Multivariate General Linear Models (MGLM) on Riemannian Manifolds

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http://pages.cs.wisc.edu/~hwkim/projects/riem-mglm/

Poster session ID: O-3C-384

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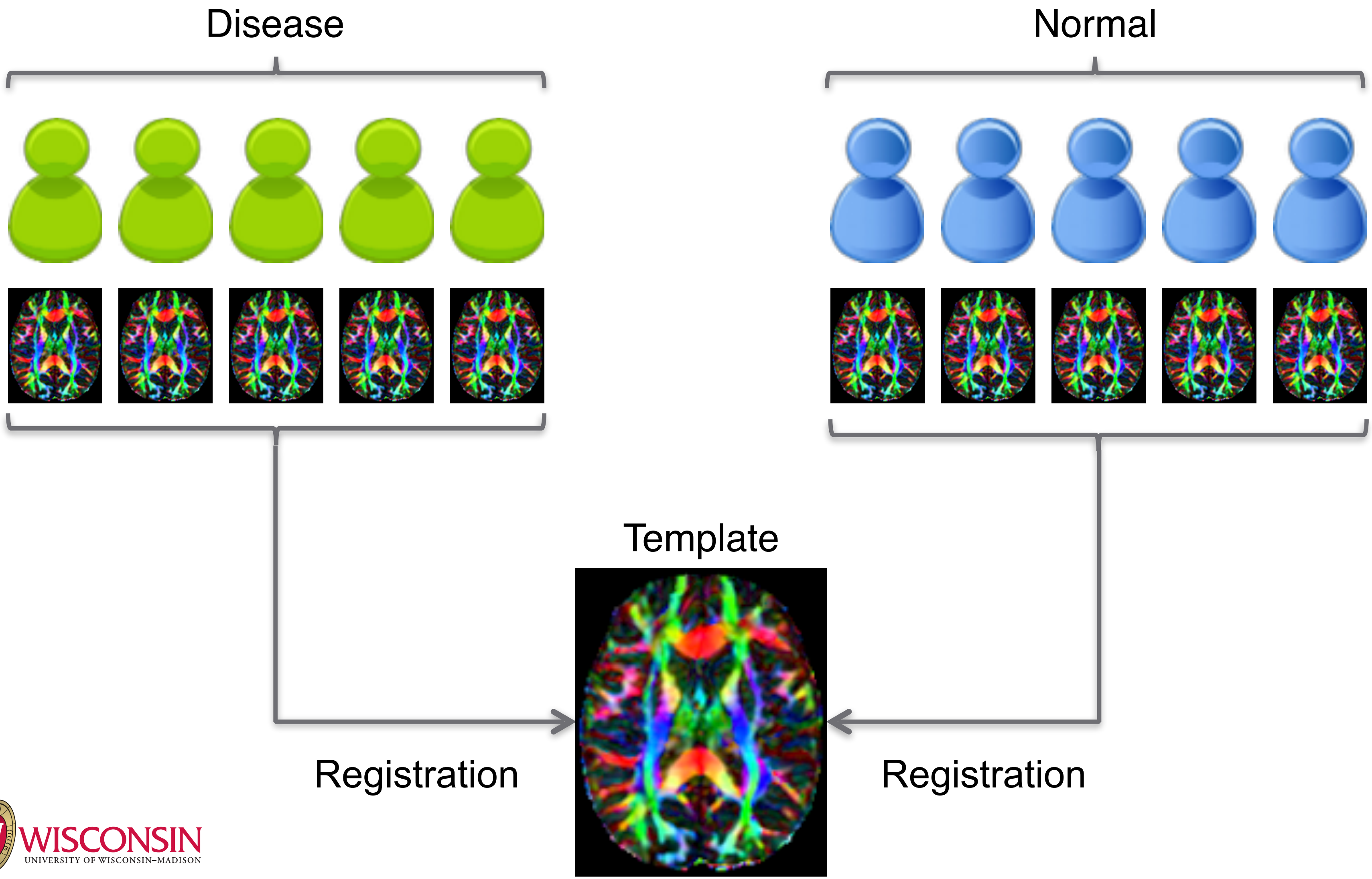
General Linear Model for Group Analysis



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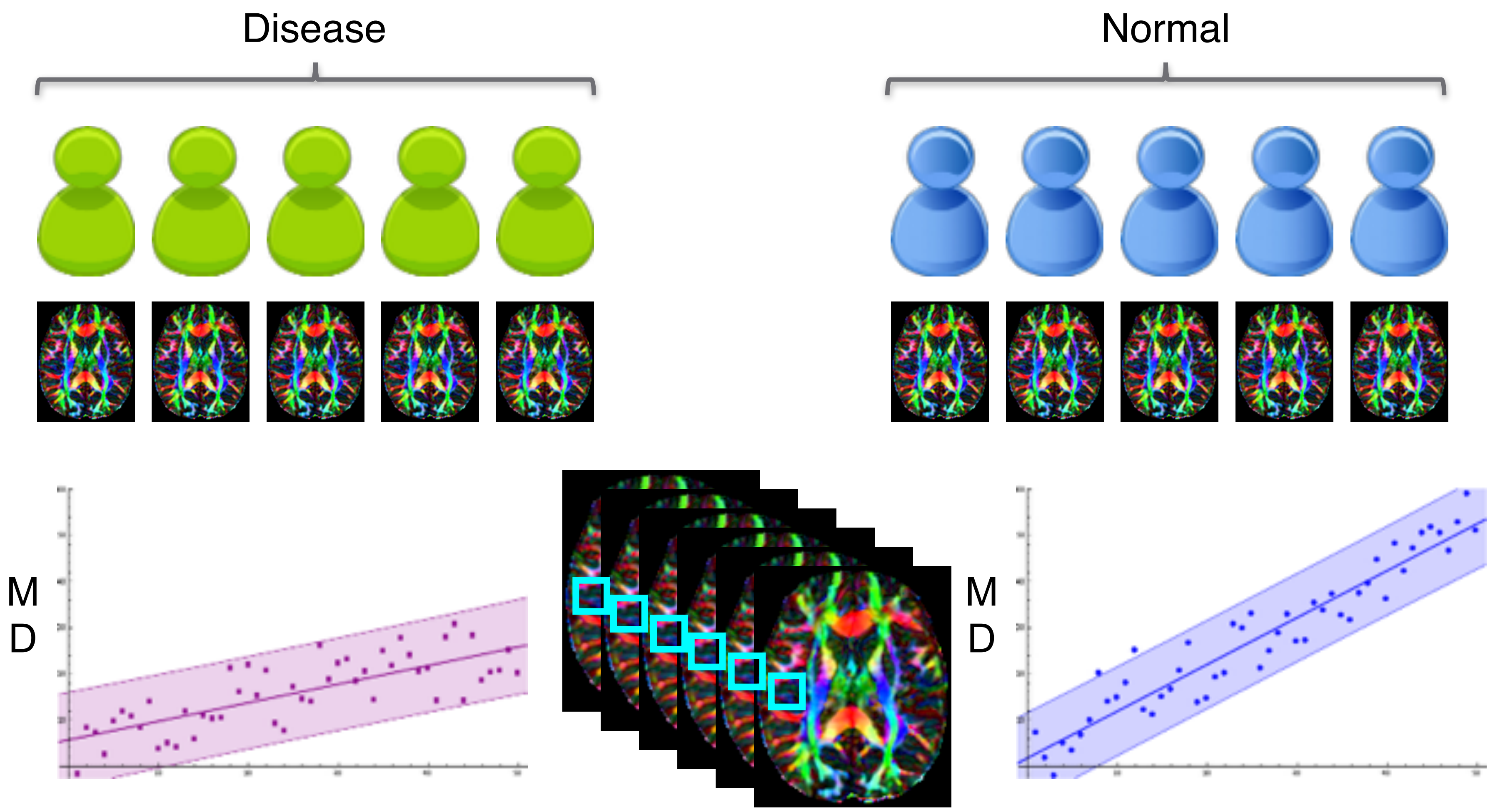

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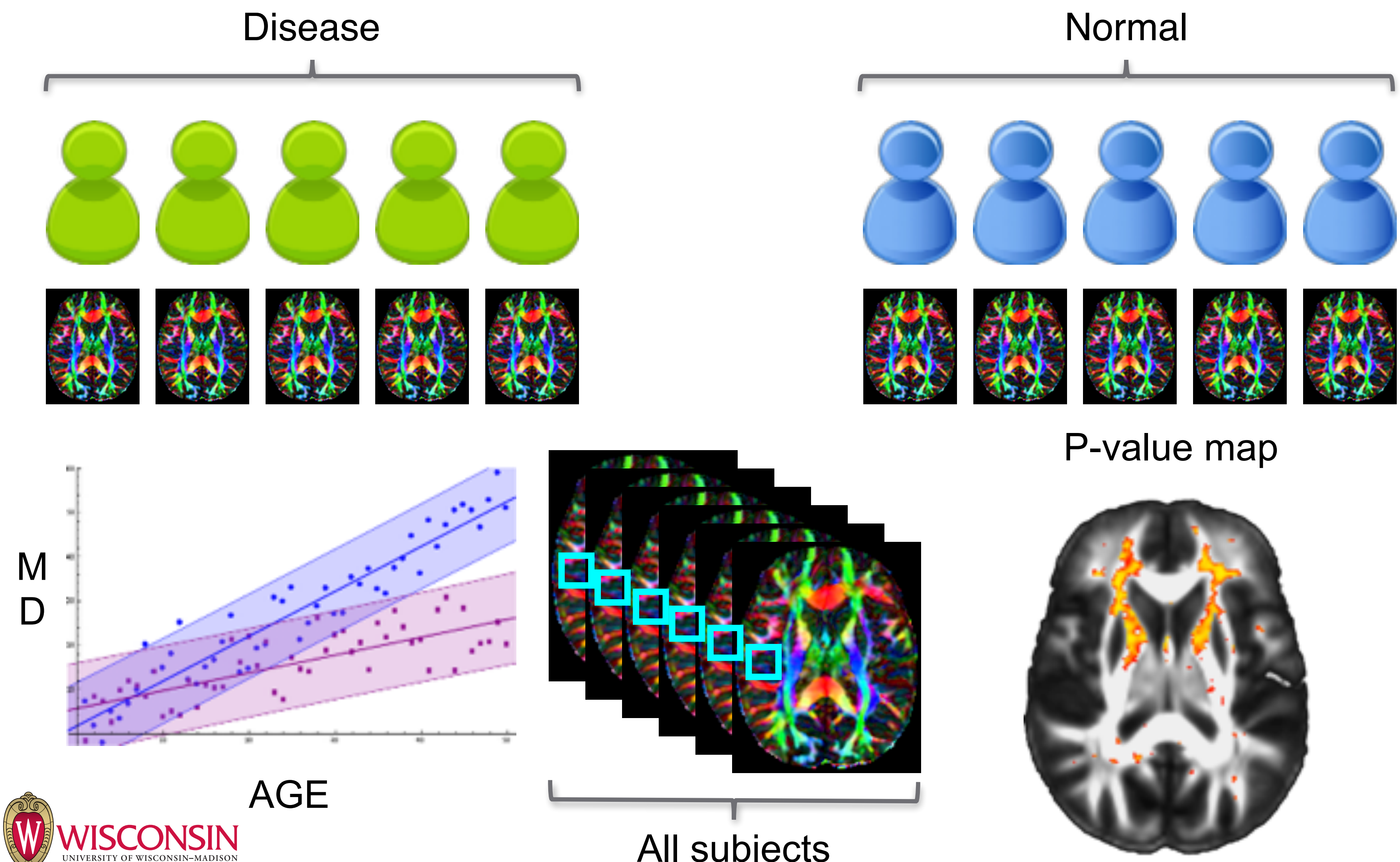
General Linear Model for Group Analysis



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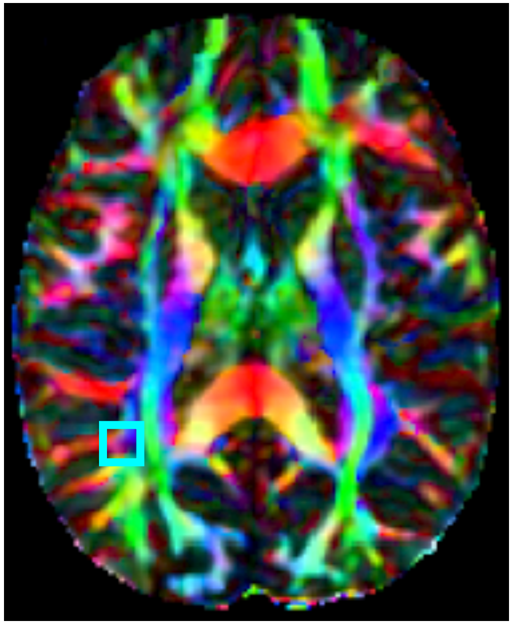
General Linear Model (GLM)

$$f : R \to R$$



$$y = \alpha + \beta^1 x^1 + \epsilon$$

x^1: Group (patient or normal)

General Linear Model (GLM)

$$f : R^2 \rightarrow R$$

$$y = \alpha + \beta^1 x^1 + \beta^2 x^2 + \epsilon$$

x^1: Group (patient or normal)

x^2: Age

General Linear Model (GLM)

$$f : R^3 \to R$$

$$y = \alpha + \beta^1 x^1 + \beta^2 x^2 + \beta^3 x^3 + \epsilon$$

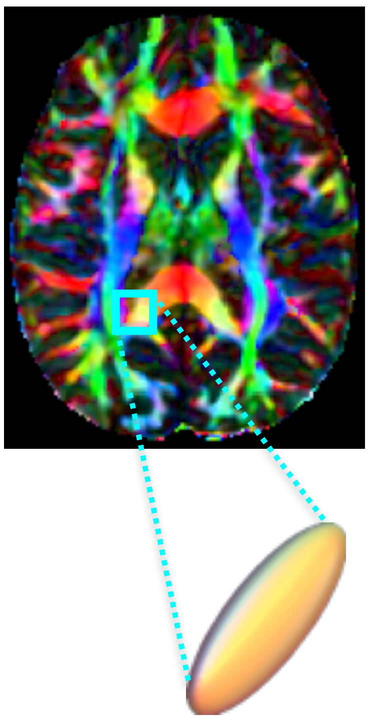
x^1: Group (patient or normal)

x^2: Age

x^3: Gender

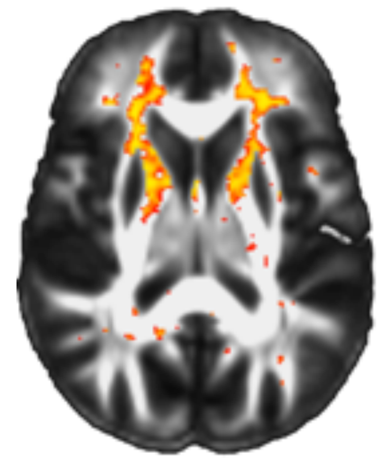
GLM on scalar valued summaries

DTI

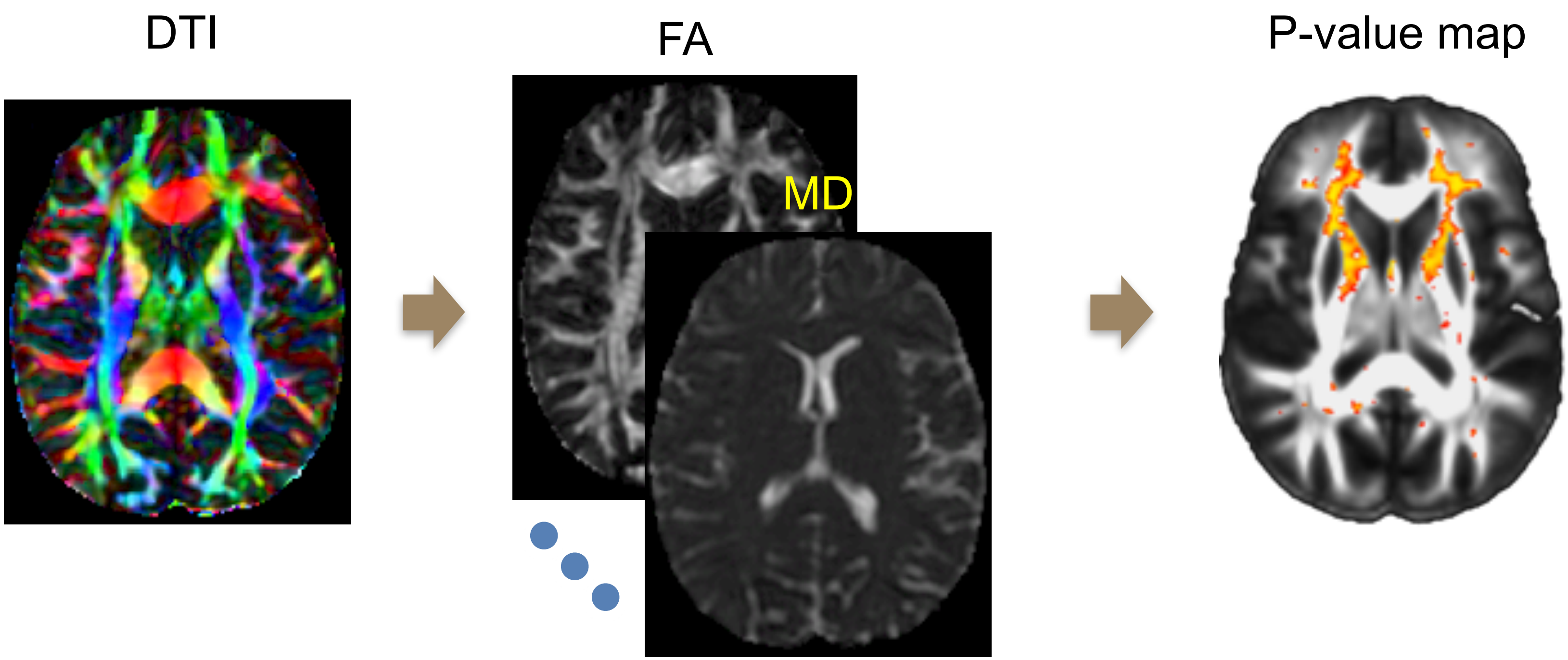


$$D = \begin{pmatrix} 1.53 & 1.38 & 0.65 \\ 1.38 & 1.33 & 0.70 \\ 0.65 & 0.70 & 1.06 \end{pmatrix}$$

P-value map



GLM on scalar valued summaries



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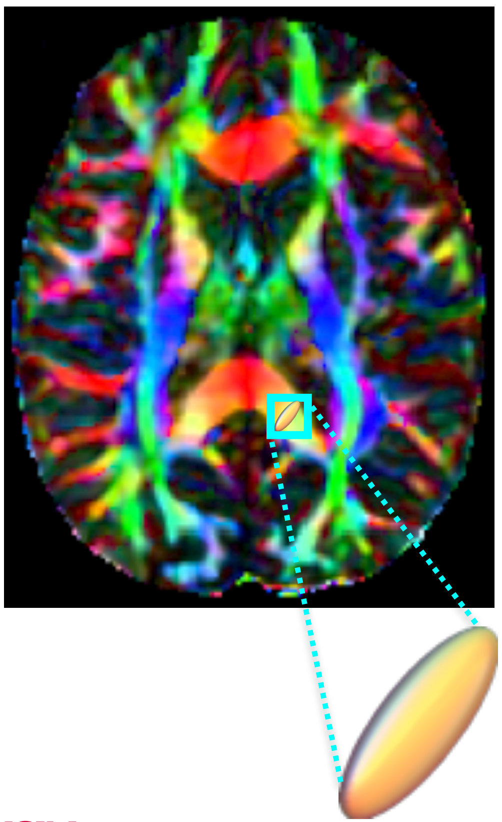
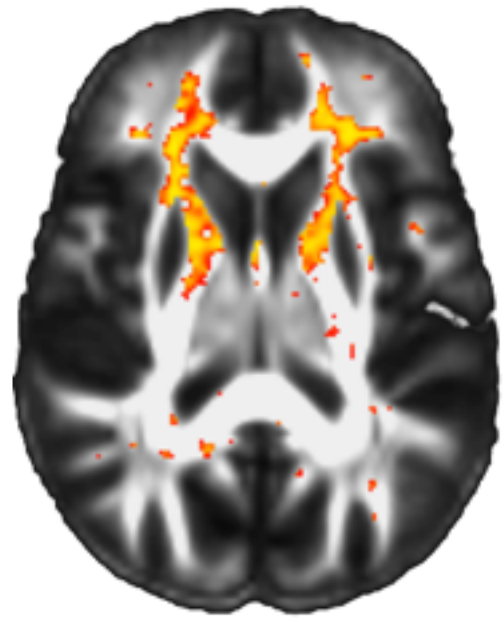
GLM on scalar valued summaries

DTI



?

P-value map



$$D = \begin{pmatrix} 1.53 & 1.38 & 0.65 \\ 1.38 & 1.33 & 0.70 \\ 0.65 & 0.70 & 1.06 \end{pmatrix} \quad x^T D x > 0, x \neq 0$$

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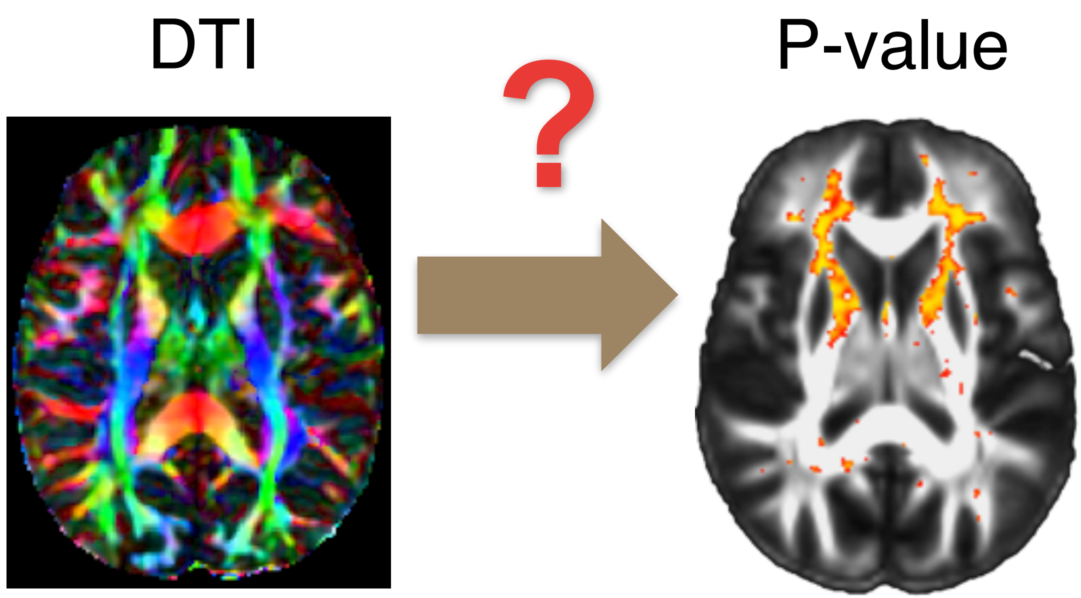
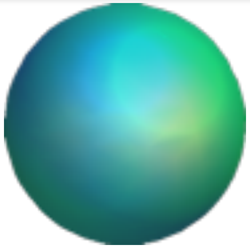
GLM on scalar valued summaries



Response Y is manifold-valued

Manifold-valued data

- Unit sphere, quotient spaces of spheres



- SPD matrix i.e covariance matrix, diffusion tensors

- Probability density functions (PDFs), orientation density functions (ODFs)

- Lie groups i.e. O(n), SO(n), GL(n), SL(n)
- Kendall shape manifolds

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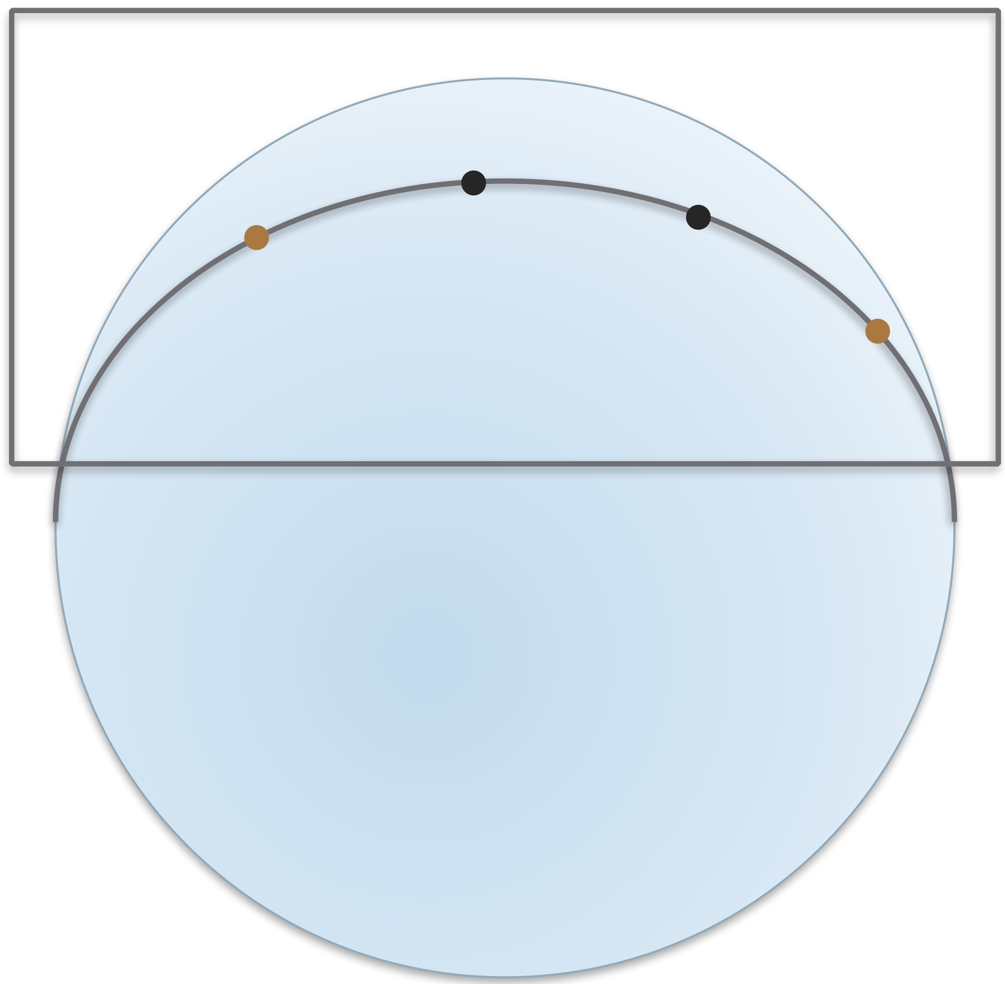
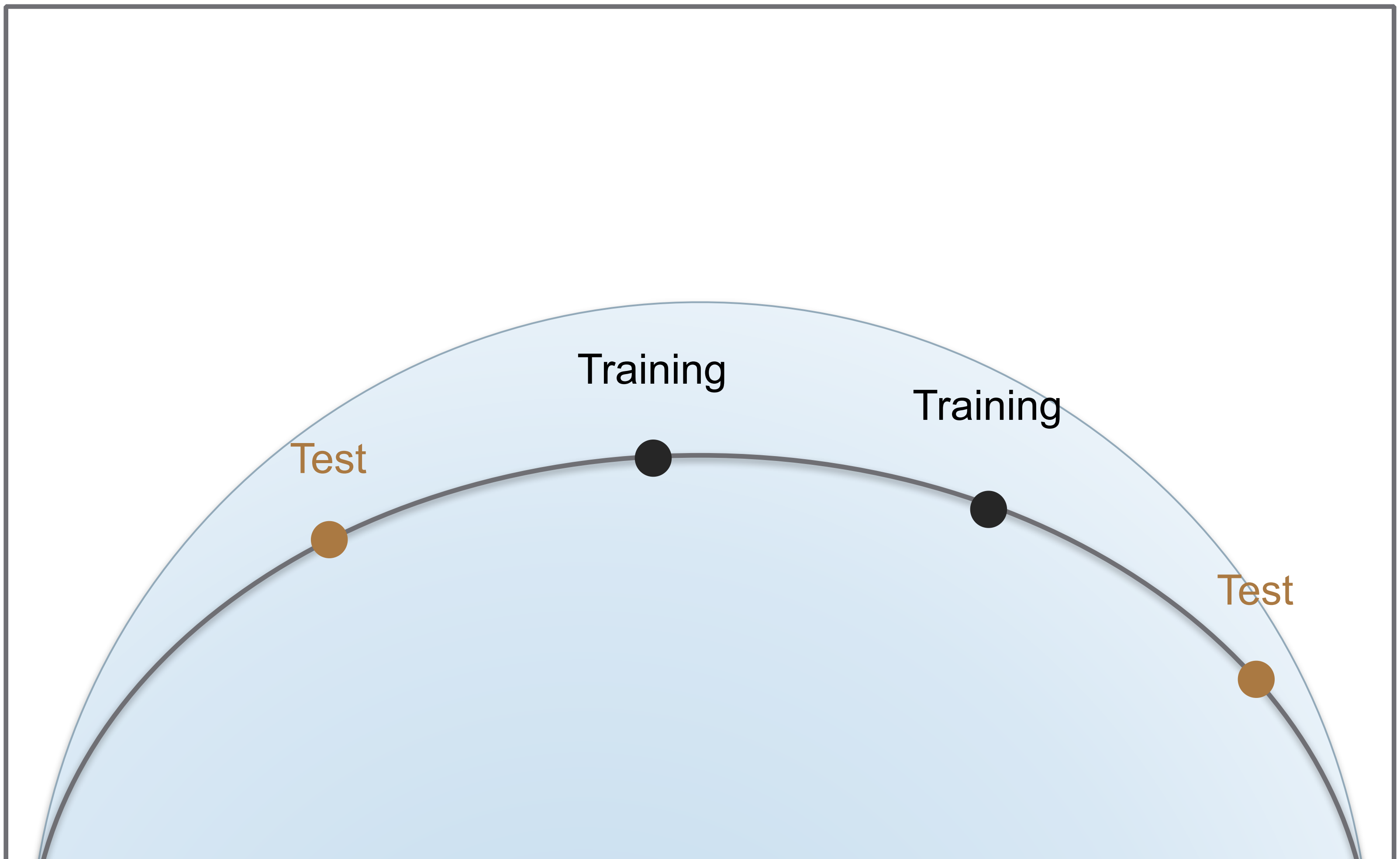
Can we directly use the ordinary general linear model with manifold data?

Euclidean model for manifolds



Euclidean model for manifolds

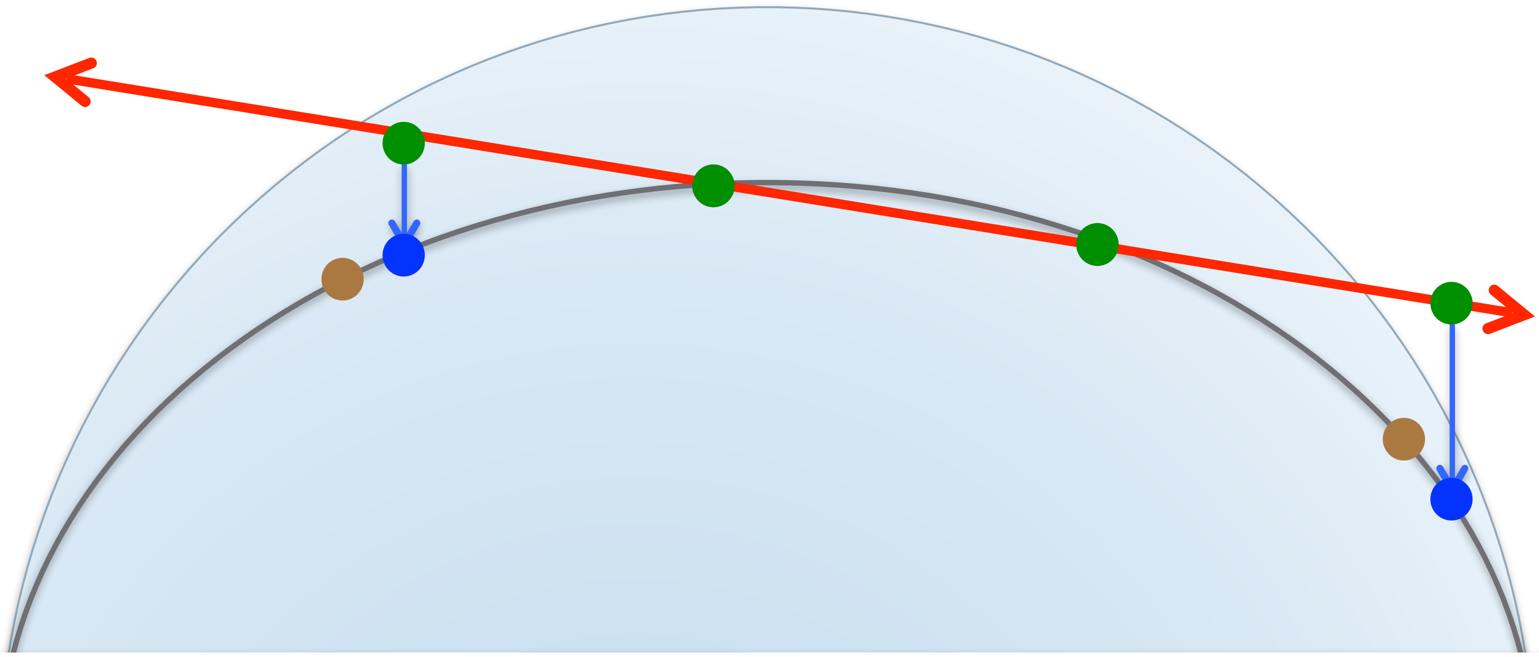


Euclidean model for manifolds

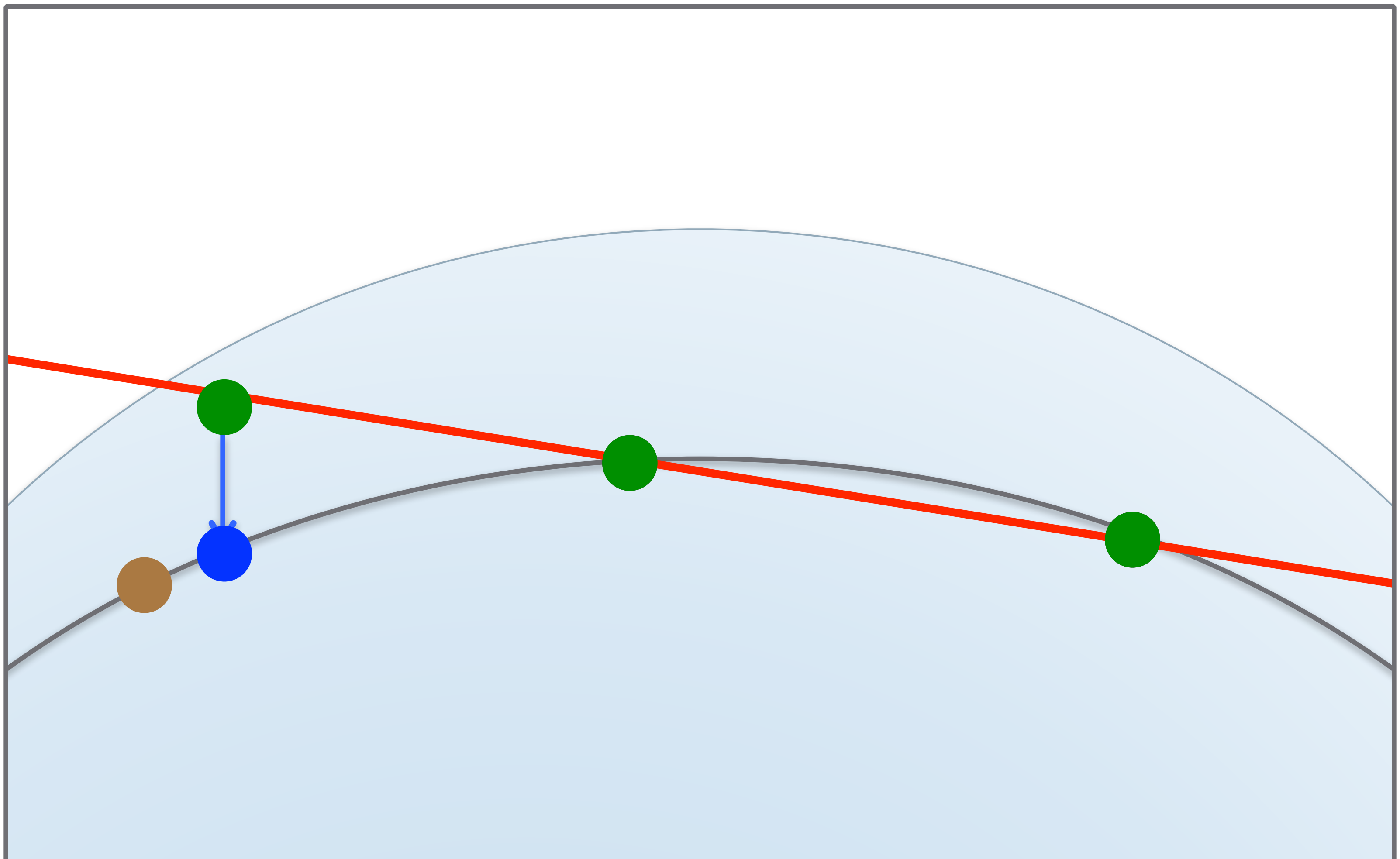
Problem 1: Model in ambient space

Predictions in ambient space

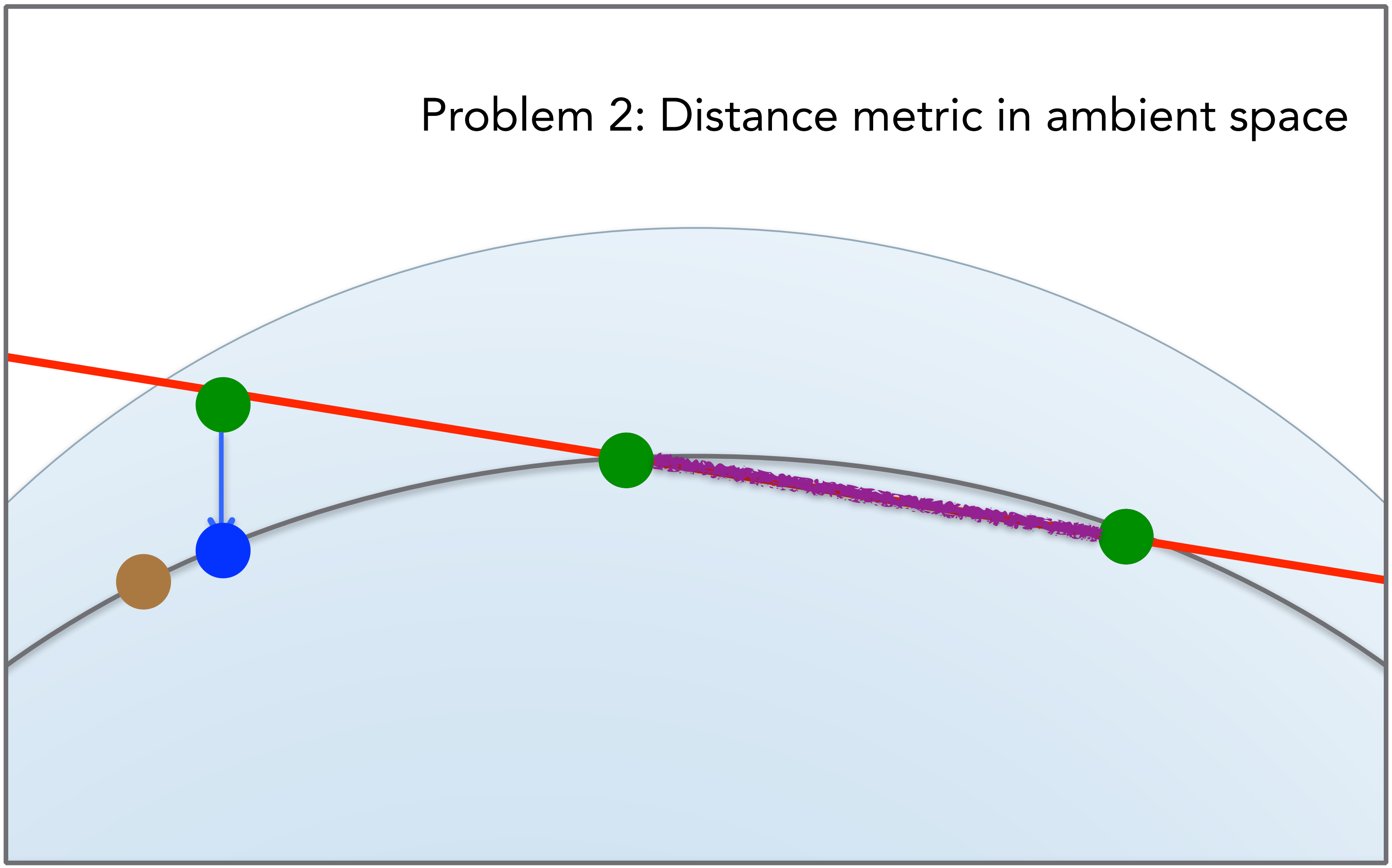
Predictions need to be projected onto manifolds



Euclidean model for manifolds



Euclidean model for manifolds



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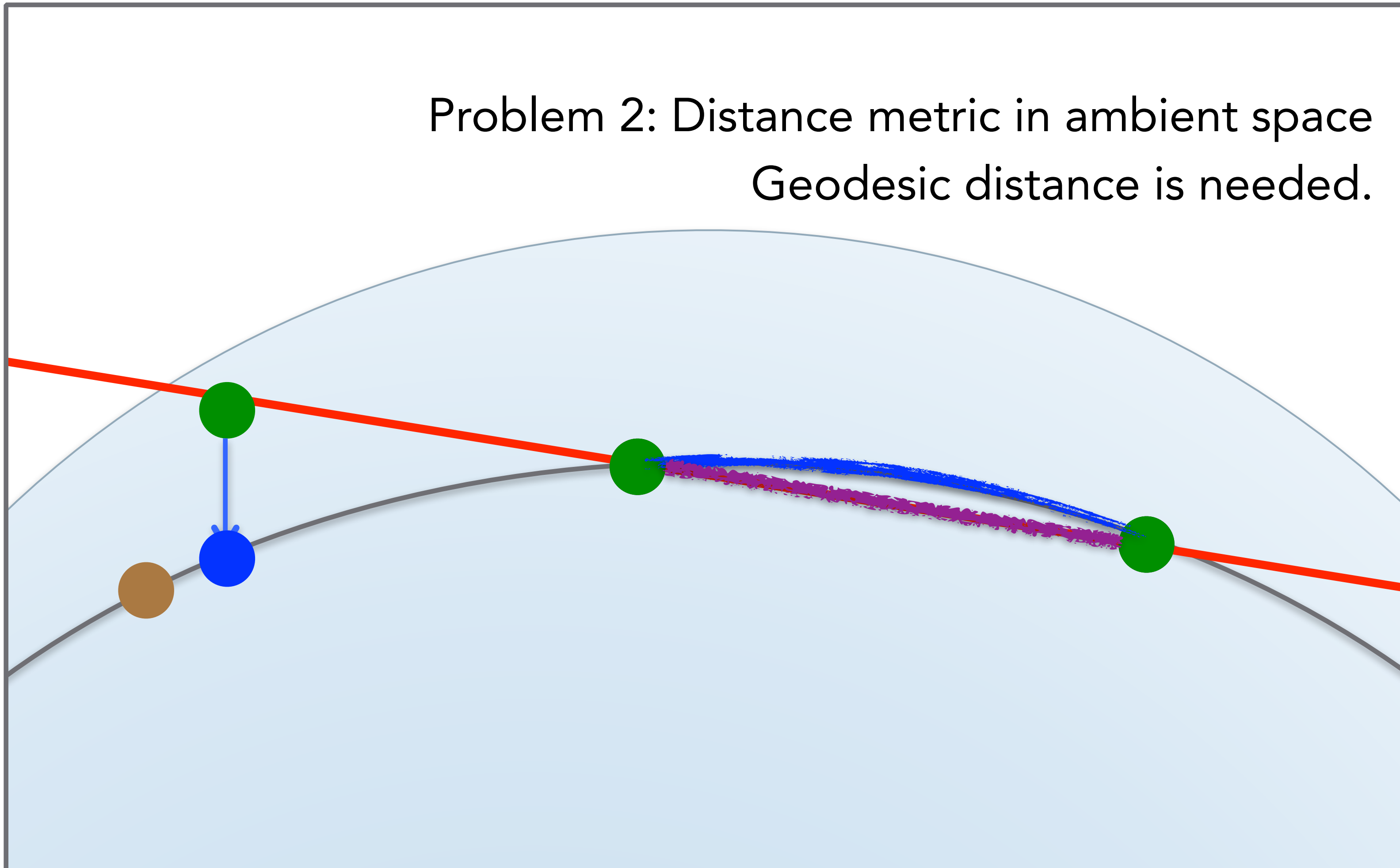
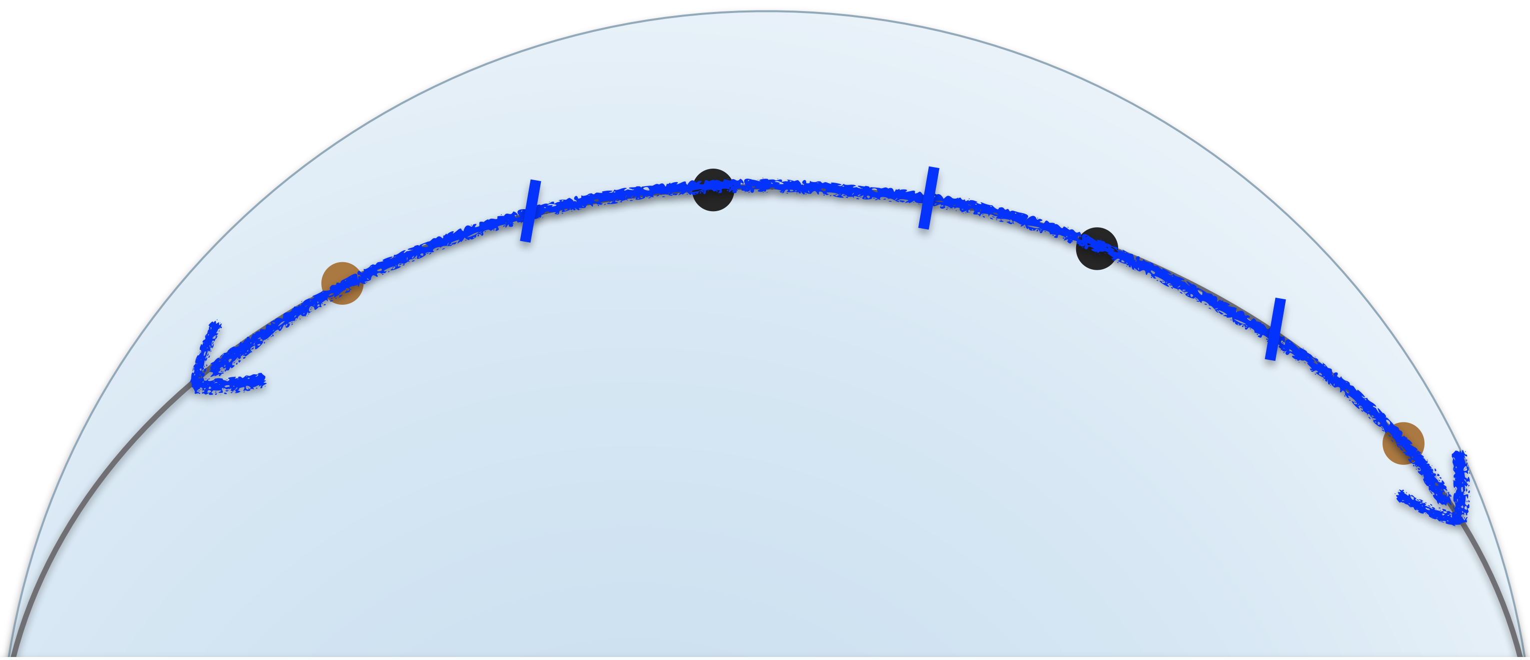


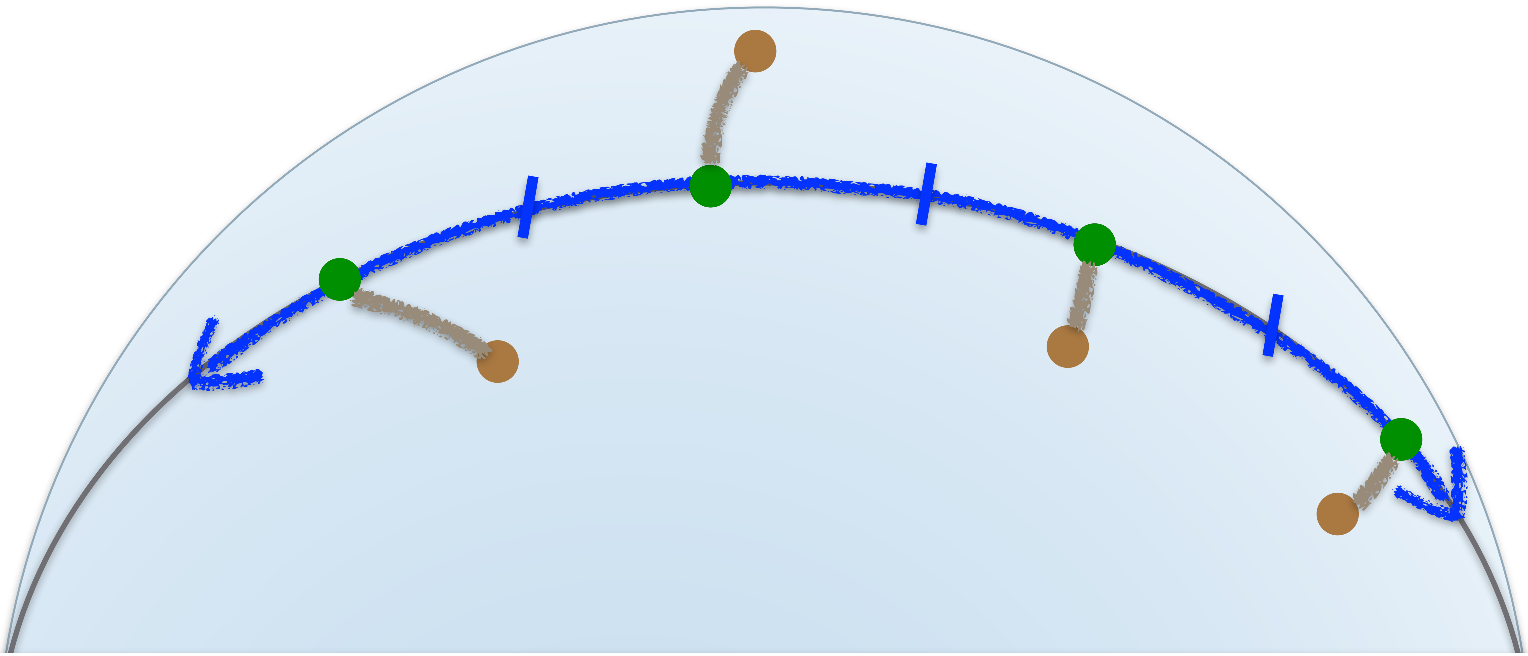
Model on manifolds



Regression with a Single Covariate

$f : R \to \mathcal{M}$

Fletcher, P. Thomas. "Geodesic regression and the theory of least squares on Riemannian manifolds." IJCV, 2013.



Model on manifolds

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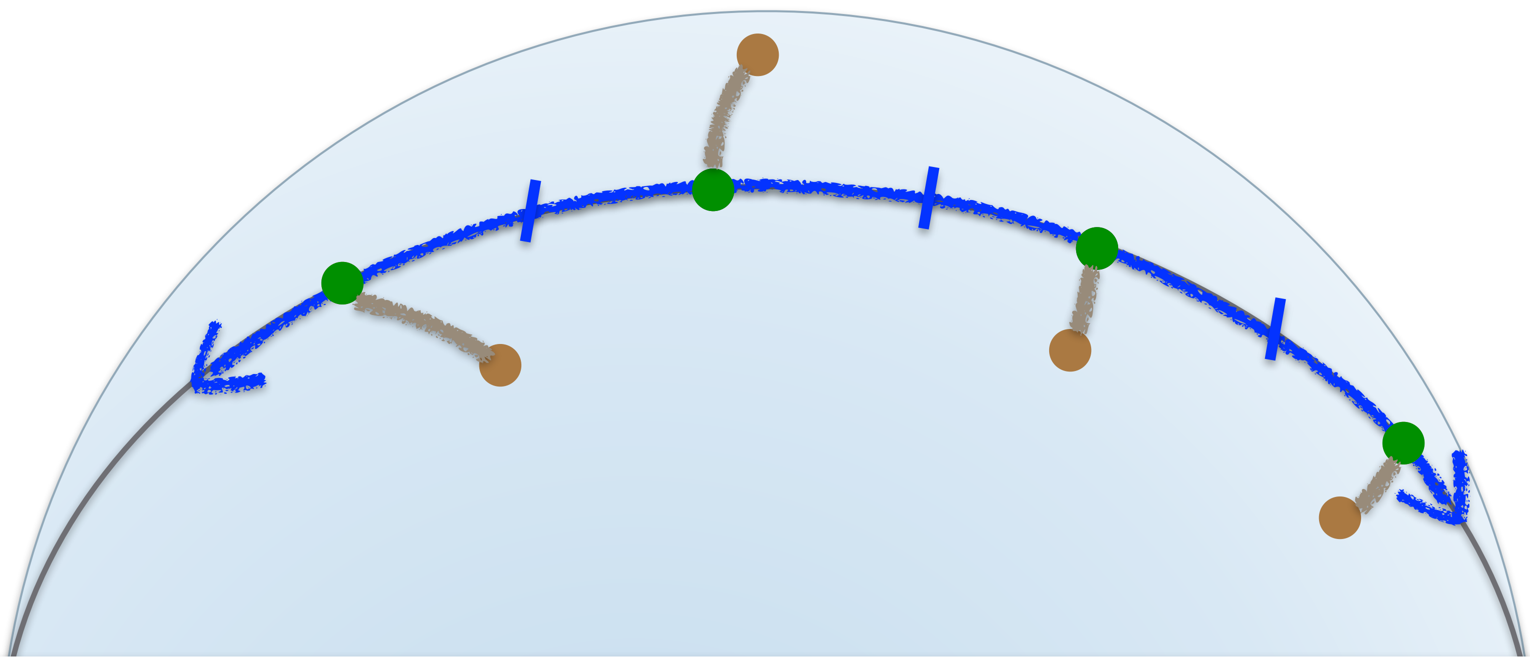

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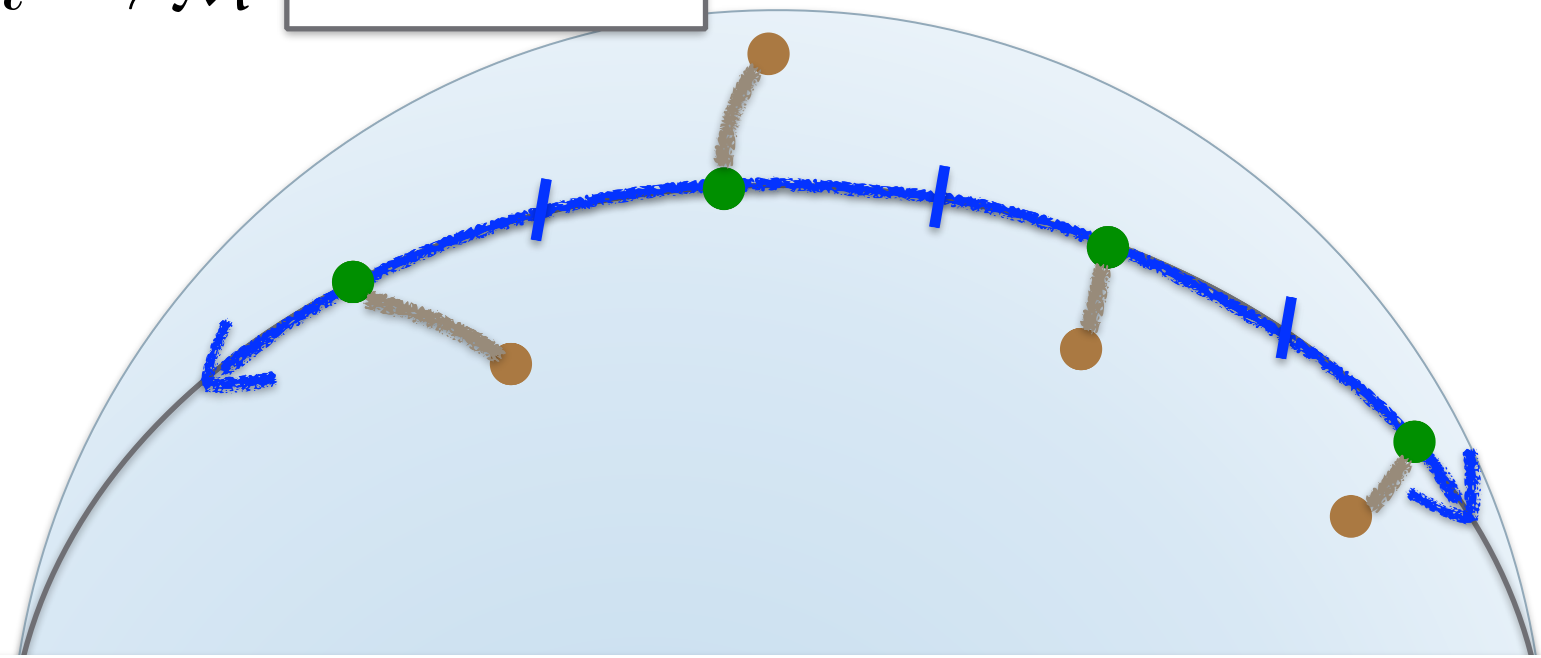
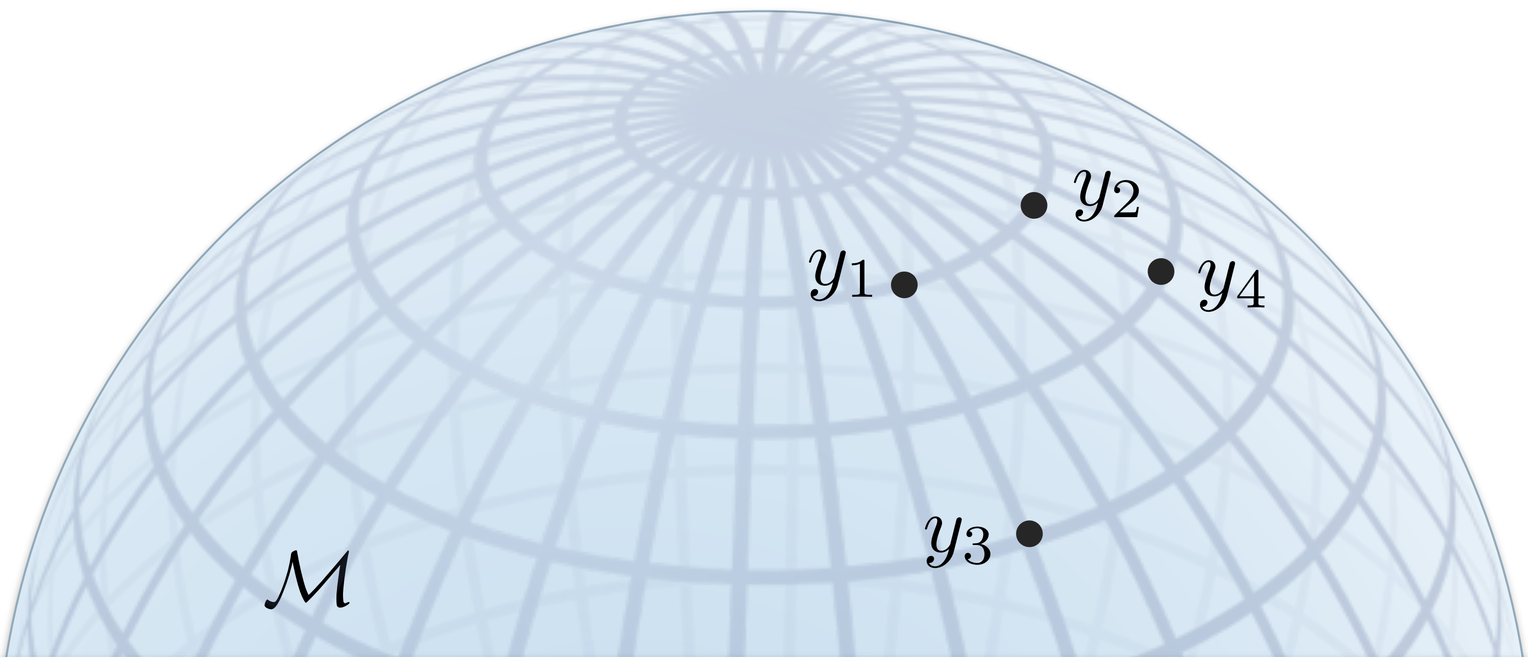
$f : R^n \rightarrow \mathcal{M}$

Our MGLM



Single covariate

$$\mathrm{E}(p, v) = \frac{1}{2}\sum_{i=1}^{N} \mathrm{d}(\mathrm{Exp}(p, x_i v), y_i)^2$$



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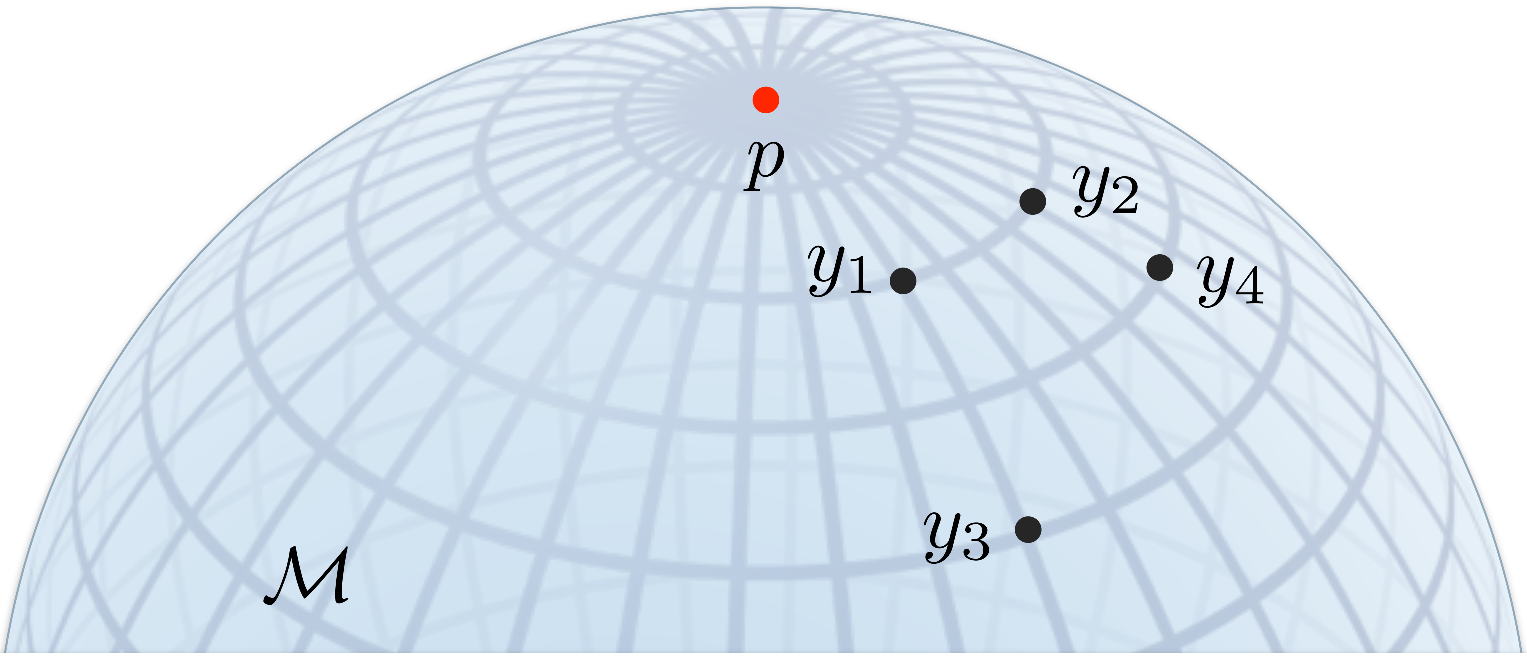

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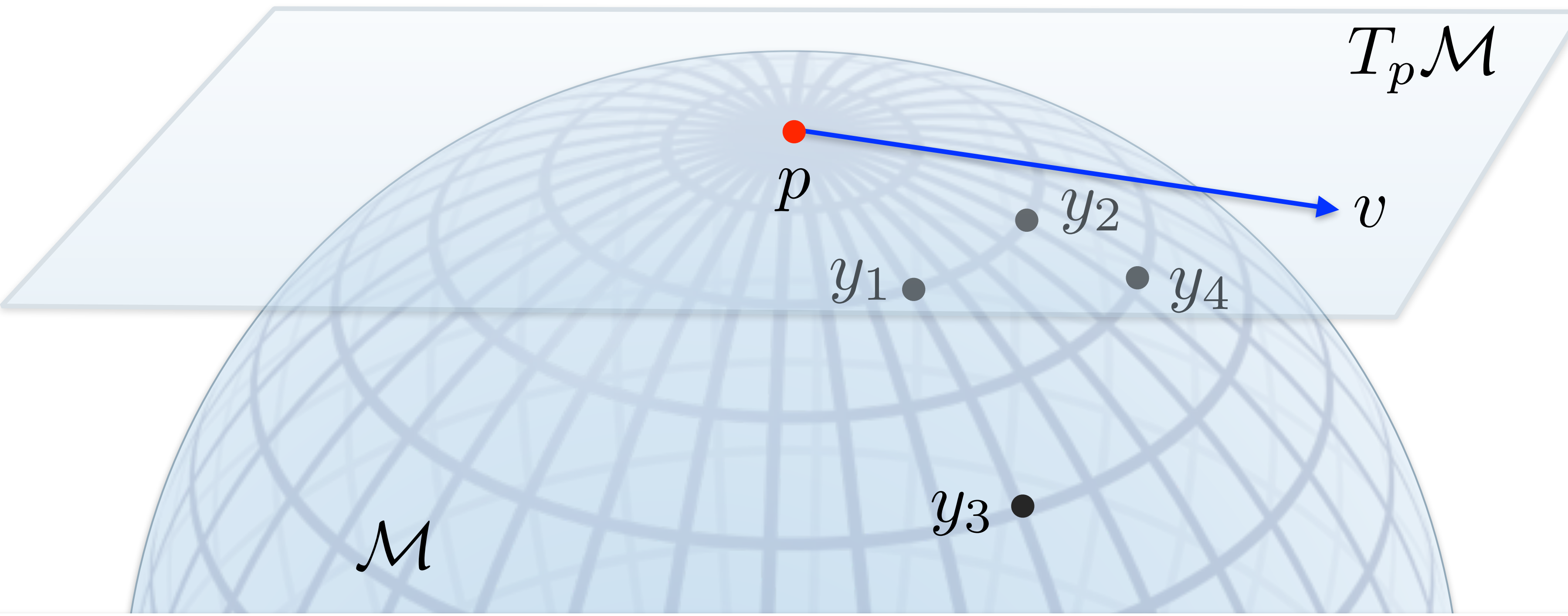

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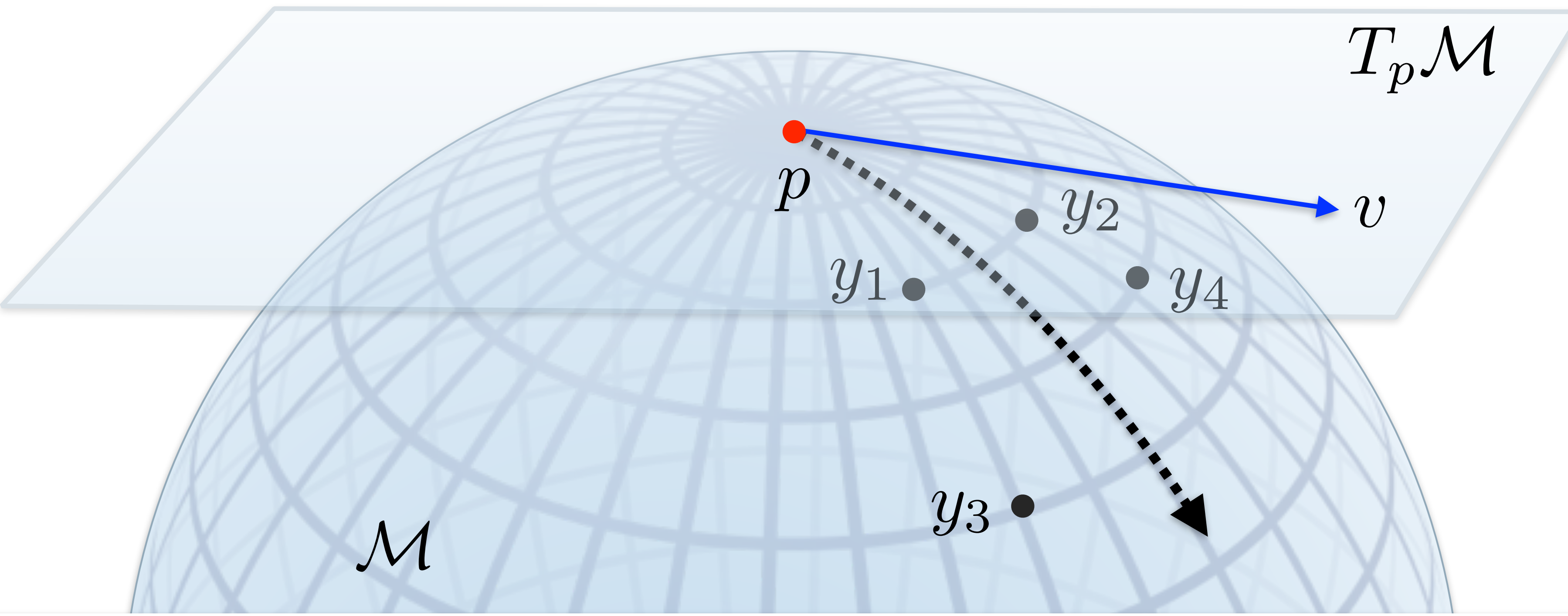

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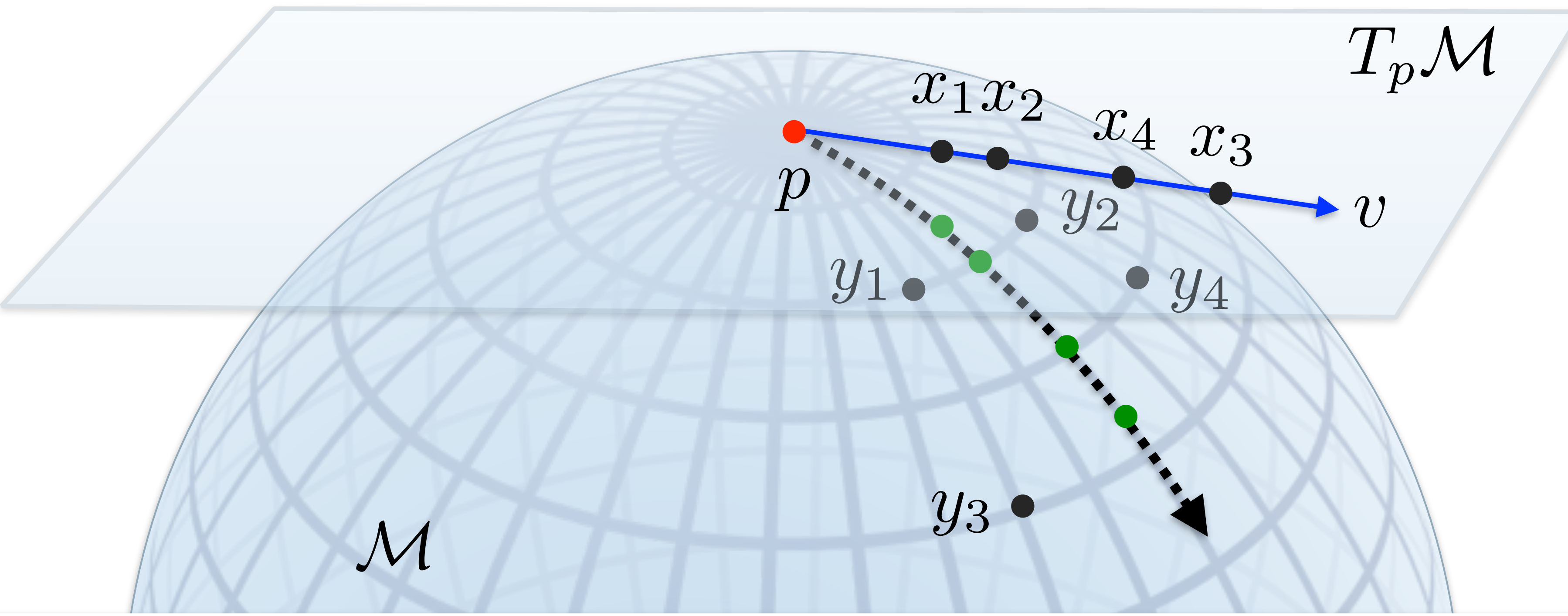

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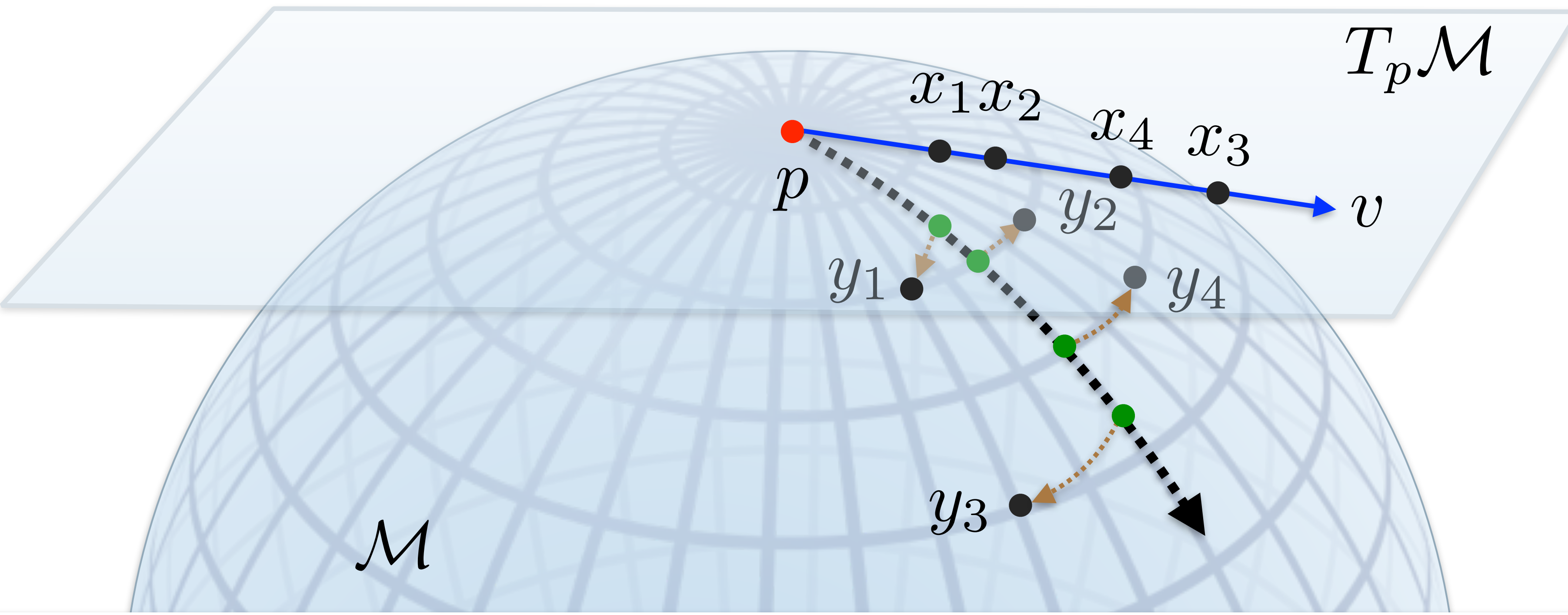
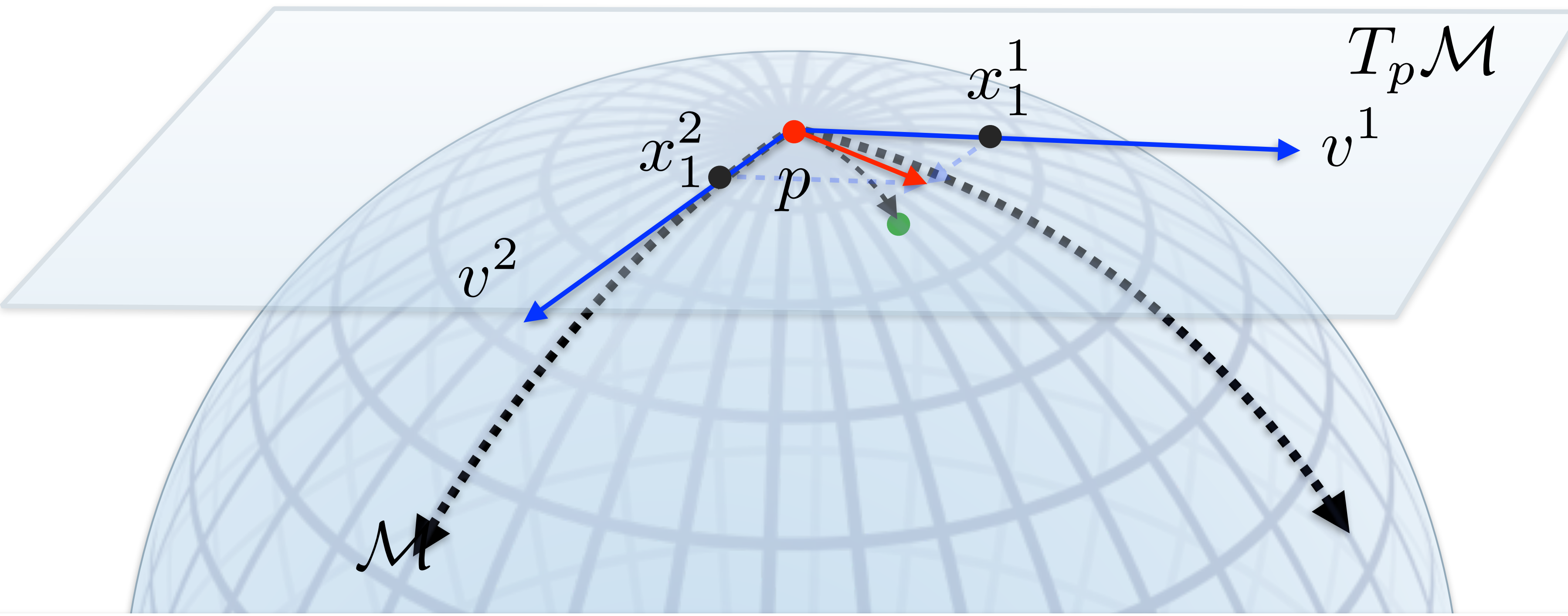


Multiple covariates

$$\mathrm{E}(p, \boldsymbol{V}) = \frac{1}{2}\sum_{i=1}^{N} \mathrm{d}(\mathrm{Exp}(p, \sum_j x_i^j v^j), y_i)^2$$



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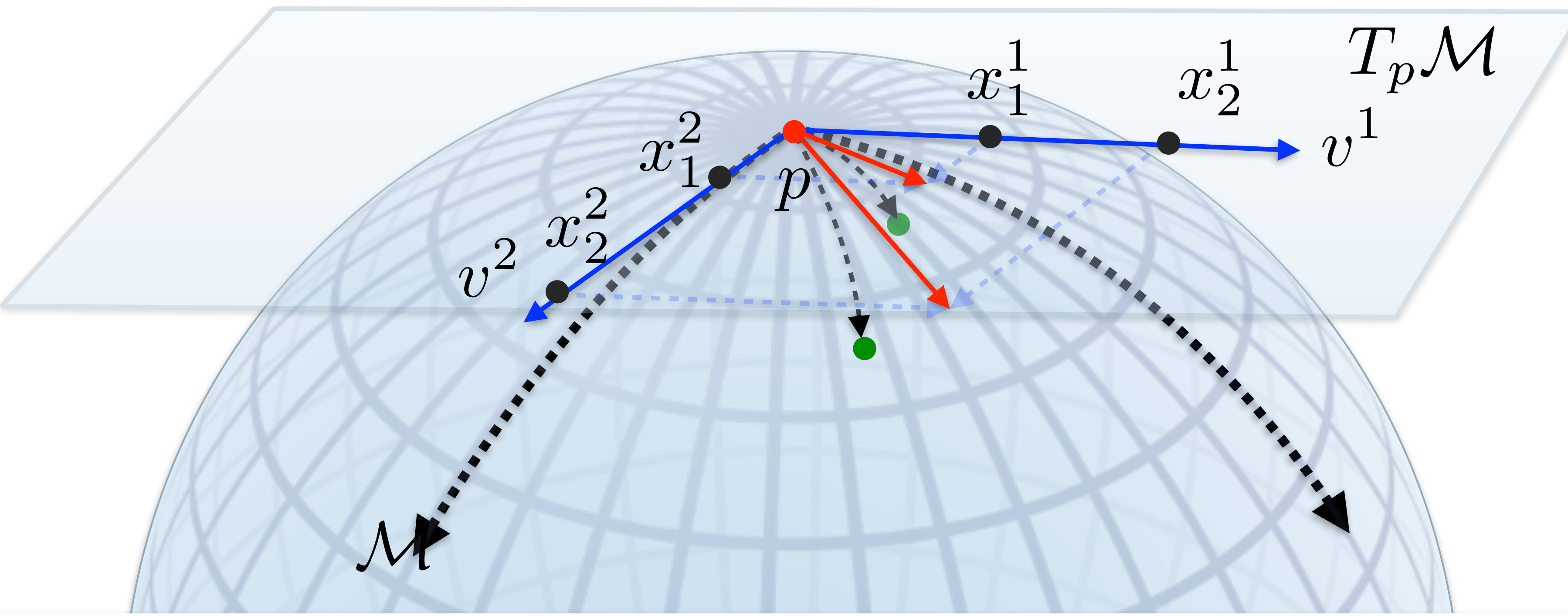
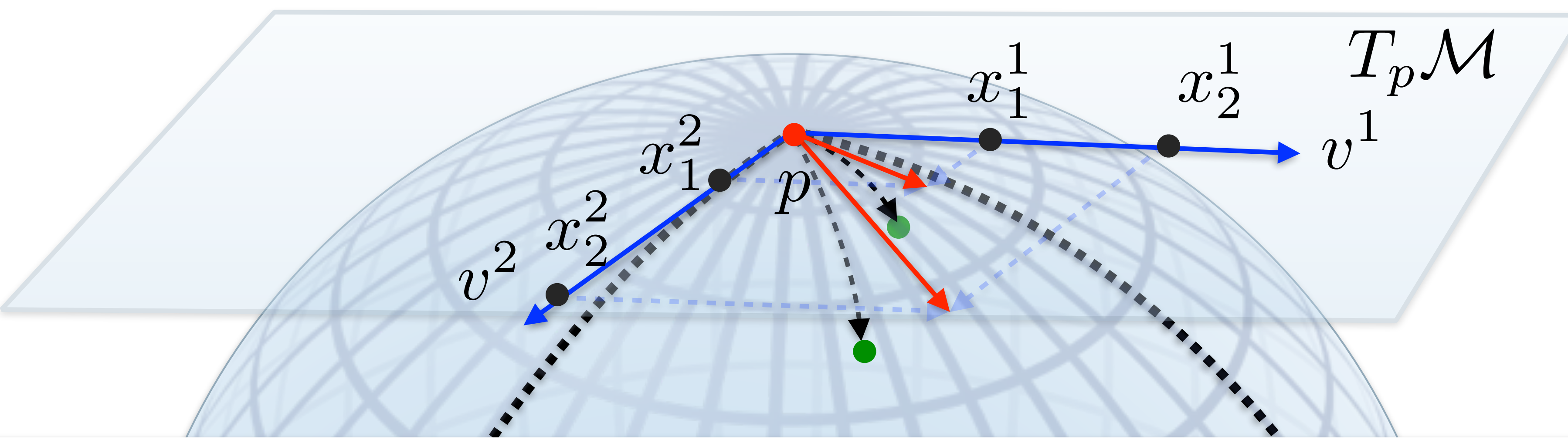


Multiple covariates

$$E(p, \boldsymbol{V}) = \frac{1}{2}\sum_{i=1}^{N} d(\mathrm{Exp}(p, \sum_j x_i^j v^j), y_i)^2$$

$$y = \alpha + \beta^1 x^1 + \beta^2 x^2 + \beta^3 x^3 + \epsilon$$



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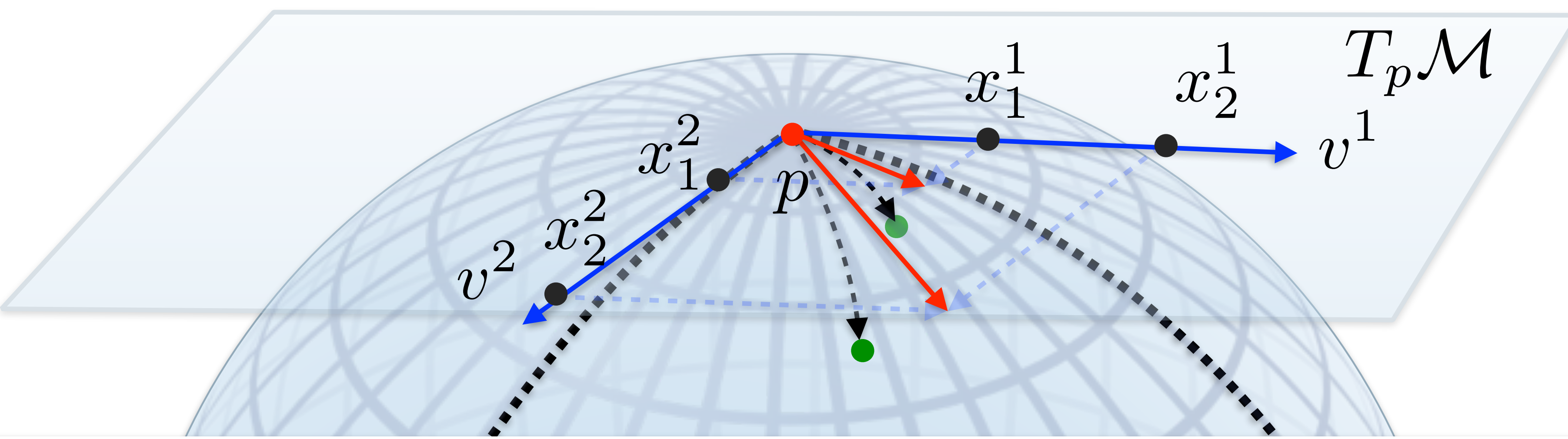

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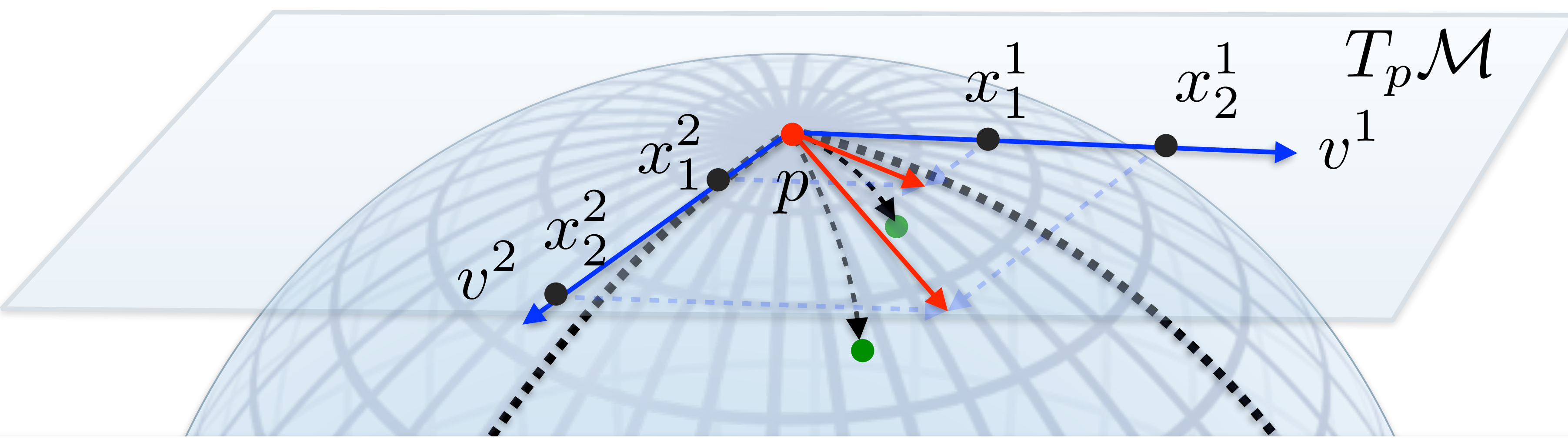
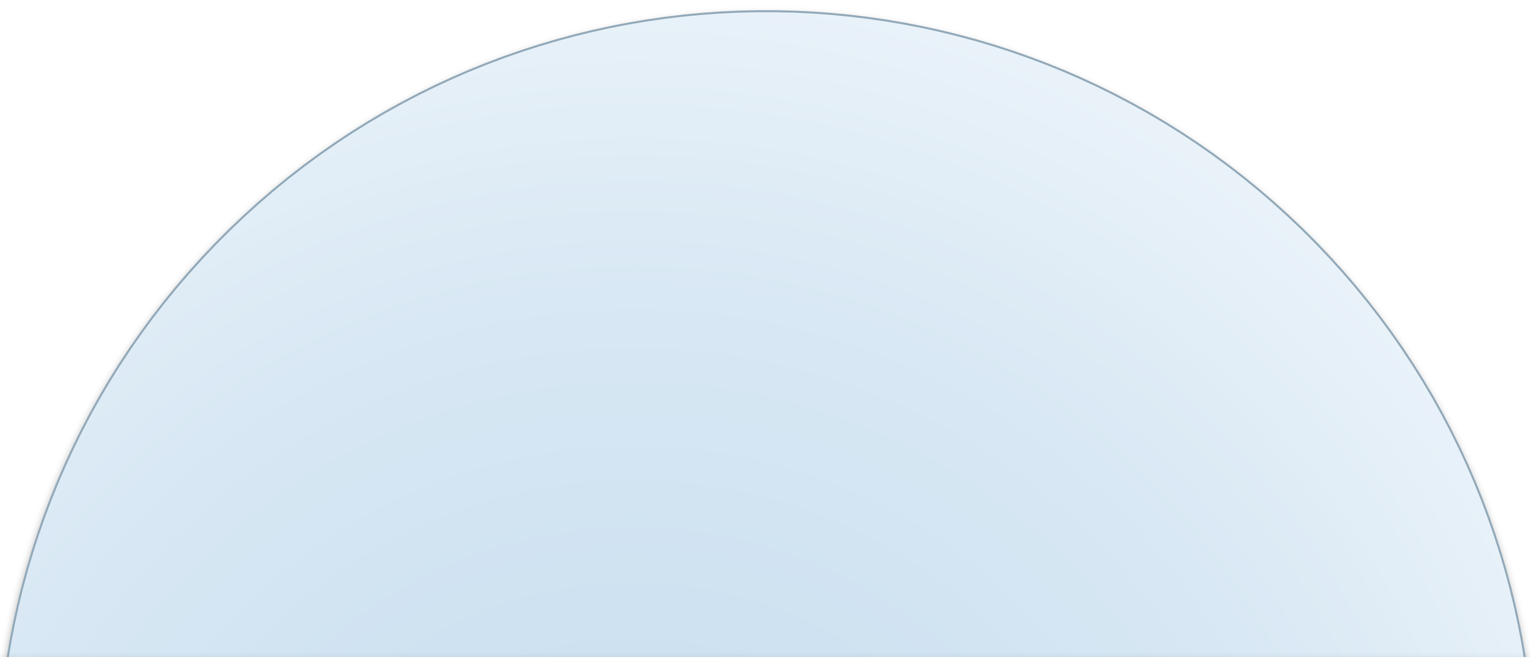


Optimization (iterative method)

$$\mathrm{E}(p, \boldsymbol{V}) = \frac{1}{2} \sum_{i=1}^{N} \mathrm{d}(\mathrm{Exp}(p, \sum_{j} x_i^j v^j), y_i)^2$$



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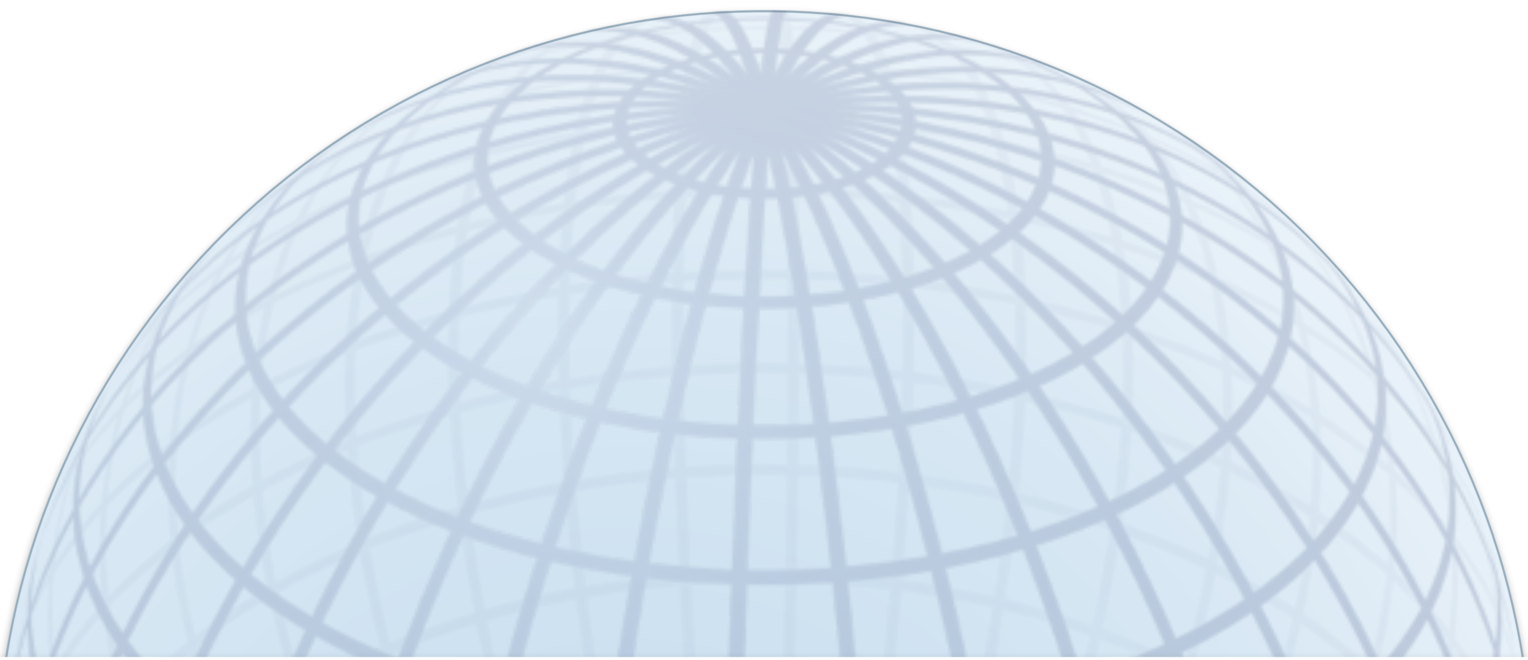
Step 1 : $p_{k+1} \leftarrow \mathrm{Exp}\left(p_k, \alpha \Sigma_i \Gamma_{\hat{y}_i \to p_k} \mathrm{Log}(\hat{y}_i, y_i)\right)$



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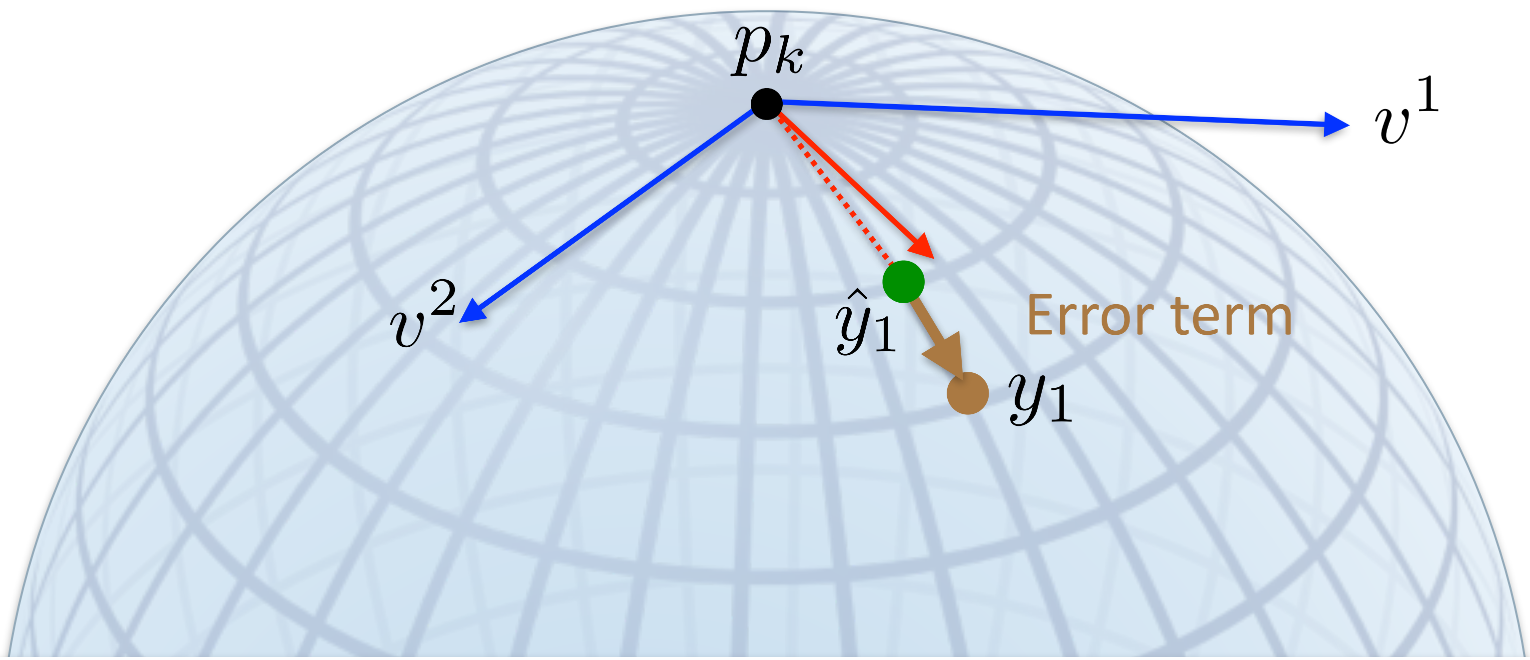
Step 1 : $p_{k+1} \leftarrow \mathrm{Exp}\left(p_k, \alpha \Sigma_i \Gamma_{\hat{y}_i \to p_k} \underline{\mathrm{Log}(\hat{y}_i, y_i)}\right)$ Error term



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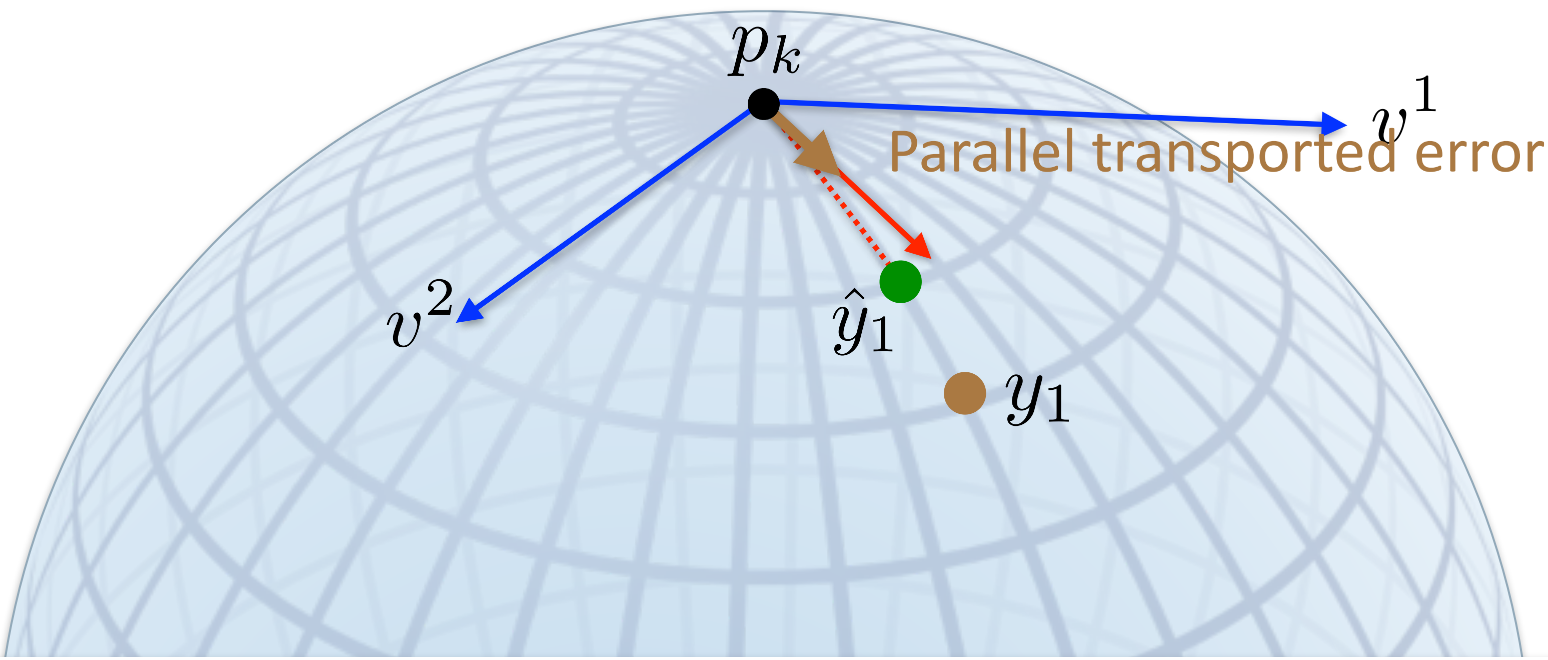
Parallel transported error



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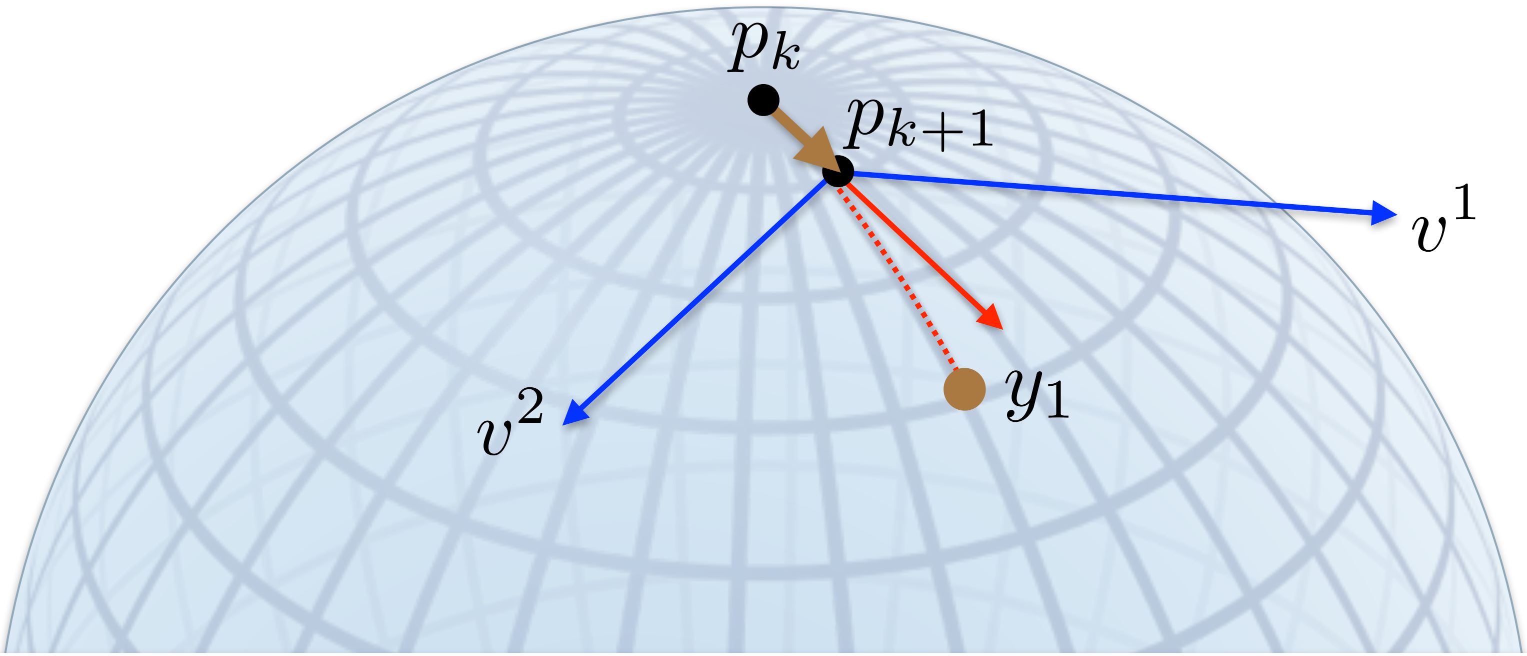

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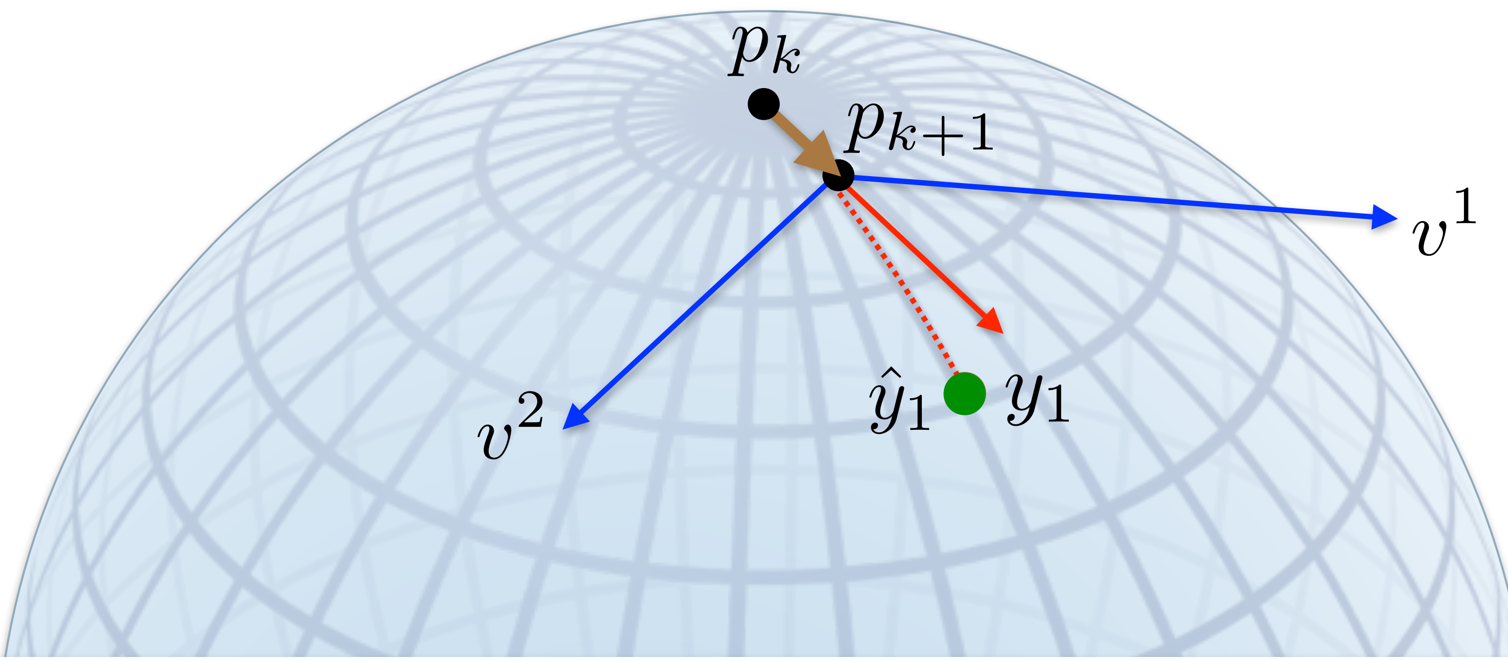
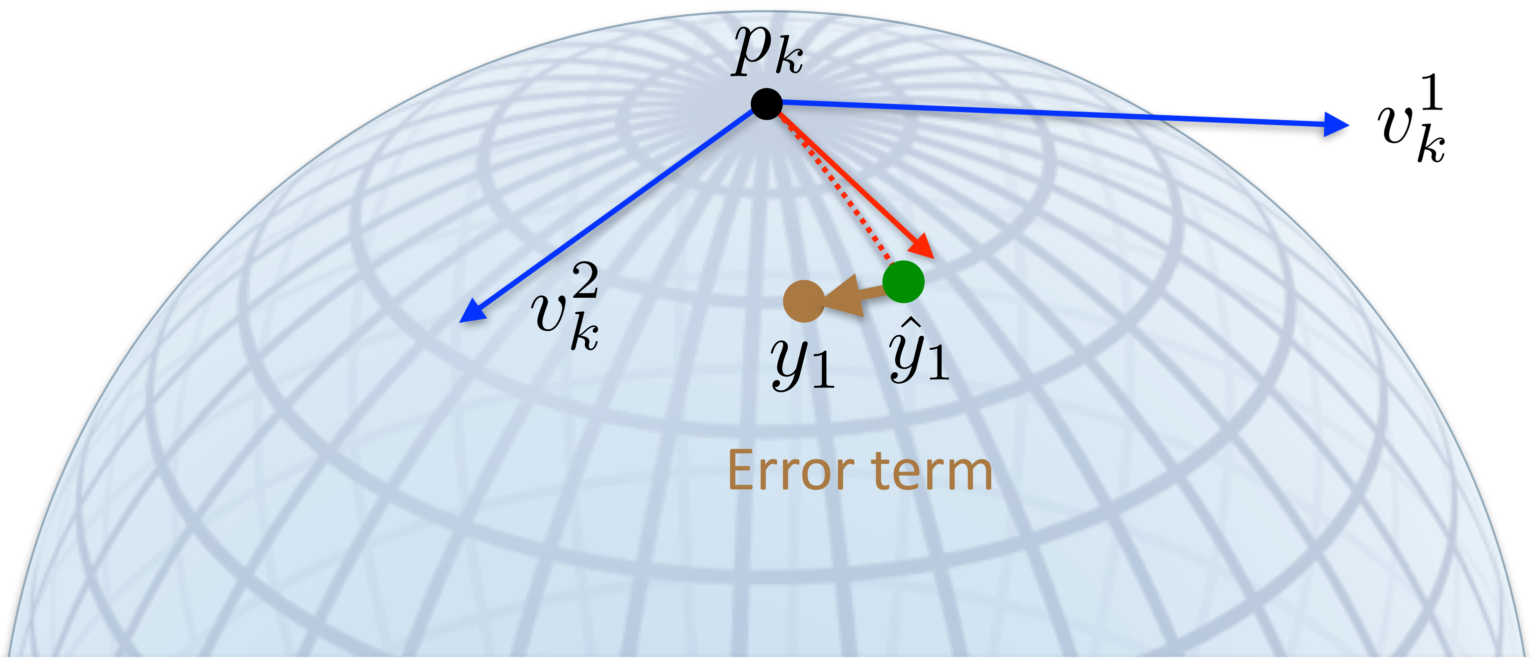


Optimization (iterative method)

$$\mathrm{E}(p, \boldsymbol{V}) = \frac{1}{2} \sum_{i=1}^{N} \mathrm{d}(\mathrm{Exp}(p, \sum_j x_i^j v^j), y_i)^2$$

Step 2 : $v^j_{k+1/2} = v^j_k + \alpha \sum_i x^j_i \Gamma_{\hat{y}_i \to p} \underline{\mathrm{Log}(\hat{y}_i, y_i)}$ Error term



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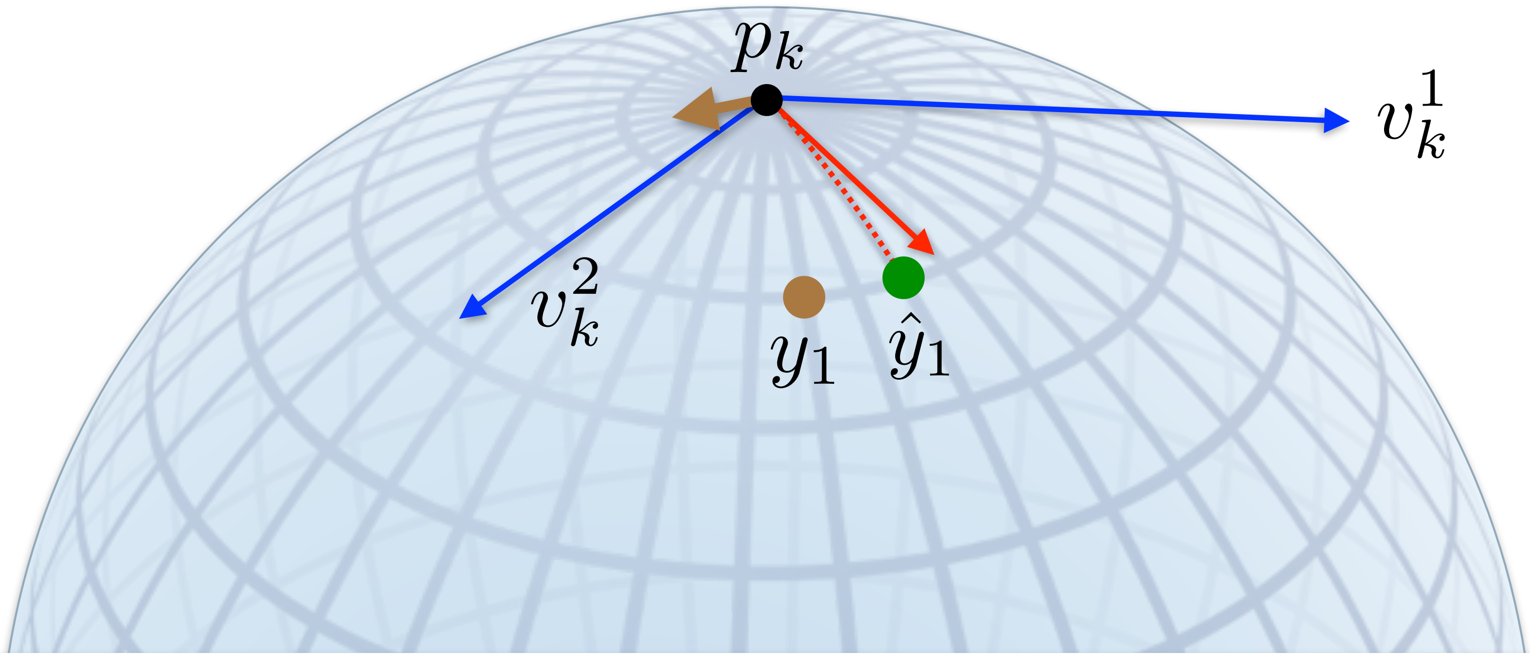
Parallel transported Error



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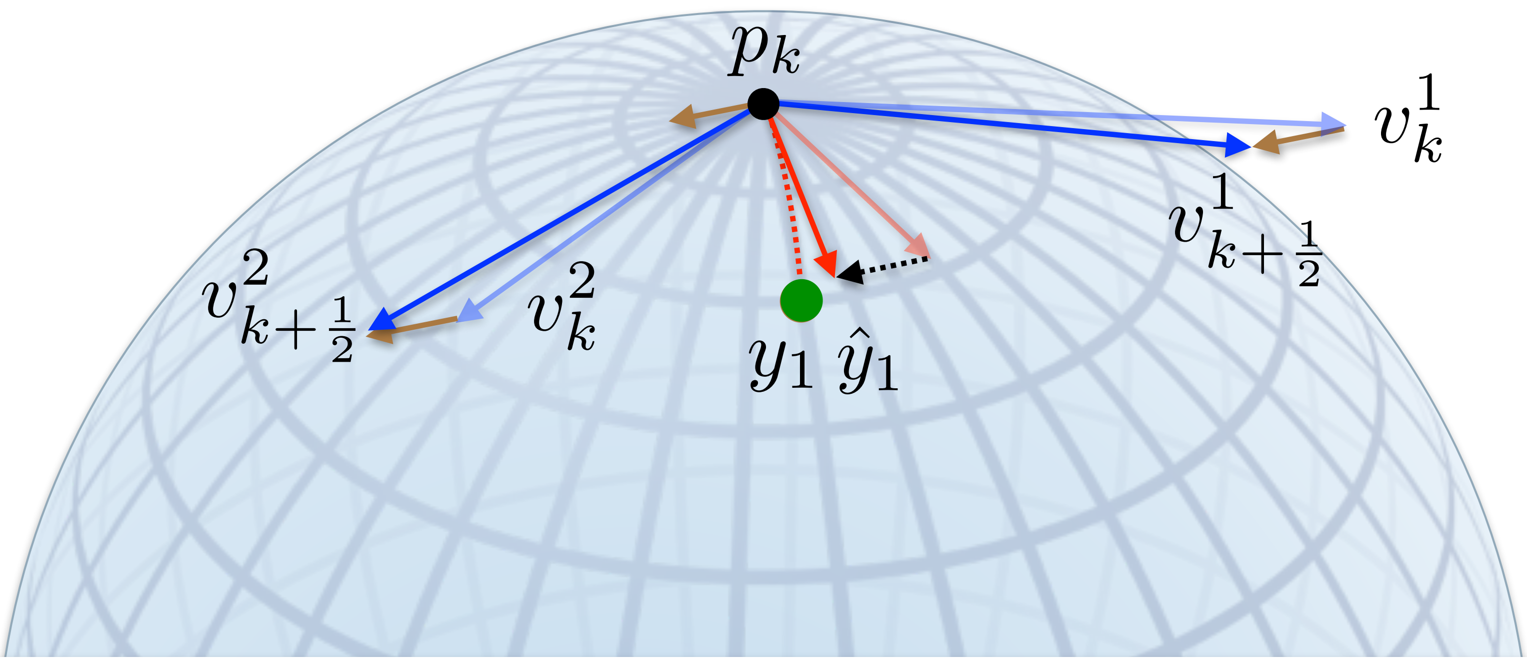

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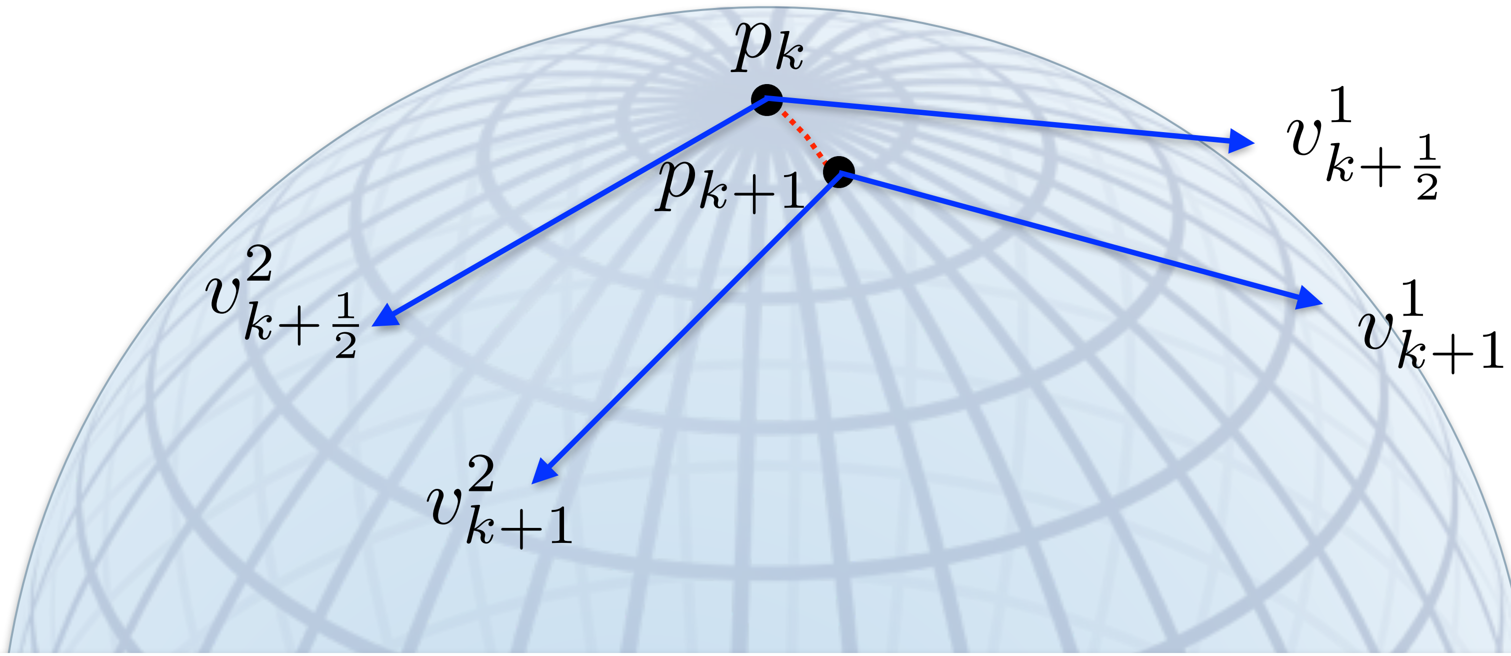
Step 3 : $$\boldsymbol{V}_{k+1} = \Gamma_{p_k \to p_{k+1}}(\boldsymbol{V}_{k+1/2})$$



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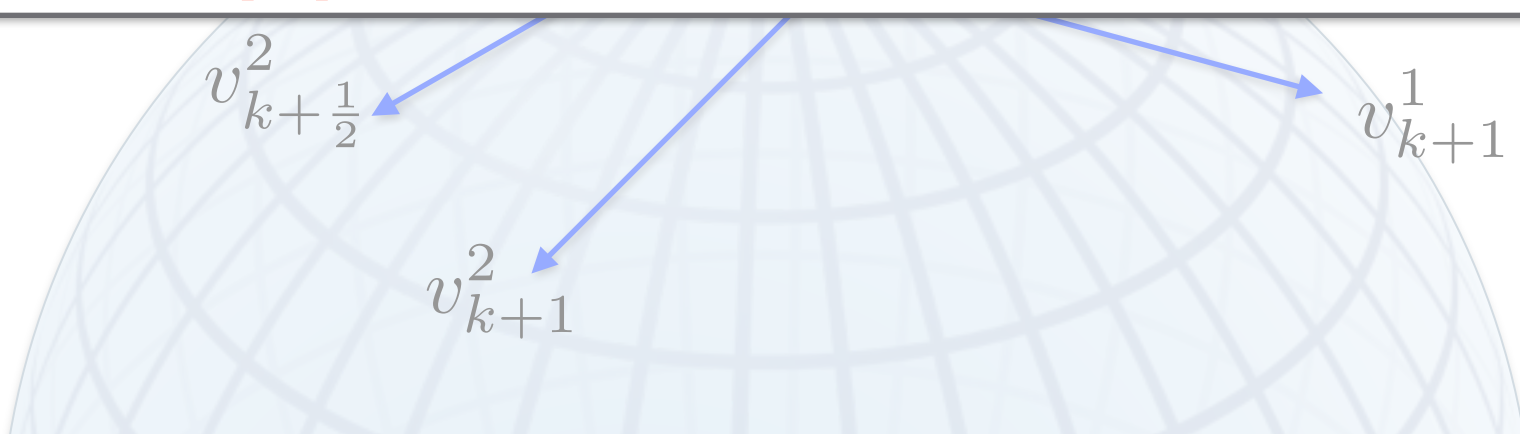
$$\mathrm{E}(p, \boldsymbol{V}) = \frac{1}{2} \sum_{i=1}^{N} \mathrm{d}(\mathrm{Exp}(p, \sum_{j} x_i^j v^j), y_i)^2$$

Step 3 : $\boldsymbol{V}_{k+1} = \Gamma_{p_k \rightarrow p_{k+1}}(\boldsymbol{V}_{k+1/2})$

Must we solve for p or can we approximate it?



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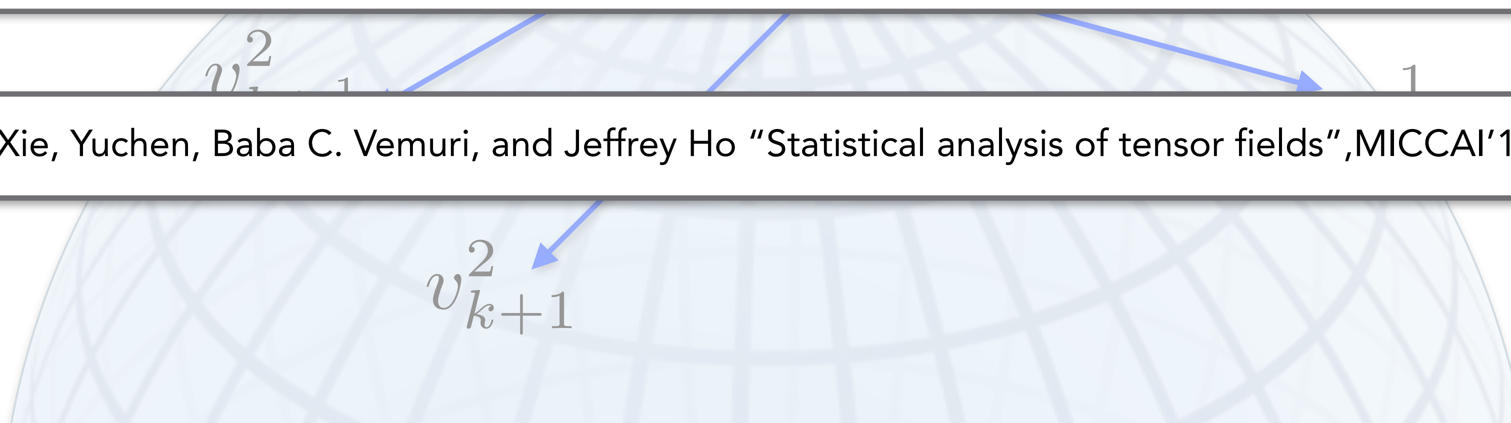
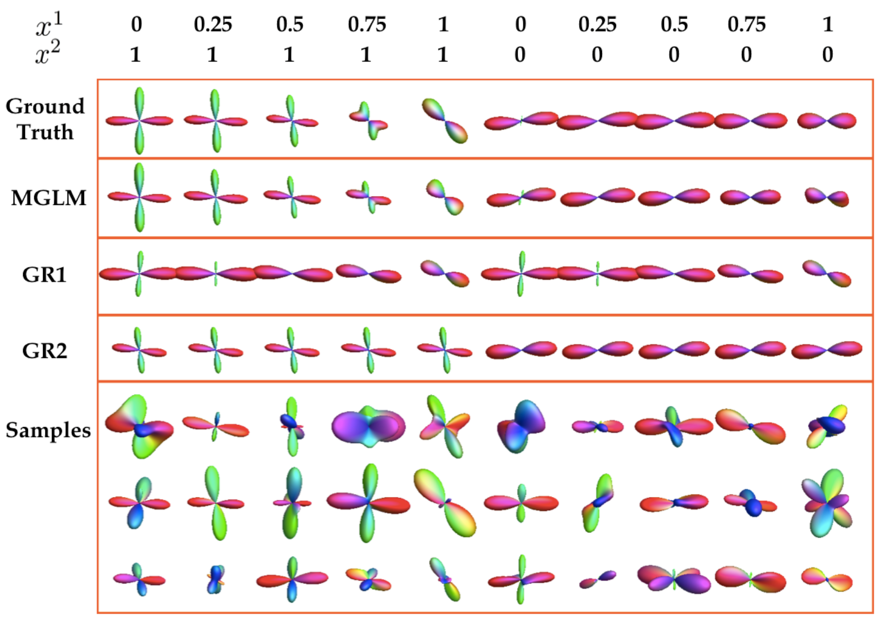


Xie, Yuchen, Baba C. Vemuri, and Jeffrey Ho “Statistical analysis of tensor fields”,MICCAI’10

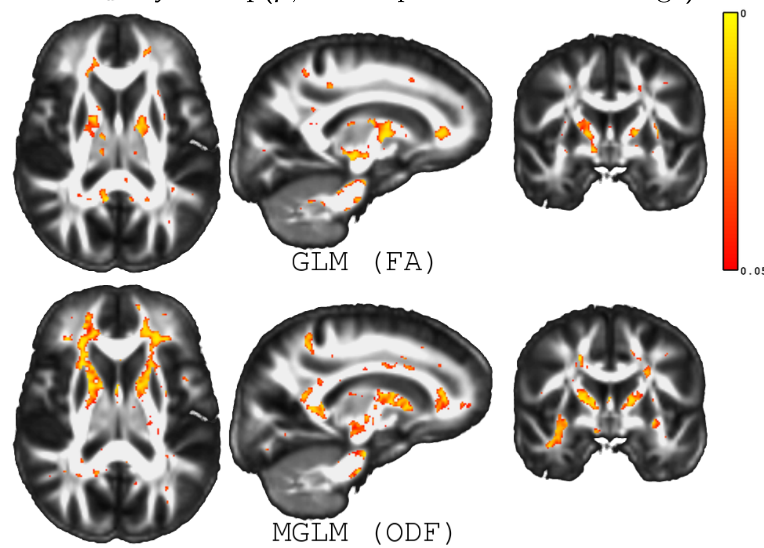
Experiments

Synthetic data



Neuroimaging data

$\text{GLM}_{\text{Full}} : y = \text{Exp}(p, v^1\text{Group} + v^2\text{Gender} + v^3\text{Age})$



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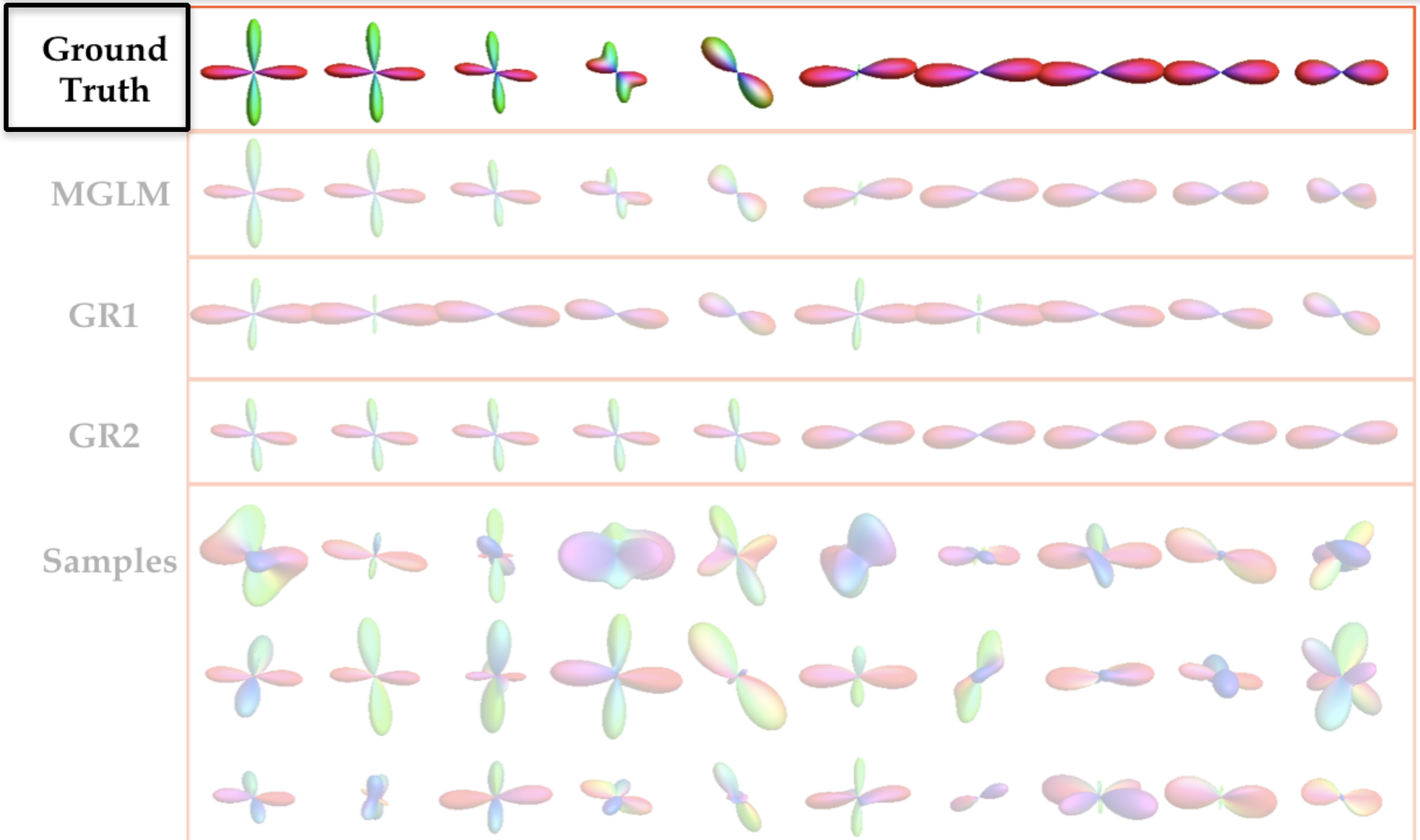
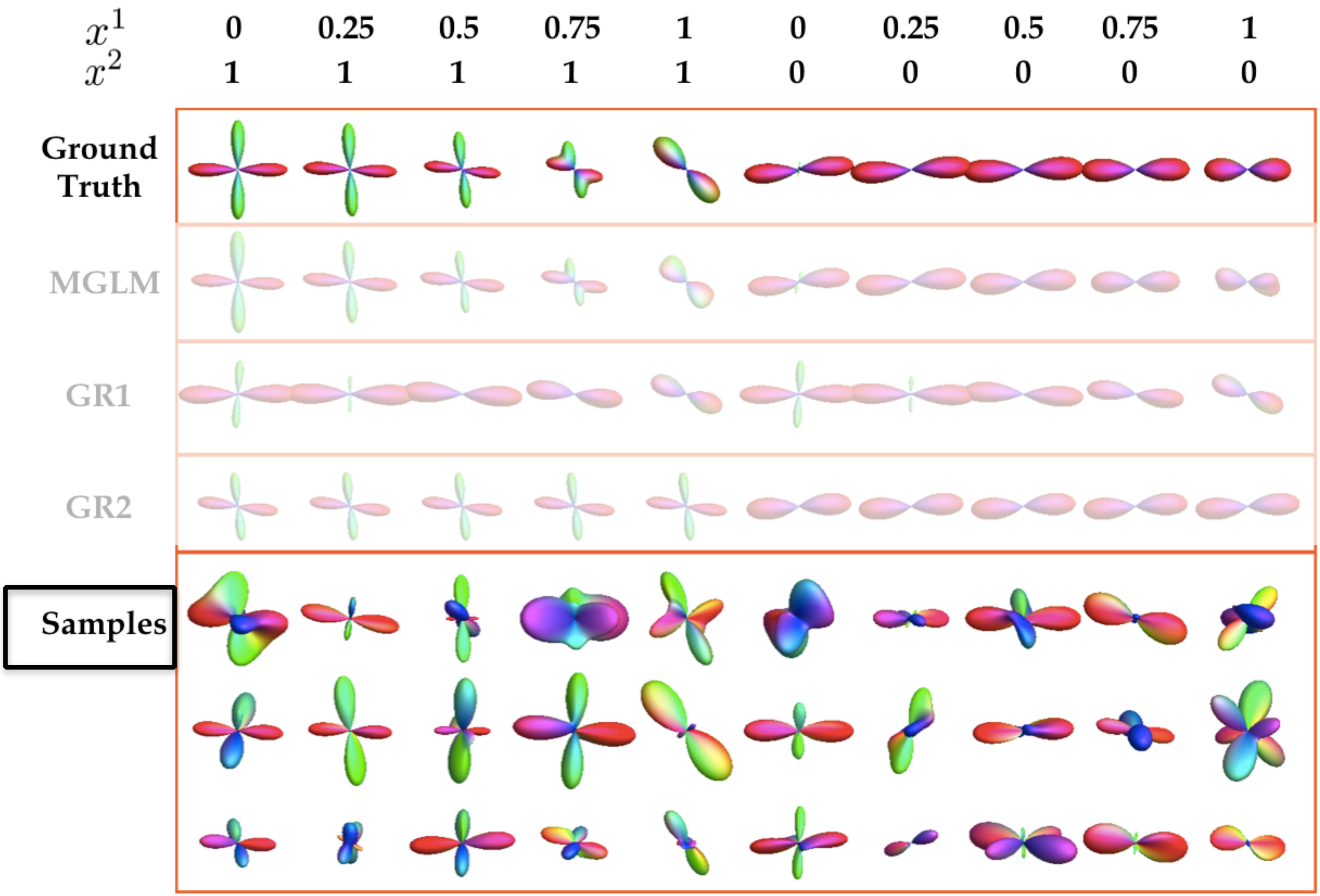
Experiments (synthetic ODF)

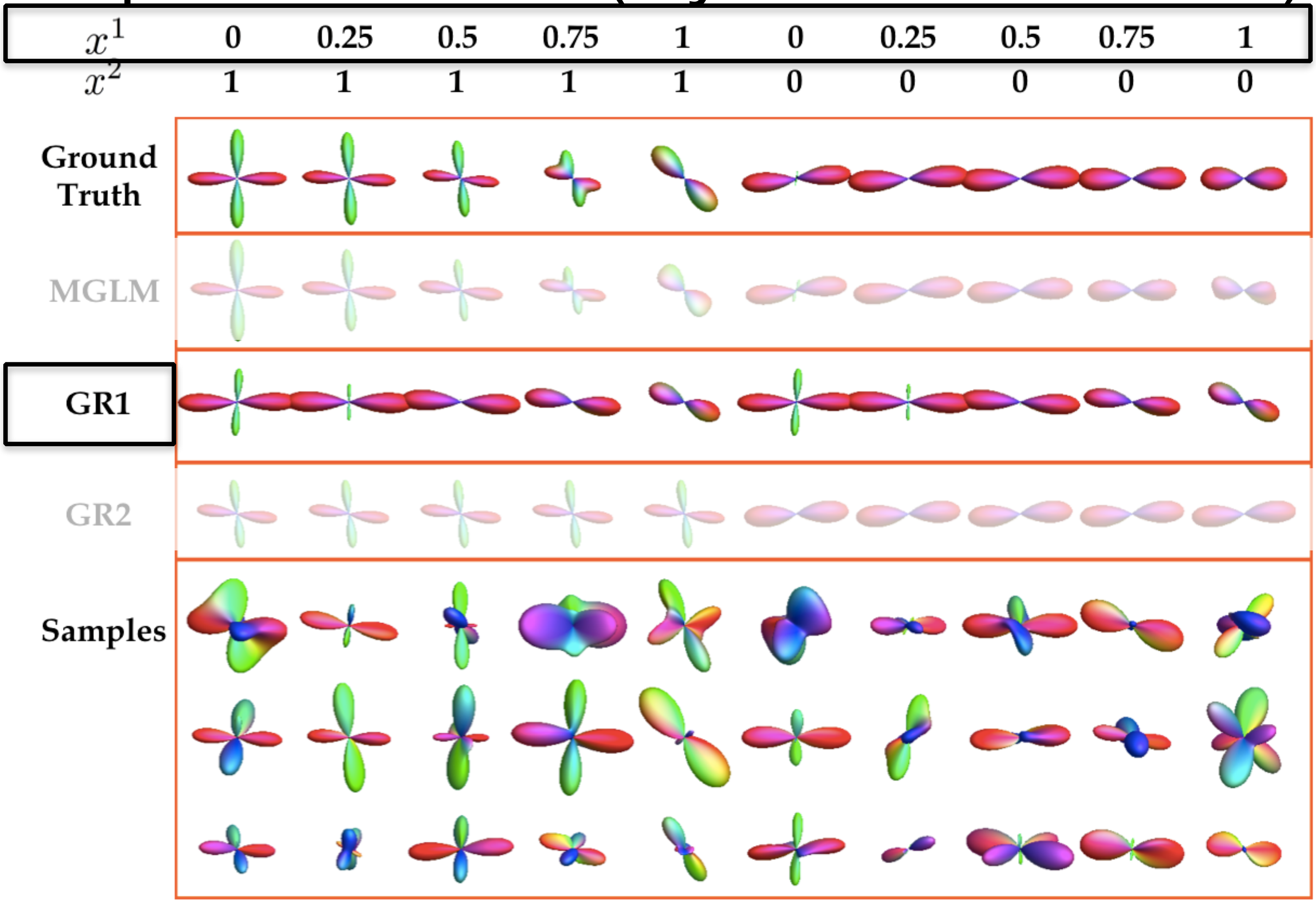
x^1	0	0.25	0.5	0.75	1	0	0.25	0.5	0.75	1
x^2	1	1	1	1	1	0	0	0	0	0



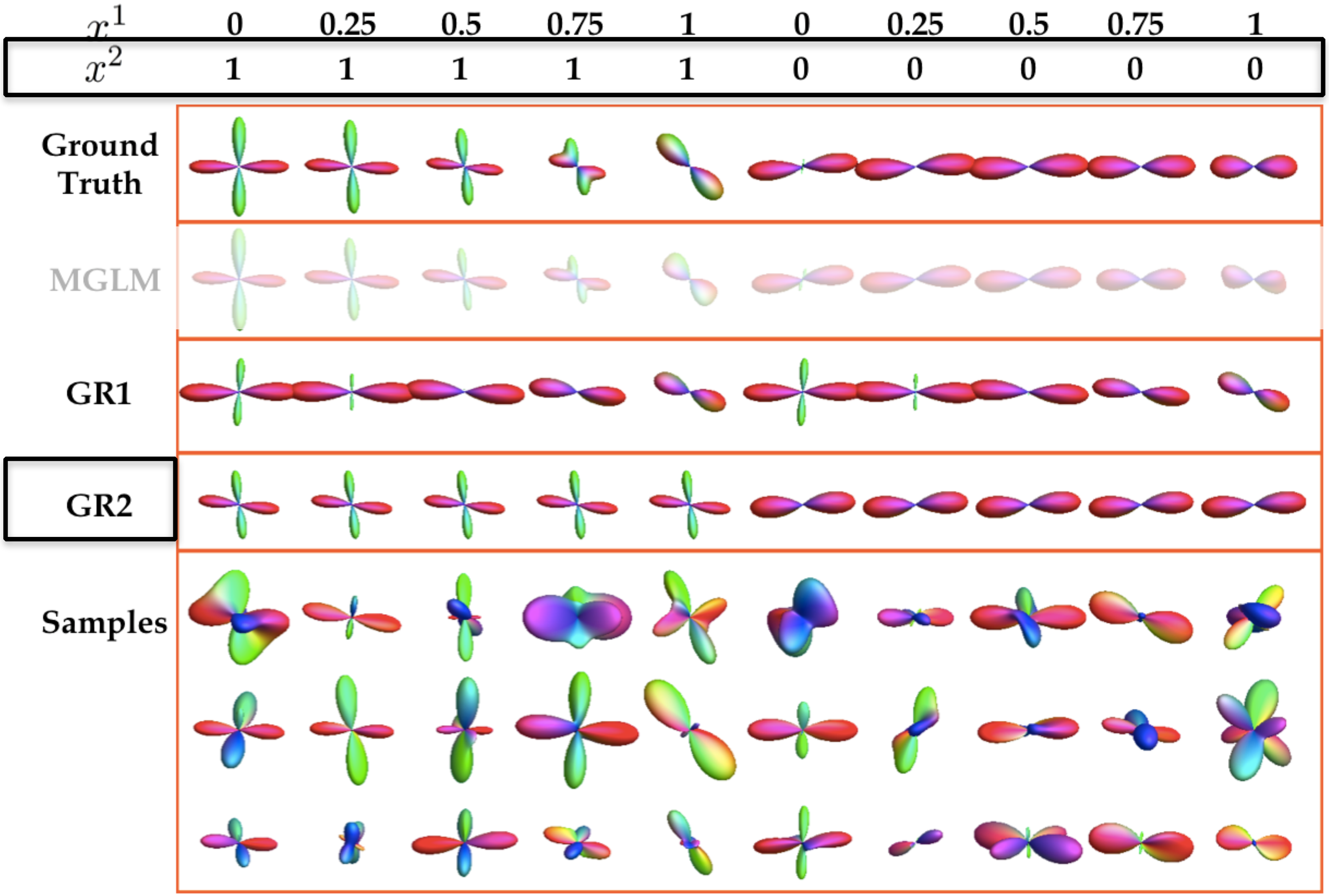
Experiments (synthetic ODF)



Experiments (synthetic ODF)



Experiments (synthetic ODF)



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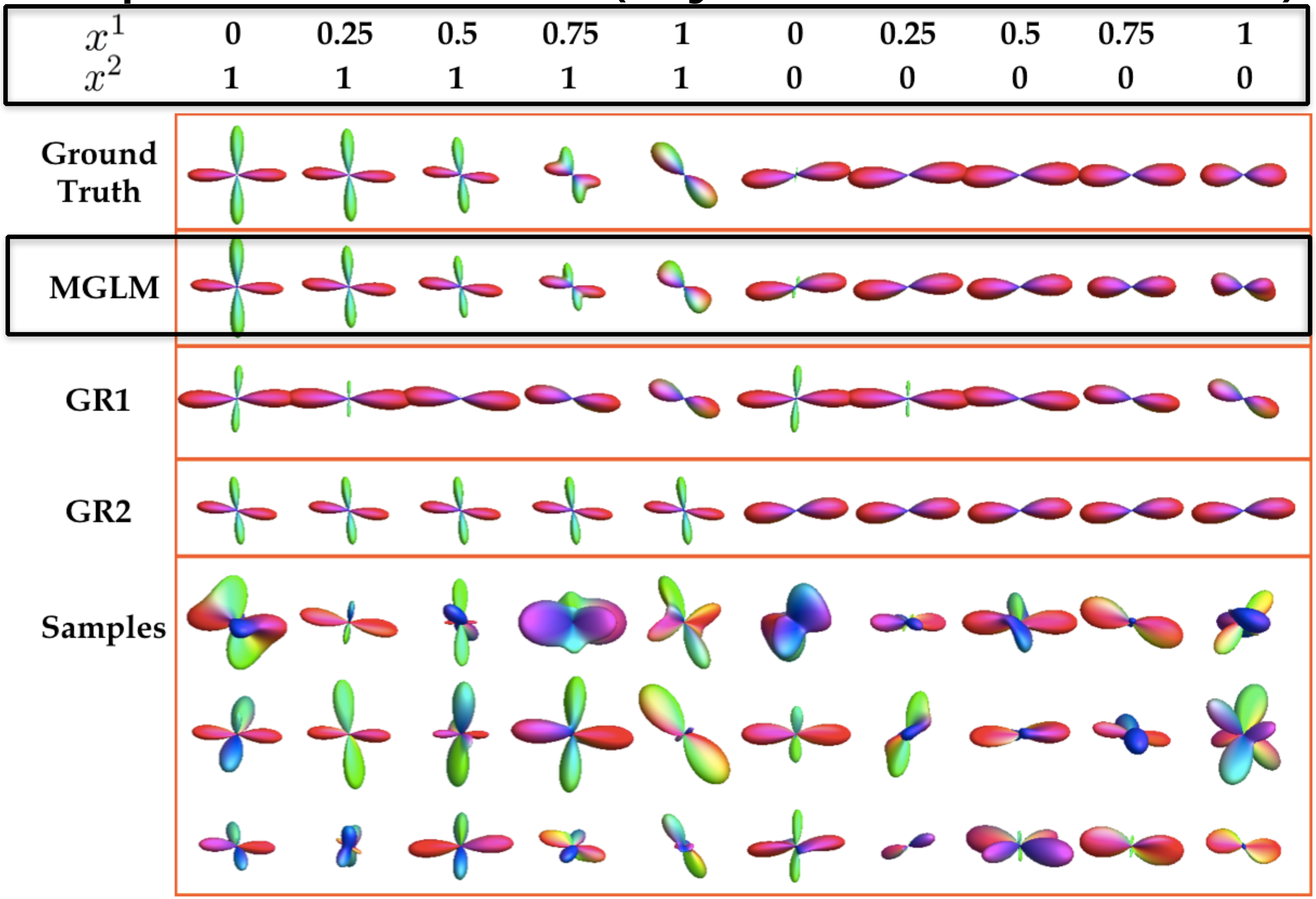
Experiments (synthetic ODF)



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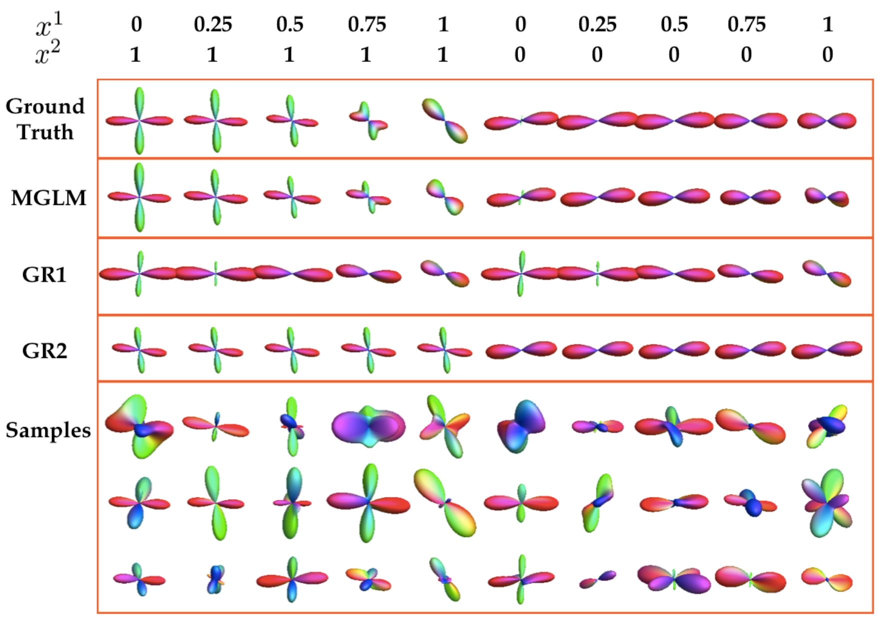
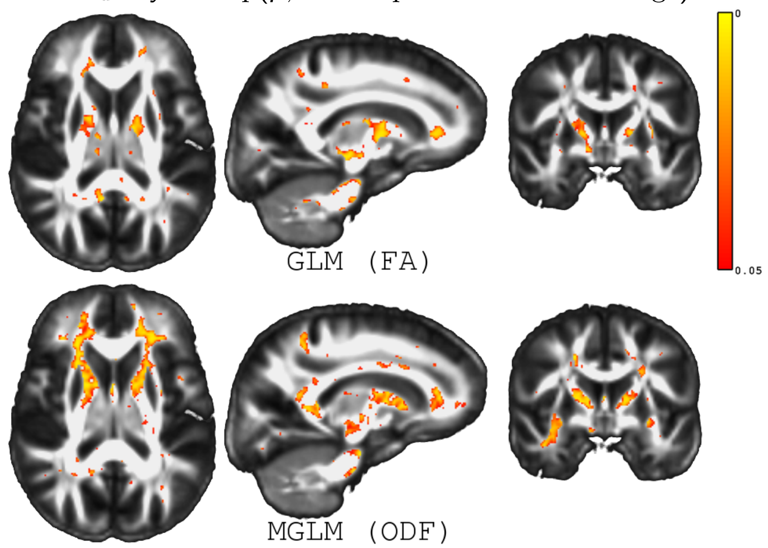
Experiments

Synthetic data



Neuroimaging data

$$\mathrm{GLM}_{\mathrm{Full}} : y = \mathrm{Exp}(p, v^1 \mathrm{Group} + v^2 \mathrm{Gender} + v^3 \mathrm{Age})$$



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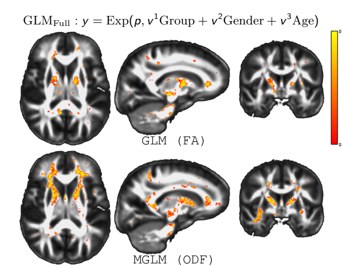
Experiments (neuroimaging study)

	LTM (ODF)		AD risk (DTI)	
Subjects	49		343	
Group	LTM	WLC	APOE4+	APOE4-
Gender	Female	Male	Female	Male
Age	28 - 65		43 - 75	

$$GLM_{Full} : y = \text{Exp}(p, v^1\text{Group} + v^2\text{Gender} + V^3\text{Age})$$

$$GLM_{Age} : y = \text{Exp}(p, v^2\text{Gender} + v^3\text{Age})$$

$$GLM_{Group} : y = \text{Exp}(p, v^1\text{Group} + v^2\text{Gender}$$

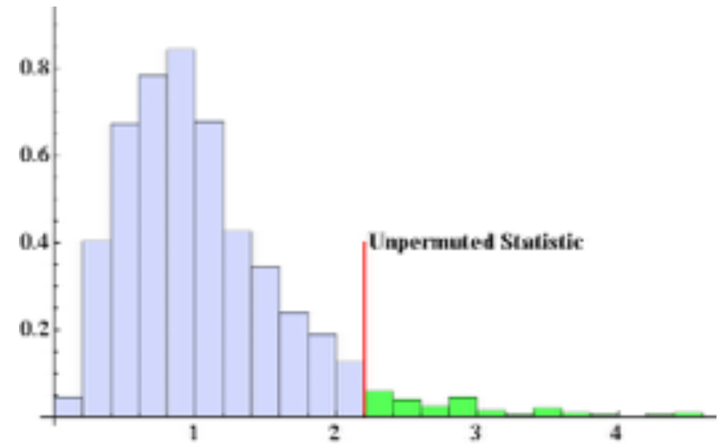


Experiments (neuroimaging study)

- *p*-values maps and histograms for effect of age and group computed from simulating the Null distribution of the F ratio statistic using 20,000 permutations.

- *F* ratio statistic is defined for a pair of nested GLMs as

$$F = \frac{\frac{\mathrm{RSS}_1 - \mathrm{RSS}_2}{p_2 - p_1}}{\frac{\mathrm{RSS}_2}{N - p_2}}$$



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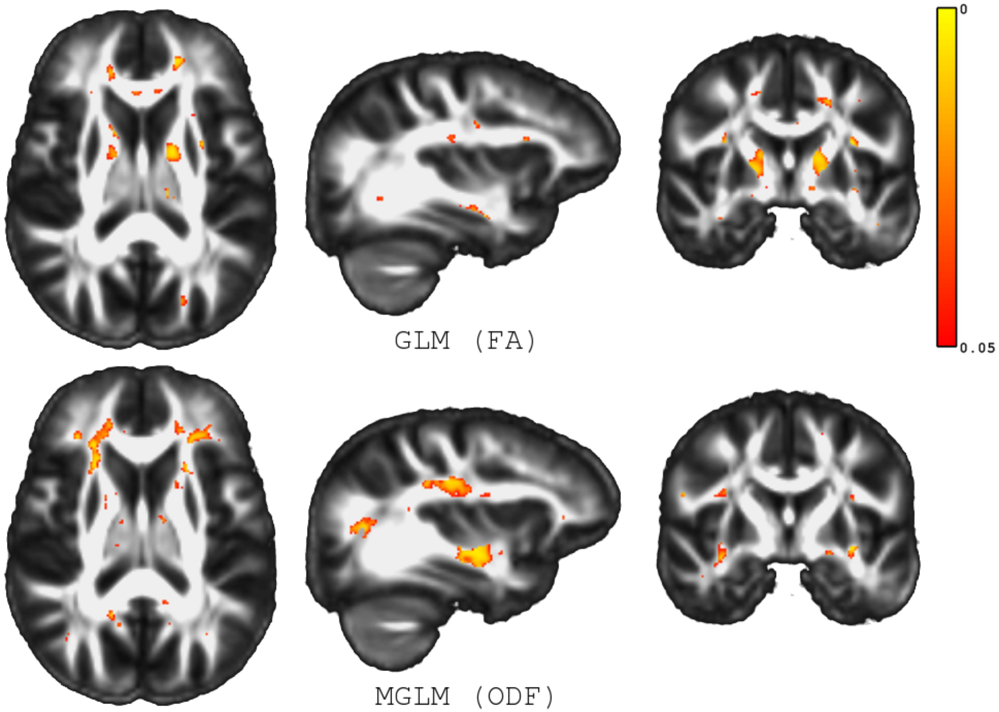
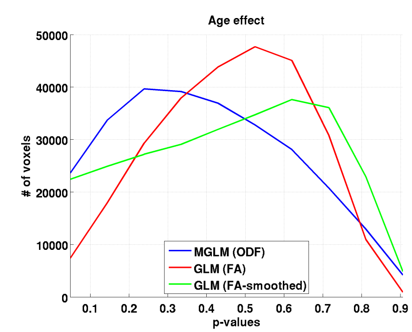
Neuroimaging study 1

- Age effect in study 1 (long-term meditators, ODFs)

$$\mathrm{GLM}_{\mathrm{Full}} : y = \mathrm{Exp}(p, v^1 \mathrm{Group} + v^2 \mathrm{Gender} + v^3 \mathrm{Age})$$
$$\mathrm{GLM}_{\mathrm{Group}} : y = \mathrm{Exp}(p, v^1 \mathrm{Group} + v^2 \mathrm{Gender})$$

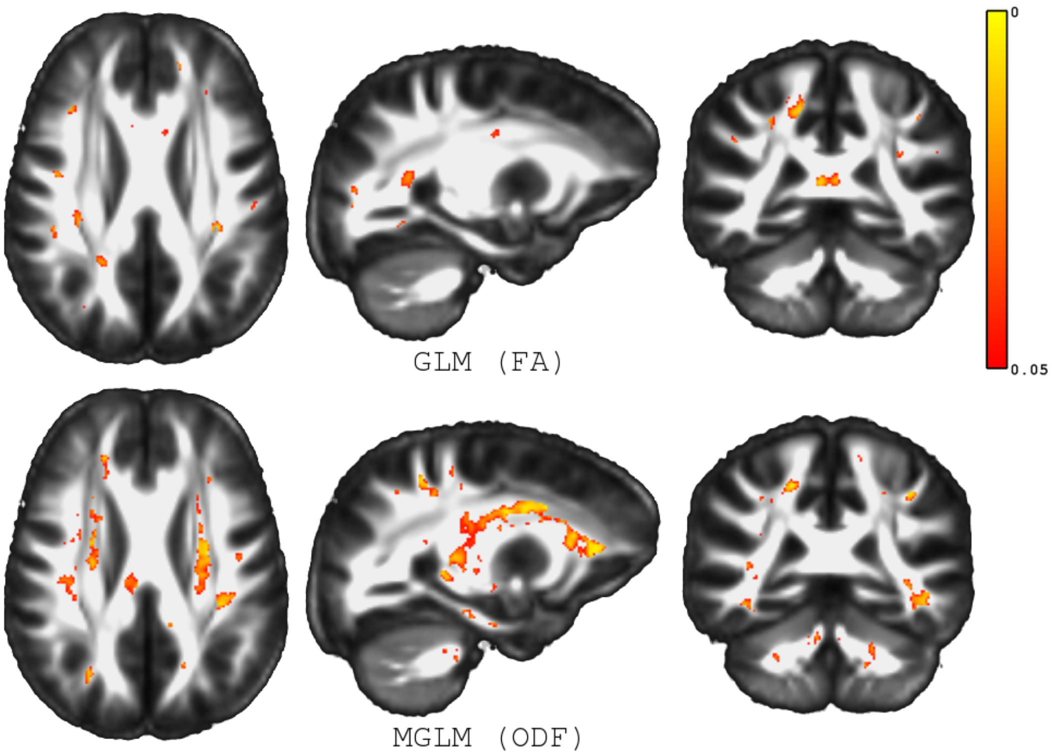


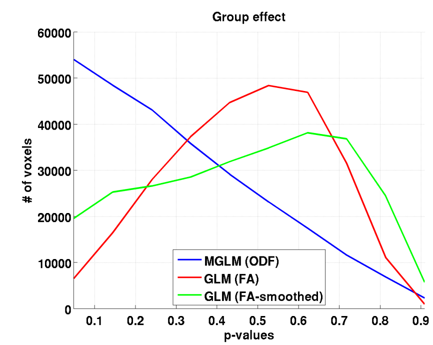


Neuroimaging study 1

- Group effect in study 1 (long-term meditators, ODFs)

$$\mathrm{GLM}_{\mathrm{Full}} : y = \mathrm{Exp}(p, v^1\mathrm{Group} + v^2\mathrm{Gender} + v^3\mathrm{Age})$$
$$\mathrm{GLM}_{\mathrm{Age}} : y = \mathrm{Exp}(p, v^2\mathrm{Gender} + v^3\mathrm{Age})$$





$\mathsf{Group} \in \{\mathsf{LTM}, \mathsf{WLC}\}.$

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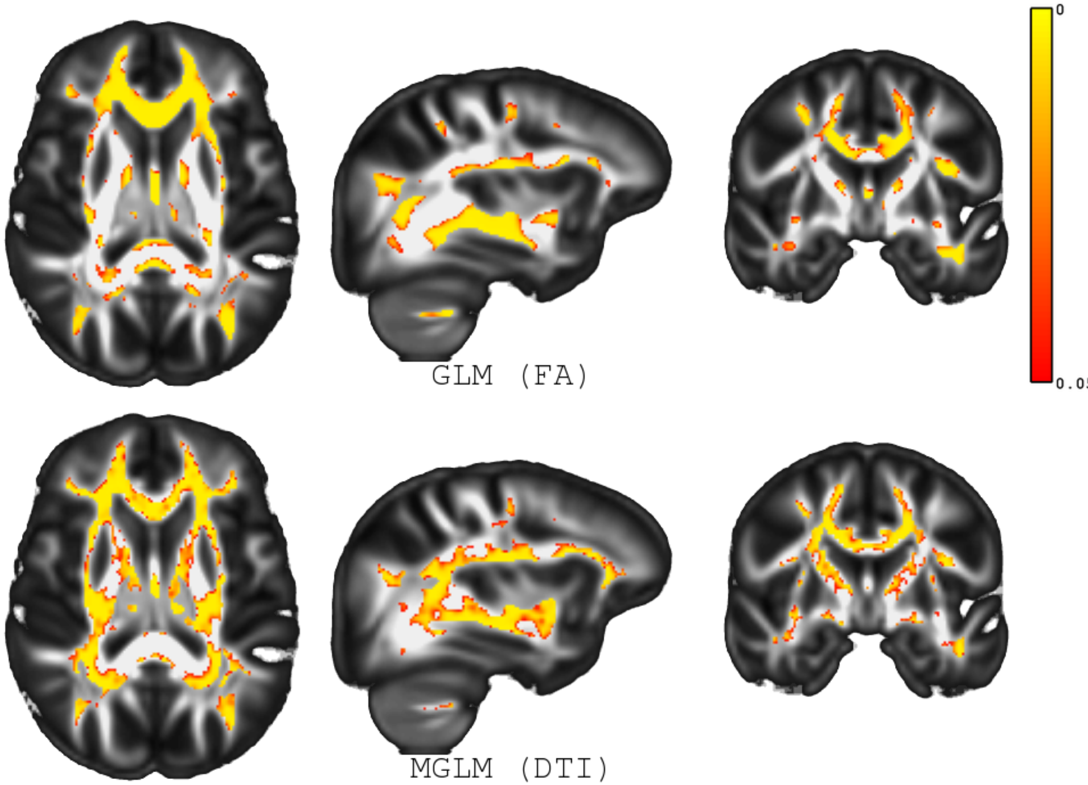
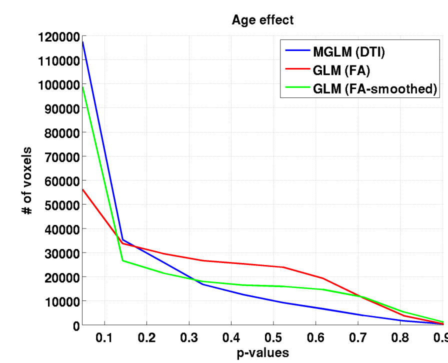
Neuroimaging study 2

- Age model of study 2 (AD, DTI)

$$\mathrm{GLM}_{\mathrm{Full}} : y = \mathrm{Exp}(p, v^1\mathrm{Group} + v^2\mathrm{Gender} + v^3\mathrm{Age})$$
$$\mathrm{GLM}_{\mathrm{Group}} : y = \mathrm{Exp}(p, v^1\mathrm{Group} + v^2\mathrm{Gender})$$





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Conclusion

- Generalization of ***multivariate general linear model*** (MGLM) to Riemannian manifolds
- Especially useful when response is manifold valued and we want to control for one or more covariates. Here, the analysis obtains significantly improved statistical power
- Applicable to other manifold-valued statistical inference problems
- Code is available. See me at the posters!

Poster session ID: O-3C-384

Thank you

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