Optimization Problem and Inequality Constraints

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Goal: Optimize profit/production/revenue given linear constraints.

Type of problems:

1) Identifying the constraints provided in the problem
   Example: (Diag. exam 2001, Question 3) You are a broker who buys junk bonds, repackages them into portfolios and resells them directly to investors. Suppose that you have recently purchased 600 high risk bonds and 2400 low risk bonds. You have determined that you have a strong market for two types of portfolios. Portfolio A combines 30 high risk bonds with 40 low risk bonds. Portfolio B packages 10 high risk bonds with 80 low risk bonds. Suppose that you can earn 2000 dollars profit on Portfolio A and 1500 on portfolio B. Suppose you decide to sell X of type A and Y of type B. What are the constraints on X and Y?
2) Finding the right combination of resources to achieve the optima
   Example: (Diag. exam 2001, Question 3) Find the values of X and Y that maximize your profit?
3) Changing conditions of the question and asking (1) and/or (2)
   Example: (Diag. exam 2001, Question 3) How would your answer change if your employer kicks in an extra 3000 dollars profit for each high yielding portfolio type A that you sell?

Strategies for each problem:

1) Identifying the constraints provided in the problem
   a) *Make a table* where "resources" go across columns and "products" that use these "resources" go down rows. Each cell in this table contains the number of resource <BLANK> needed to create product <BLANK>. On the very last row, write down the total amount of resources that are available. An example is provided below:

	Resource A (e.g. High Risk Bonds)	Resource B (e.g. Low Risk Bonds)
Product A' (e.g. Portfolio A)	30	40
Product B' (e.g. Portfolio B)	10	80
…		
Total Available	600	2400

   b) For each resource, *add down the rows* and write an inequality that relates to the total amount available: (e.g. $30X + 10Y \leq 600$ and $40X + 80Y \leq 2400$)
   c) *Identify other constraints*: (e.g. maximum amount of production possible)
   d) *"Dummy" conditions*: (e.g. $X, Y \geq 0$ (aka we can't produce negative units))
2) Finding the right combination of resources to achieve the optima
   a) *Replace the inequalities in 1b with equalities and solve for the unknown variables*
      Example: $30X + 10Y \leq 600 \;\Rightarrow 30X + 10Y = 600 \;\Rightarrow 240X + 80Y = 4800$
      $40X + 80Y \leq 2400 \Rightarrow 40X + 80Y = 2400 \Rightarrow\; 40X + 80Y = 2400 \xRightarrow{Subtract\ eqn.} 200X = 2400 \Rightarrow \boldsymbol{X = 12}$
      Plug $X = 12$ into one of the equations. $30(12) + 10Y = 600 \Rightarrow 360 + 10Y = 600 \Rightarrow 10Y = 240 \Rightarrow \boldsymbol{Y = 24}$
   b) *Solve for cases when none of product <BLANK> is produced. Do this for all products!*
      Example: When $X = 0$, we are left with $10Y \leq 600 \Rightarrow Y \leq 60$ and $80Y \leq 2400 \Rightarrow Y \leq 30$ as constraints. Since we have to satisfy both constraints, we can produce, at most, 30 Ys. Similarly, when $Y = 0$, we can produce, at most, 20 Xs.
   c) *Plug in solutions you found in (a) and (b) to the profit/revenue equation and pick the optimal solution*
      Example: $< X = 12, Y = 24 >$, $< X = 0, Y = 30 >$, and $< X = 20, Y = 0 >$ are solutions from (a) and (b). Plugging each solution into $2000X + 1500Y$, you find $< X = 12, Y = 24 >$ be the optimal solution. Note that in almost all cases, your solution in 2a will be the answer.
3) Changing conditions of the question and asking (1) and/or (2)
   a) *Identify the changes and repeat (1) and/or (2)*

Additional Information: Slides 40-57 (Waterman's lecture slides) & Class02.pdf, Section 3 (Wyner's lecture notes)

Problems:

1) Google has been custom building its servers since 2005. Google makes two types of servers for its own use. A high capacity server requires 100 hard drives and 10 switches. A low capacity server requires 5 hard drives and 5 switches. On average, a high capacity server can serve 1 million users while a low capacity server can serve 200,000 users. Google would like to optimize building servers.
   a. Suppose Google only has 500 hard drives and 100 switches. What are some constraints that Google server engineers face?
   Answer:

	Hard Drives	Switches
High Capacity Servers	100	10
Low Capacity Servers	5	5
Total Available	500	100

   Let H be the number of high capacity servers and L be the number of low capacity servers. Then, we have $100H + 5L \leq 500$ and $10H + 5L \leq 100$. We also have "dummy" conditions $H, L \geq 0$
   b. How many high and low capacity servers should Google make?
   Answer:
   We solve the equation $100H + 5L = 500$ and $10H + 5L = 100$ and obtain $L = \frac{100}{9} = 11.111$ and $H = \frac{40}{9} = 4.444$. Because we can't build a fraction of servers***, we choose to build 11 low capacity servers and 4 high capacity servers. Also, when $H = 0$, we can build 20 low capacity servers. Similarly, when $L = 0$, we can build 5 high capacity servers. Plugging the three possible decisions into the equation $1{,}000{,}000H + 200{,}000L$, $H = 4$ and $L = 11$ give you the maximum number of users.
2) A mango smoothie startup sells two types of smoothies. A regular mango smoothie costs $5 and special mango smoothie costs $7. To make a regular mango smoothie, you need 1 mango and 2 cups of milk. To make a special mango smoothie, you need 2 mangos and 1 cup of milk. The startup can only make 100 smoothies per day. Also, for public health reasons, the company only has 49 mangos and 2 gallons of milk per day. (Note: 16 cups = 1 gallon)
   a. What are some limitations the mango company face each day?
   Answer:

	Mangos	Milk
Regular	1	2
Special	2	1
Total Available	49	$16 * 2 = 32$

   Let R be the number of regular smoothies and let S be the number of special smoothies. Then, we have $1R + 2S \leq 49$ and $2R + 1S \leq 32$. In addition, we can only produce 100 smoothies (aka $R + S \leq 100$). Finally, there are the "dummy" conditions $R, S \geq 0$.
   b. Suggest the best combination of regular and special smoothies.
   Answer:
   Solving $R + 2S = 49$ and $2R + S = 32$ we get $R = 5$ and $S = 22$. Also, when we only make special smoothies, we can make 32 of them. Similarly, when we only make regular smoothies, we can make 49 of them. Comparing each decision's profit, we see that $R = 5$ and $S = 22$ gives the best sales.

***To be <u>really, really, really</u> precise, you actually have three possible H and L combinations when you round. You have H=4 and L=11 combination as seen above. You can also "round-down" H and "round-up" L to get H=4, L=12. Similarly, you can "round-up" H and "round-down" L to get H=5, L=11. First, you should try each condition to see if it satisfies the constraints; if a condition does not satisfy the constraint, throw them away (in our case, only H=4 and L=11 satisfy the constraints). Second, if you have more than one condition that meet the constraints, plug each condition into the "profit" equation and see which condition gives you the highest profit! Yes…it's tedious, but I want to cover my bottom in case if one of you decides to sue Wharton and the dept. ☺