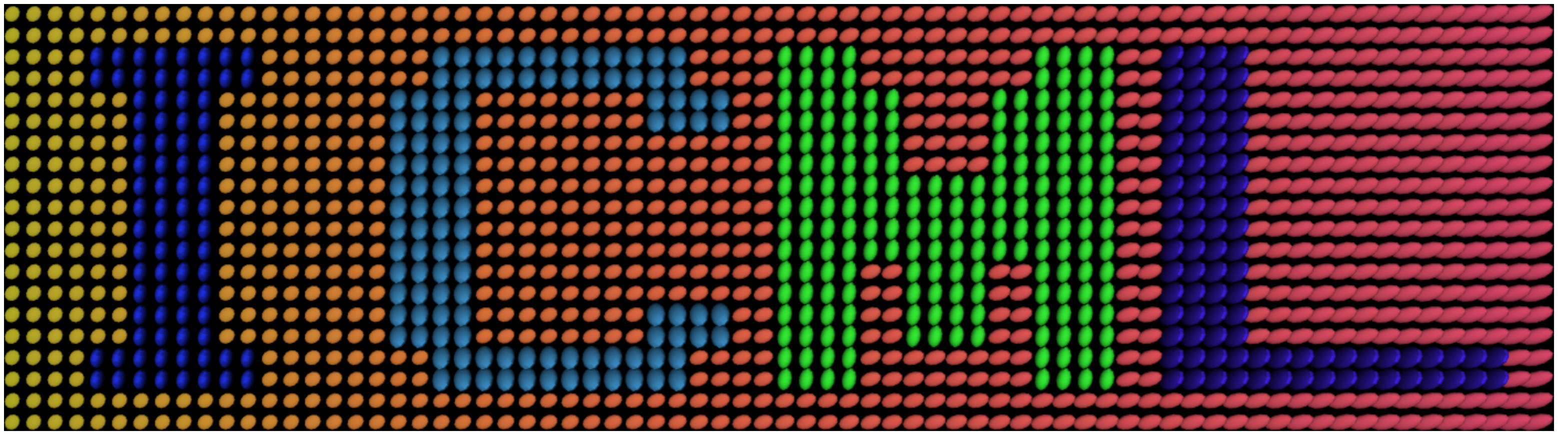
Clustering result of patch 2

Experiment 2
Clustering on SPD(3)

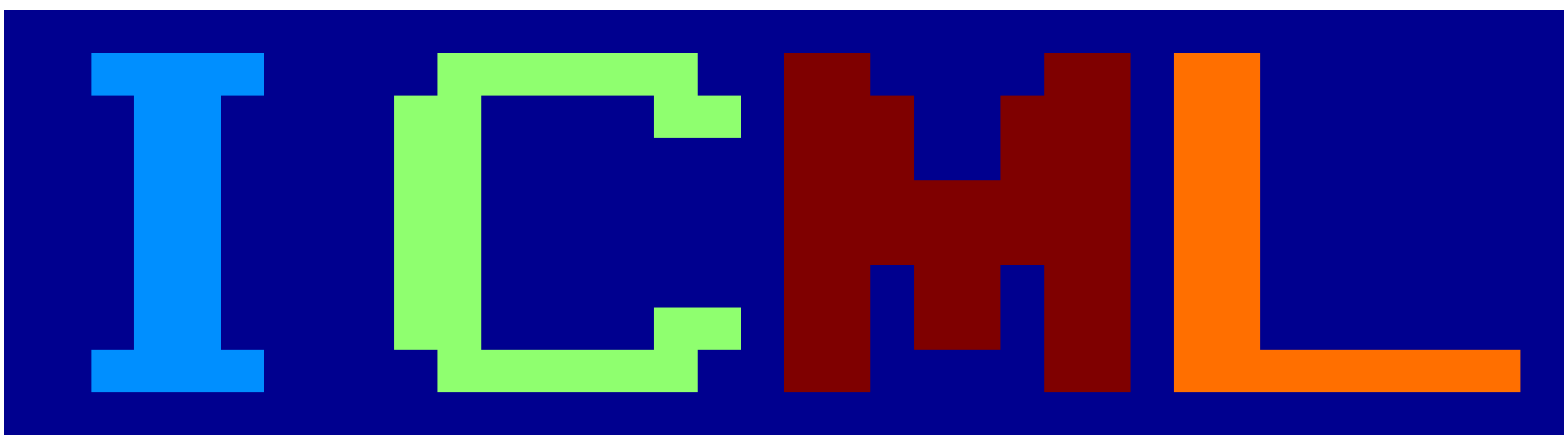


3D ellipsoid patch of ICML

Experiment 2
Clustering on SPD(3)



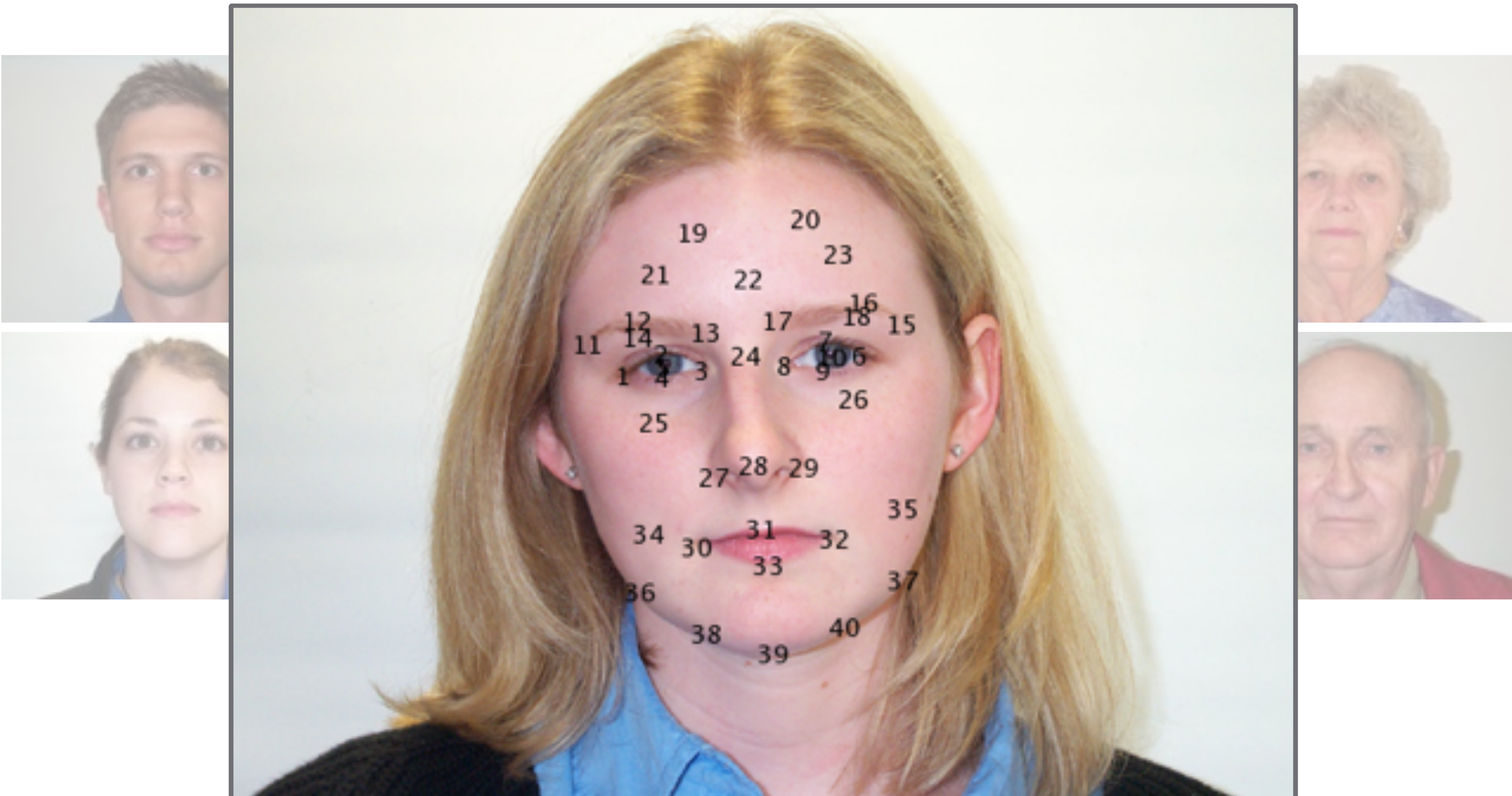
3D ellipsoid patch of ICML



Clustering results

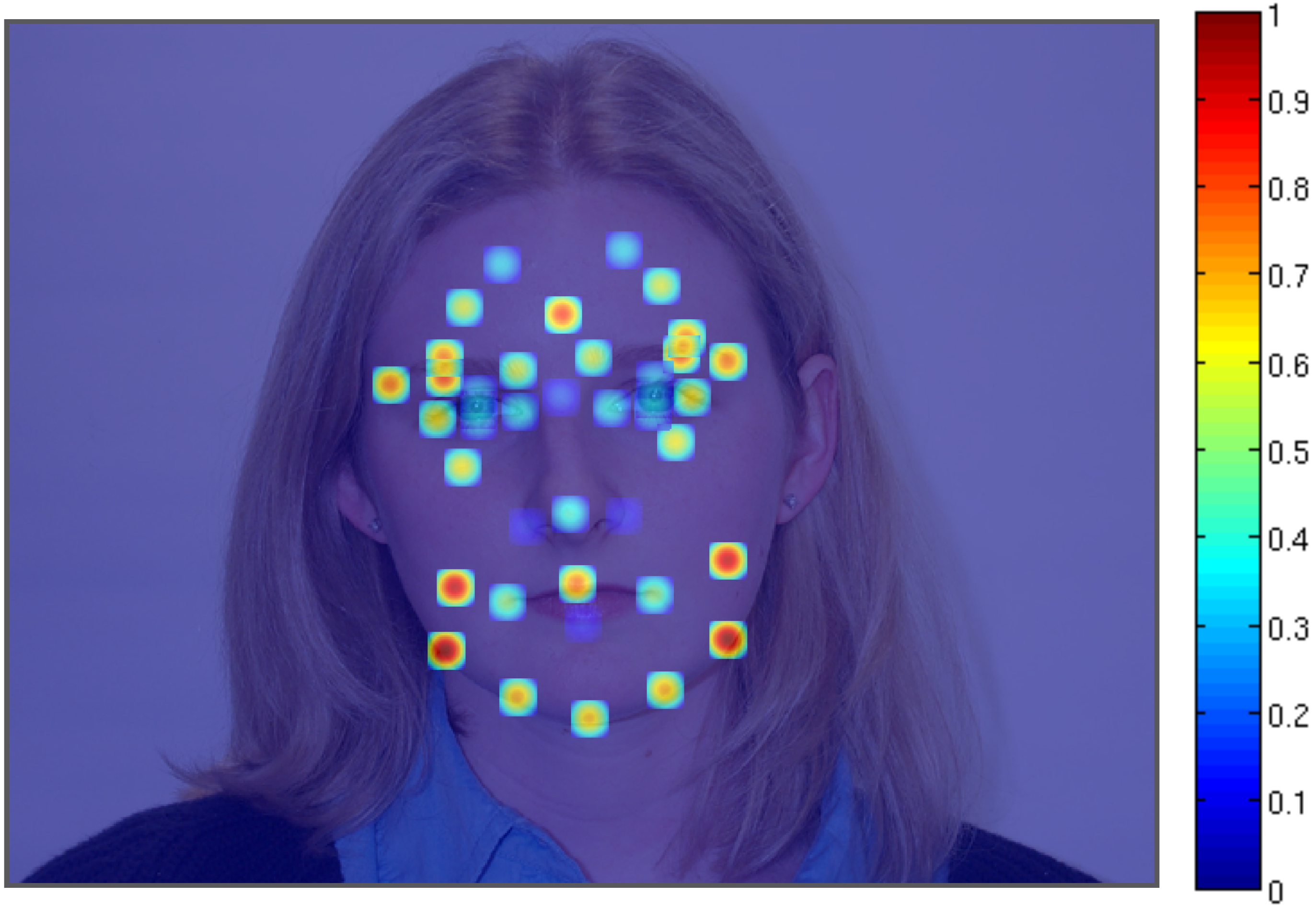
Experiment 3
Age versus landmark appearance

Experiment 3
Age versus landmark appearance



Landmarks

Experiment 3
Age versus landmark appearance



Correlation Magnitude

Conclusion

- DP-MGLM (Dirichlet process multivariate general linear model) for Riemannian manifolds (SPD(n)) learns more complicated models than MGLM
- Clustering based on nonlinear correlation between Euclidian covariates and manifold-valued response
- New distribution (over base point and tangent vectors) and HMC algorithm for DP-MGLM on SPD(n).

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Thank you!