Top Arm Identification in Multi-Armed Bandits with Batch Arm Pulls

Kwang-Sung Jun
UW-Madison

Kevin Jamieson
UC Berkeley

Robert Nowak
UW-Madison

Xiaojin Zhu
UW-Madison

Abstract

We introduce a new multi-armed bandit (MAB) problem in which arms must be sampled in batches, rather than one at a time. This is motivated by applications in social media monitoring and biological experimentation where such batch constraints naturally arise. This paper develops and analyzes algorithms for batch MABs and top arm identification, for both fixed confidence and fixed budget settings. Our main theoretical results show that the batch constraint does not significantly affect the sample complexity of top arm identification compared to unconstrained MAB algorithms. Alternatively, if one views a batch as the fundamental sampling unit, then the results can be interpreted as showing that the sample complexity of batch MABs can be significantly less than traditional MABs. We demonstrate the new batch MAB algorithms with simulations and in two interesting real-world applications: (i) microwell array experiments for identifying genes that are important in virus replication and (ii) finding the most active users in Twitter on a specific topic.

1 Introduction

We consider the top-$k$ pure-exploration problem for stochastic multi-armed bandits (MAB). Formally, we are given $n$ arms that produce a stochastic reward when pulled. The reward of arm $i$ is an i.i.d. sample from a distribution $\mu_i$ whose support is in $[0,1]$. The bounded-support assumption can be generalized to the $\sigma$-sub-Gaussian assumption. Denote by $\mu_i = E_{X \sim \mu_i} X$ the expected reward of the arm $i$. We assume a unique top-$k$ set; i.e., $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_k > \mu_{k+1} \geq \ldots \geq \mu_n$.

In this work, we introduce a new setting in which arms must be pulled in batches of size $b$. We also consider the additional constraint that any one arm can be pulled at most $r \leq b$ times in a batch (if $r = b$ then there is no constraint). Said another way, the action space is defined as $\mathcal{A} := \{a \in \{0,1,\ldots,r\}^n \mid \sum_{i=1}^n a_i \leq b\}$, where $a \in \mathcal{A}$ indicates the number of arm pulls for each arm. We call this the $(b,r)$-batch MAB setting. Note that this encompasses the standard MAB setting, which corresponds to $b = 1, r = 1$. The general $(b,r)$-batch setting is fundamentally different, since sampling decisions must be made one batch at a time, rather than one sample at a time. This loss in flexibility could hinder performance, but our main theoretical results show that in most cases there is no significant increase in sample complexity.

There are many real-world applications where the batch constraint arises. For example, suppose the arms represent Twitter users and the goal is to find users who tweet the most about a topic of interest. Using Twitter’s free API, one can follow up to $b = 5000$ users at a time, which amounts to a batch of that size. Also, a user is either observed or not, so $r = 1$ in this application. The batch is the fundamental sampling unit in such applications, and thus this paper is interested in methods that aim to reduce the batch complexity: the total number of batches, or rounds, to correctly identify the top-$k$ arms. As another example, we consider the problem of using microwell array experiments to identify the top-$k$ genes involved in virus replication processes. Here the arms are genes (specifically single knockdown cell strains) and a batch consists of $b = 384$ microwells per array. In this case, we may repeat the same gene (cell strain) in every microwell, so $r = 384$.

Returning to the discussion of theory and algorithms, we consider two settings: fixed confidence and fixed budget. In the fixed confidence setting, given a target failure rate $\delta$, the goal is to identify the top-$k$ arms with probability at least $1 - \delta$ using the fewest possible number of batches. In the fixed budget setting, given a budget number of batches $B$, the goal is to identify the top-$k$ arms with as high probability as possible. For the fixed confidence (budget) setting, we propose...
the BatchRacing (BatchSAR) algorithm and prove its theoretical guarantee in Section 3 (Section 4). Our analysis shows that batch MABs have almost the same sample complexity as unconstrained MABs as long as \( r \) is not too small relative to \( b \).

An alternative to our batch MAB algorithms is to use standard MAB algorithms as follows. One could use each batch to obtain a sample from a particular arm of interest, and effectively ignore the rest of the data from each batch. This would allow one to use standard MAB algorithms. For example, in the Twitter application we could follow just one user at a time while we can follow 5000 for the same cost (free). Our results show that in most cases this (obviously naive) reduction to the standard MAB would have a batch complexity \( b \) times greater than that of our batch MAB algorithms.

In section 5 we validate the analysis of the proposed algorithms with a toy experiment, and evaluate the algorithms on real-world data from the two applications introduced above.

## 2 Related Work

The pure exploration problem in MAB has a long history dating back to the '50s with the work of [4] and [20]. The top-\( k \) exploration, or subset selection, in stochastic MAB has received much attention recently. In the fixed confidence setting including the Probably Approximately Correct (PAC) setup, the Racing algorithm for the top-1 exploration problem was proposed by Maron and Moore [19] and by Even-Dar et al. [0] independently. The Racing algorithm performs uniform sampling on surviving arms while deactivating infeasible arms based on confidence intervals. Racing was generalized to top-\( k \) identification by Heidrich-Meisner and Igel [9], and its sample complexity was analyzed in [17]. Median Elimination (ME) [14] runs in stages where each stage eliminates the bottom half of the surviving arms. Under the PAC setup, the sample complexity of ME matches the lower bound. Going beyond elimination-based algorithms, LUCB proposed by Kalyanakrishnan et al. [15] adaptively chooses which arm to pull based on confidence intervals without explicitly removing arms. Kaufmann et al. improved both Racing and LUCB with confidence intervals based on Chernoff information, though restricted to exponentially distributed rewards [17].

In the fixed budget setting, Successive Accepts and Rejects (SAR) proposed by Bubeck et al. [3] runs in \( n - 1 \) phases. Each phase performs uniform sampling for a predetermined number of rounds. At the end of each phase, SAR either accepts the empirically best arm or rejects the worst arm. UCB-E by Audibert et al. [3] and LUCB-E by Kaufmann et al. [17] run in a more adaptive way with confidence intervals without eliminations, although the problem hardness parameter must be given to the algorithms, which is unrealistic.

Our proposed \((b,r)\)-batch setting overlaps with various existing settings, but is, to the best of our knowledge, never subsumed by any. One popular setting, though studied under the regret setting, is semi-bandit feedback [2] where an action \( a \in A \subseteq \{0,1\}^n \) indicates coordinate-wise whether or not \( a_j = 1 \) or \( 0 \) one pulls an arm. \((b,r=1)\)-batch is a special case of semi-bandit feedback where the action space \( A \) is defined to be all \( b \)-sized subsets of arms. This is called multiple plays in [1]. A variant of the multiple plays was studied in [18] where the player pulls \( b \) times in a round, but is allowed to pull an arm more than once, up to \( b \) times. However, the authors assume that pulling the same arm more than once in a round produces the same reward, whereas in our \((b,r=b)\)-batch setting repeated pulls produce independent rewards.

In the delayed feedback setting [23, 13], the reward at time \( t \) is revealed after \( \tau_t \) time steps. One can relate this setting to our \((b,r)\)-batch setting by assuming that rewards are revealed in blocks after every \( b \) rounds: \( \tau_t = b - 1 - (t - 1 \mod b) \). If \( b \) is known to the player, this construction is exactly the \((b,r=b)\)-batch setting. Nevertheless, delayed feedback has only been considered in the regret setting to the best of our knowledge.

Recently, Perchet et al. have proposed a generalized notion of batch arm pulls [22] where each batch can be of different sizes. However, the authors only consider two-armed bandits \((n = 2)\) and assume no limitation on the repeated arm pulls \((b=r)\). More importantly, the authors only consider the regret setting. Wu et al. [24] proposed a pure exploration MAB framework for combinatorial action space of which \((b,r=1)\)-batch is an instance. However, repeated arm pulls \((r > 1)\) are not allowed in their framework.

## 3 The Fixed Confidence Setting

In the fixed confidence setting, given a target failure rate \( \delta \), one tries to find the top-\( k \) arms with probability at least \( 1 - \delta \) in the fewest number of batches as possible. An algorithm must output the correct top-\( k \) arms w.p. \( \geq 1 - \delta \), and its performance is measured in the batch complexity.

Denote by \( \max(k) F \) the \( k \)-th largest member of a set \( F \). Let \( X_{i,j} \) be the \( j \)-th reward of arm \( i \). Define the empirical mean of arm \( i \) up to \( \tau \) samples as \( \hat{\mu}_{i,\tau} = \frac{1}{\tau} \sum_{j=1}^{\tau} X_{i,j} \). The key success of an algorithm in the fixed confidence setting often relies on the confidence bound on the true mean \( \mu_i \); the tighter the bounds are,
the less arm pulls we spend. We adopt and simplify a
confidence bound proposed in [11] that resembles the law
of the iterated logarithm.

Lemma 1. (non-asymptotic law of the iterated log-
arithm) Let \( X_1, X_2, \ldots \) be i.i.d. zero-mean sub-
Gaussian random variables with scale \( \sigma > 0 \); i.e.,
\[ \Pr(\forall \tau \geq 1, \sum_{s=1}^{\tau} X_s \leq 4\sigma \sqrt{\tau \log(\log_2(2\tau)/\omega)}) \geq 1 - 6\omega^2. \] \tag{1}

The proof of Lemma 1 is in the supplementary
material. Note that a bounded random variable \( X \in [a, b] \)
is a sub-Gaussian with scale \( \sigma = (b - a)/2 \). In our
case, \( \sigma = \frac{1}{2} \). Define a deviation function
\[ D(\tau, \omega) := \sqrt{\frac{4\log(\log_2(2\tau)/\omega)}{\tau}}. \]

Let \( T_i(t) \) be the number of arm pulls of arm \( i \) at round
\( t \). We define the lower confidence bound (LCB) \( L_i(t, \delta) \)
and the upper confidence bound (UCB) \( U_i(t, \delta) \) of arm
\( i \) at round \( t \) as
\[ L_i(t, \delta) := \hat{\mu}_{i, T_i(t)} - D(T_i(t), \sqrt{\delta/(6n)}) \quad \text{and} \quad U_i(t, \delta) := \hat{\mu}_{i, T_i(t)} + D(T_i(t), \sqrt{\delta/(6n)}). \] \tag{2}

Now that the environment allows the \((b, r)\)-batch arm
pull, can we propose an algorithm whose batch com-
plexity achieves \( b \) factor reduction from the sample
complexity of the state-of-the-art algorithms? Inspired
by the Racing algorithm [19, 9], we propose BatchRacing
algorithm for the \((b, r)\)-batch setting; see Algorithm 2.
Algorithm 2. The algorithm maintains a set of surviving
arms that is initialized as \( S_1 = [n] \) before the first
round. At round \( t \), the algorithm calls RoundRobin
(Algorithm 1) to choose \( b \) arm pulls that keeps the
pull count of each arm in the surviving set \( S_t \) as
uniform as possible. Then, the algorithm checks if there
is any arm that is confidently top-\( k \) or confidently
not top-\( k \) using the LCB and UCB as follows. Let \( A_t \) \((R_t)\)
be the set of accepted (rejected) arms at round \( t \) and
\( A_1 = R_1 = \emptyset \). Let \( k_t = k - |A_t| \), the remaining
number of top arms to identify. Any arm \( i \) whose LCB is
greater than the UCB of \( |S_t| - k_t \) arms is moved to
the accept set: \( A_{t+1} \leftarrow A_t \cup \{i\} \). Symmetrically,
any arm \( i \) whose UCB is smaller than the LCB of \( k_t \) arms
is moved to the reject set \( R_{t+1} \leftarrow R_t \cup \{i\} \). Those
accepted or rejected arms are removed from the
surviving set \( S_t \). The process is repeated until it accepts
\( k \) arms \((|A_t| = k)\), and finally it outputs \( A_t \).

We define the gap \( \Delta_i \) of arm \( i \) and denote by \( \sigma(j) \)
the arm with \( j \)-th smallest gap as follows:
\[ \Delta_i := \begin{cases} 
\mu_i - \mu_{k+1} & \text{if } i \leq k \\
\mu_k - \mu_i & \text{if } i > k 
\end{cases} \] \tag{3}
where \( \Delta_{\sigma(1)} = \Delta_{\sigma(2)} \) by definition. Let \( \mathcal{E}_i(\delta) := \{ \forall \tau \geq 1, L_i(t, \delta) \leq \mu_i \leq U_i(t, \delta) \} \) be the event that the LCB
and UCB of arm \( i \) defined in (2) traps the true mean
\( \mu_i \) for all \( t \geq 1 \). Let \( \omega = \sqrt{\delta/(6n)} \). Define \( T_i := 1 + [64\Delta_i^{-2}\log((2/\omega)\log_2(192\Delta_i^{-2}/\omega))] \). Lemma 2 shows
how many arm pulls are sufficient to classify an arm to
either reject set \( R_t \) or accept set \( A_t \).

Algorithm 2. Assume \( \cap_{t=1}^n \mathcal{E}_i(\delta) \). In Algorithm 2 let
\( T'(t) = \min_{i \in S_t} T_i(t) \) and \( k_t = k - |A_t| \). Then,
\[ \forall t, \forall i > k, \left( T'(t) \geq T_i \implies U_i(t, \delta) \leq \max_{j \in S_t} L_j(t, \delta) \right) \] \tag{4}
\[ \forall t, \forall i < k, \left( T'(t) \geq T_i \implies L_i(t, \delta) \geq \max_{j \in S_t} U_j(t, \delta) \right). \] \tag{5}

Proof. We use \( L_i(t) \) and \( U_i(t) \) as shorthands for
\( L_i(t, \delta) \) and \( U_i(t, \delta) \) respectively. It suffices to show
for the case where \( A_t \) and \( R_t \) are empty since other-
wise the problem is equivalent to removing rejected or
accepted arms from consideration and starting a new
problem with \( n \leftarrow (n - |A_t| - |R_t|) \) and \( k \leftarrow (k - |A_t|) \)
while maintaining the samples collected so far.
To prove [4], let $i > k$ and $\hat{k}$ be the arm with $k$-th largest empirical mean at $t$-th round. We prove the contrapositive. Assume the RHS is false: $U_i(t) \not\geq \max_{j \in S_k} L_j(t)$. Note that using $D(T_i(t), \omega) \leq D(T'(t), \omega)$,

$$U_i(t) \leq \hat{\mu}_{i,T_i(t)} + D(T'(t), \omega) \leq \mu_i + 2D(T'(t), \omega)$$

and

$$U_i(t) \geq \max_{j \in S_k} L_j(t) = \hat{\mu}_{k,T_k(t)} - D(T'(t), \omega),$$

which implies $\mu_i + 2D(T'(t), \omega) \geq \hat{\mu}_{k,T_k(t)} - D(T'(t), \omega)$ that is due to Lemma[3] (see the supplementary material),

$$\mu_i + 2D(T'(t), \omega) \leq \mu_k - 2D(T'(t), \omega)$$

$$\Delta_i \leq 4D(T'(t), \omega)$$

$$= 4\sqrt{(4/T'(t)) \log(\log_2(2T'(t))/\omega)}$$

$$T'(t) \leq 64\Delta_i^{-2} \log(\log_2(2T'(t))/\omega).$$

Invert this using

$$\tau \leq c \log \left( \frac{\log_2 (2\tau)}{\omega} \right) \Rightarrow \tau \leq c \log \left( \frac{2}{\omega} \log \left( \frac{3c}{\omega} \right) \right)$$

with $c = 64\Delta_i^{-2}$ to have $T'(t) \leq 64\Delta_i^{-2} \log((2/\omega) \log_2(192\Delta_i^{-2}/\omega))$.

Then, $T'(t) < 1 + [64\Delta_i^{-2} \log((2/\omega) \log_2(192\Delta_i^{-2}/\omega))] = T_i$. This completes the proof of [4]. By symmetry, one can show [5] as well.

Theorem 4 states the batch complexity of the BatchRacing algorithm. Hereafter, all proofs can be found in the supplementary material. Note that even if $r > [b/2]$, BatchRacing pulls the same arm at most $[b/2]$ times in a batch. Thus, our analysis hinges on the effective repeated pull limit $r' = \min\{r, [b/2]\}$ instead of $r$.

**Theorem 1.** If $b \geq 2$, with probability at least $1 - \delta$, Algorithm 3 outputs the top-$k$ arms $\{1, \ldots, k\}$ after at most

$$\frac{1}{r'} T_{\sigma(1)} + \frac{1}{b} \left( \sum_{i=[b/r']+1}^{n} T_{\sigma(i)} \right) + \log n + \frac{1}{b'} + \frac{1}{r'} + 2$$

$$= O \left( \frac{1}{r'} \Delta_{\sigma(1)}^{-2} \log \left( \frac{n \log(\Delta_{\sigma(1)}^{-2})}{\delta} \right) \right) +$$

$$\frac{1}{b} \left( \sum_{i=[b/r']+1}^{n} \Delta_{\sigma(i)}^{-2} \log \left( \frac{n \log(\Delta_{\sigma(i)}^{-2})}{\delta} \right) \right) + \log n$$

(7)

batches. In the case of $b = r = 1$, the algorithm does so after at most $\sum_{i=1}^{n} T_{\sigma(i)} = O \left( \sum_{i=1}^{n} \Delta_{\sigma(i)}^{-2} \log \left( \frac{n \log(\Delta_{\sigma(i)}^{-2})}{\delta} \right) \right)$ batches.

In the case of $b=r=1$, BatchRacing is exactly the Racing algorithm, and the batch complexity is equivalent to the sample complexity. The sample complexity of Racing stated in Theorem 4 is the best known insofar as the exact top-$k$ identification problem, although it does not match the best known lower bound $\Omega(\sum_{i=1}^{n} \Delta_i^{-2})$. If $r \geq [b/2]$, one can verify that the batch complexity (7) reduces to $\frac{1}{2}$ fraction of the sample complexity of Racing except the additive log $n$ term. The log $n$ term is the price to pay for performing batch arm pulls, which limits adaptivity. Note that the batch complexity is at least $n/b$ since each arm must be pulled at least once. Thus, unless $b$ is large enough to satisfy $n/b \ll \log(n)$, the additive log $n$ is negligible. For simplicity, we ignore the additive log $n$ from the discussion. If $r < [b/2]$, $r$ plays an interesting role. The $r$ factor reduction is applied to the largest term that involves $\Delta_{\sigma(i)}^{-2}$. The terms involving arm $\sigma(2), \ldots, \sigma([b/r])$ disappear. The rest of the terms enjoy $b$ factor reduction.

The contribution of $r$ depends on the gaps $\{\Delta_i\}$. If the smallest gap $\Delta_{\sigma(1)}$ is much smaller than the other gaps, the term $(1/r')T_{\sigma(1)}$ becomes dominant, thus making $r$ important. On the other hand, if the gaps are all equal, one can show that the overall $b$ fold reduction is achieved regardless of $r$ using the fact $(1/r') \Delta_{\sigma(1)}^{-2} \log(n \log(\Delta_{\sigma(1)}^{-2})/\delta) = (1/b) \sum_{i=1}^{[b/r']} \Delta_{\sigma(i)}^{-2} \log(n \log(\Delta_{\sigma(i)}^{-2})/\delta)$. We empirically verify this case with toy experiments in Section 4.

**4 The Fixed Budget Setting**

We now consider the problem of identifying the top-$k$ arms with as high a probability as possible within a given number of batches $B$ under the $(b, r)$-batch setting. An algorithm must terminate after spending no more than $B$ batches and output $k$ arms that are believed to be the top-$k$. The guarantee is typically made on the misidentification probability $P(A^* \neq \{1, \ldots, k\})$ where $A^*$ is the output of the algorithm. However, it is often convenient to look at its batch complexity that can be derived from the misidentification probability. Let $H_2 = \max_{i \in [n]} i \Delta_i^{-2}$. Bubeck et al. showed that the Successive Accepts and Rejects (SAR) algorithm for the top-$k$ identification problem under the fixed budget setting has the sample complexity of $O(H_2 \log^2 n)$ [5]. One might try running SAR by supplying SAR with $bB$ sample budget. However, such a modification breaks its theoretical guarantee. We propose BatchSAR that extends SAR to allow $(b, r)$-batch arm pull, and prove its performance guarantee.

The intuition behind BatchSAR is as follows. Imagine the $(b = n, r = 1)$-batch setting where one has
no choice but to pull every arm once at every round. There is no room for adaptivity; eliminating an arm does not yield more arm pulls in return. Let \( \hat{r} = \min \{r, [b/2]\} \) and \( \hat{n} = [b/\hat{r}] \). Note \( \hat{n} \geq 2 \) by definition. Similarly in the general \((b, r)\)-batch setting, once the surviving number of arms becomes \( \hat{n} \) or less, eliminating an arm does not help. For example, if \( r < [b/2] \) and there are \( \hat{n} - 1 \) surviving arms, the maximum number of arm pulls one can make in a batch is \( r(\hat{n} - 1) < b \), thus wasting \( b - r(\hat{n} - 1) \) arm pulls.

### Algorithm 3 BatchSAR

1. **Input:** \( n \) arms, \( k \): the target number of top arms, \( b \): batch size, \( r \): repeated pull limit, \( B \): batch budget
2. **Output:** \( k \) arms.
3. Let \( \hat{n} = \max \{[b/r], 2\} \) and \( c_1 = b + \hat{n} \min \{r, [b/2]\} + n \)
4. Define \( m_s = \left\lfloor \frac{bB - \sum_{i=\hat{n}}^{n} (i/c_1 - c_i)}{\frac{b}{\hat{r}} \sum_{i=\hat{n}+1}^{n+1} (1/i)} \right\rfloor \) for \( s \leq n - \hat{n} \)
5. Define \( m_s = \left\lfloor \frac{bB - \sum_{i=\hat{n}}^{n} (i/c_1 - c_i)}{\frac{b}{\hat{r}} \sum_{i=\hat{n}+1}^{n+1} (1/i)} \right\rfloor \) for \( s = n - \hat{n} + 1 \)
6. for \( s = 1, \ldots, (n - \hat{n} + 1) \) do
7. \( \text{while } \min_{i \in S_s} T_i(t-1) < m_s \) do
8. \( \text{a} \leftarrow \text{RoundRobin}(S_s, \{T_i(t-1)\}_{i \in S_s}, b, r) \)
9. Pull by \( a \) (pull arm \( a \) times, \( \forall i \in S_s \))
10. \( T_i(t) \leftarrow T_i(t-1) + a, \forall i \in S_s \)
11. \( t \leftarrow t + 1 \)
12. end while
13. Let \( t_s = t - 1 \), the last round in stage \( s \), and \( k' = k - |A_s| \), the remaining number of top arms to identify.
14. if \( s \leq n - \hat{n} \) then
15. Let \( \hat{\mu}_i = \hat{\mu}_i T_i(t_s) \) and \( \rho(i) \) be the arm with \( i \)-th largest empirical mean in \( S_s \) so that \( \hat{\mu}_{\rho(1)} \geq \cdots \geq \hat{\mu}_{\rho(n-s+1)} \). Let \( \hat{\Delta}_{\rho(1)}(t_s) = \hat{\mu}_{\rho(1)}(t_s) - \hat{\mu}_{\rho(k')}(t_s) \) and \( \hat{\Delta}_{\rho(n-s+1)}(t_s) = \hat{\mu}_{\rho(n-s+1)}(t_s) - \hat{\mu}_{\rho(n-s+1)}(t_s) \).
16. \( j_s \leftarrow \arg\max_{i \in [\rho(1), \rho(n-s+1)]} \hat{\Delta}_i \) (break ties arbitrarily).
17. Remove arm \( j_s \); \( S_{s+1} \leftarrow S_s \setminus \{j_s\} \).
18. if \( j_s = \rho(1) \) then
19. \( A_{s+1} \leftarrow A_s \cup \{j_s\} \)
20. else
21. \( A_{s+1} \leftarrow A_s \)
22. end if
23. (Early exit case 1) If \( |S_{s+1}| = k - |A_{s+1}| \), then accept all the remaining surviving arms \( A^* \leftarrow A_{s+1} \cup S_{s+1} \) and exit the for loop.
24. (Early exit case 2) If \( |A_{s+1}| = k \), then set \( A^* \leftarrow A_{s+1} \) and exit the for loop.
25. else
26. \( A^* \leftarrow A_s \cup \{k' \text{ max}_{i \in S_s} \hat{\mu}_i, T_i(t_s)\} \) (break ties arbitrarily)
27. end if
28. end for
29. Output \( A^* \).

### Theorem 2

Let \( A^* \) be the output of BatchSAR. Given a batch budget \( B \), the misidentification probability of BatchSAR under the \((b, r)\)-batch setting satisfies

\[
P(A^* \neq \{1, \ldots, k\}) \leq 2n^2 \exp \left( -\frac{1}{8} B - \left( \sum_{i=\hat{n}+1}^{n} [b/i] / b - c_1/b \right) \right)
\]

We can derive the batch complexity of BatchSAR as \( O \left( H_3^{<b, r>} \log(n) + \frac{1}{b} \sum_{i=\hat{n}+1}^{n} \left[ \frac{b}{i} \right] \right) \). The term \( \frac{1}{b} \sum_{i=\hat{n}+1}^{n} \left[ \frac{b}{i} \right] = O(\log(n) + n/b) \) is the price to pay for performing the \((b, r)\)-batch where the algorithm might pull an arm more than necessary. The batch complexity is upper-bounded by \( O \left( \frac{1}{b} + \frac{\log n}{b} \right) \). The additive \( \log(n) \) is negligible unless \( n/b \ll \log(n) \).

In the extreme case of \( b=r=1 \), the term \( \frac{1}{b} \sum_{i=\hat{n}+1}^{n} \left[ \frac{b}{i} \right] \) is \( n \), which can be ignored since the batch complexity is at least of order \( n \) when \( b = 1 \). Thus, we achieve \( O(H_3 \log^2 n) \), which is equivalent to the sample complexity of SAR. If \( r \geq [b/2] \), \( H_3^{<b, r>} \log n + \log n \leq \log(n) \).
Propose to replace \( m \) with \( \sigma \) and \( \beta \) \((\sum \delta)\) to verify that \( m \) reduces to uniform sampling and the batch complexity is \( O(\Delta_{\sigma}^2 \log n) \).

**Practical Considerations** In general, the algorithm finishes earlier than \( B \) batches. The reason is that \( m \) is defined to defend against the worst case. In practice, one might want to exhaust the budget \( B \) to minimize the error. Let \( \log(b|a) = \sum_{i=a+1}^{b} \delta \). We propose to replace \( m \) with

\[
m'_a = \left[ bB - \left( \sum_{j=1}^{n-1} T_{j,j}(t_{j}) \right) - \left( \sum_{i=n+1}^{n+1} \left[ \frac{1}{b} \right] \right) - c'_1 \right] \div (n/2 + \log(n-s+1|n)) (n-s+1) \tag{10}
\]

where \( c'_1 = b + n\delta + n - s + 1 \). For the last stage \( s = n - n\delta + 1 \), we sample the surviving arms until the budget is exhausted. One can show that \( m'_a \geq m \) after using (8) to verify \( \sum_{i=a+1}^{n+1} \left[ \frac{1}{b} \right] + c'_1 \leq \left( \sum_{i=n+1+1}^{n+1} \left[ \frac{1}{b} \right] \right) + c_1 + \frac{(bB-\left( \sum_{i=n+1}^{n+1} \left[ \frac{1}{b} \right] \right)-c_1)}{(n/2 + \log(n|n))} \log(n|n-s+1). \) Thus, Theorem 2 still holds true with \( m'_a \).

**5 Experiments**

We verify our theoretical results with simulation experiments in the fixed confidence setting, then present two important applications of the \((b,r)\)-batch MAB problem. For applications, we focus on the fixed budget setting due to its practicality.

**5.1 Toy Experiments**

The batch size \( b \) and repeated pull limit \( r \) contribute to the batch complexity in an intricate way in both the fixed confidence and budget settings. We would like to verify how \( b \) and \( r \) change the batch complexity empirically and compare it to our theory. Note that the batch complexity shown in the analysis considers the worst case; the number of batches suggested by the theory is often more than what is experienced in practice. However, if such slack is independent of \( b \) and \( r \), we can remove it by monitoring the “speedup”, see below.

We consider two bandit instances with \( n = 100 \) arms: (i) Bernoulli arms with expected rewards that linearly decrease, i.e., \( \mu_i = (99 - i + 1)/99 \), and (ii) sparsely distributed Bernoulli arms where the expected rewards of 10 arms are 0.5 and the rest are 0.3. We call the former Linear and the latter Sparse. Note we focus on the fixed confidence setting here due to space constraints; the same result can be obtained in the fixed budget setting. We run BatchRacing 10 times with \( \delta = 0.1 \), \( k = 10 \), \( b \in \{1, 4, 16, 64\} \) and \( r \in \{1, 2, 4, 8, 16, 32\} \). For each pair \((b', r')\), we compute the empirical speedup by dividing the average number of batches in \((b=1, r=1)\) by the average number of batches in \((b', r')\). Let \( M_{b,r} \) be the batch complexity computed by (7). For each pair \((b', r')\), we also compute the theory-based speedup by \( \frac{\sum_{i=1}^{n} T_{\sigma(i)}}{M_{b', r'}} \).

Table 4 shows the the speedup of each \((b, r)\)-batch. Overall, the theory-based speedup matches extremely well with the empirical speedup. This implies that the rather complicated contribution of \( b \) and \( r \) in the batch complexity well explains the reality. Note that \( r \) plays an important role in Linear; it is only after \( r = b/2 \) that the speedup achieves \( b \). This is because \( \Delta_{\sigma}^2 = (\mu_k - \mu_{k+1})^2 \) dominates, which boosts the batch complexity’s reliance on the factor \( 1/r \). On the other hand, in Sparse the gaps \( \{\Delta_i\} \) are uniformly 0.2, and we achieved an overall \( b \) fold speedup (almost) regardless of \( r \) as predicted in Section 3.

**5.2 Application 1: Virus Replication Experiments**

Virologists are interested in identifying which genes are important in virus replication. For instance, Hao et al. tested 13k genes of drosophila to find which genes promote influenza virus replication [8]. The authors perform a sequence of batch experiments using a plate with 384 microwells. Each well contains a single-gene knock-down cell strain to which the fluorescing virus is added. They measured the fluorescence levels after an incubation period and obtained an indication of how important the knock-down gene is in virus replication. Since the measurements are noisy, we formulate the problem as a top-\( k \) identification problem in MAB. Each arm is a gene, and the reward is its knock-down fluorescence level. The batch arm pull is performed with \( b = 384 \) wells where the repeated pull limit \( r \) is equal to \( b \) since one can place the same gene knock-down in multiple wells.

**Dataset** We simulate the microwell experiments by assuming a Gaussian distribution on each arm, where the mean and variance are estimated from the actual data. Specifically, we use two measurements from each of the \( n = 12,979 \) gene knock-downs from [8]. For the \( j \)-th measurement of arm/gene \( i \), \( x_{i,j} \), we remove the plate-specific noise by dividing it by the average control measurement from the same plate. Then, we take the logarithm to effectively make the variance of each fluorescence measurement similar, following [8]. We also flip the sign to call it \( z_{i,j} \) since a low fluorescence level implies that the knock-down gene pro-

\( \frac{1}{2}(2 + \log n)H_2 \log n + \log n = O(\frac{1}{2}H_2 \log^2(n) + \log n) \), which achieves an overall \( b \) factor reduction from the sample complexity of SAR except for the additive \( \log n \). In the other extreme case where \( b = n \) and \( r = 1 \), BatchSAR reduces to uniform sampling and the batch complexity is \( O(\Delta_{\sigma}^2 \log n) \).
Table 1: Toy experiment result: the speedup in the number of batches in the fixed confidence setting.

<table>
<thead>
<tr>
<th>$b$</th>
<th>$r$</th>
<th>Linear Theory</th>
<th>Actual</th>
<th>Sparse Theory</th>
<th>Actual</th>
</tr>
</thead>
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<tr>
<td>4</td>
<td>1</td>
<td>2.71</td>
<td>2.74</td>
<td>4.00</td>
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</tr>
<tr>
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<td>3.14</td>
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<td>16.00</td>
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</tr>
<tr>
<td>16</td>
<td>2</td>
<td>6.08</td>
<td>6.10</td>
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</tr>
<tr>
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</tr>
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<td>16.00</td>
<td>16.00</td>
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<tr>
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<td>1</td>
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<td>63.97</td>
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</tr>
<tr>
<td>64</td>
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<td>6.41</td>
<td>63.97</td>
<td>61.88</td>
</tr>
<tr>
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<td>4</td>
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</tr>
<tr>
<td>64</td>
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<td>64.00</td>
<td>63.99</td>
<td>63.97</td>
<td>63.90</td>
</tr>
</tbody>
</table>

Sparse notations the replication. We set the mean of arm $i$ as $\mu_i = (z_{i,1} + z_{i,2})/2$. We compute the unbiased estimate of the variance $\hat{\sigma}_i$ of each gene and compute the average $(1/n) \sum_{i=1}^n \hat{\sigma}_i$ across all arms, which is an unbiased estimate of the variance if all arms have an equal variance. In the end, we simulate measurements for arm $i$ from $\mathcal{N}(\mu_i,0.1)$. We plot the arm distribution in Figure 1(a), where we see that relatively few genes have large rewards (that facilitate virus replication).

Algorithm 4 Halving

1: Input: $n$ arms, $k$: the target number of top arms, $b$: batch size, $r$: repeated pull limit, $B$: batch budget
2: Output: $k$ arms.
3: $t \leftarrow 1$, $S_t \leftarrow [n]$, $T_i(0) \leftarrow 0$, $\forall i$
4: for $s = 1$ to $\lceil \log_2 \frac{k}{B} \rceil$ do
5: for $i = 1$ to $\left\lceil \frac{B}{\log_2(n/k)} \right\rceil$ do
6: $a \leftarrow \text{RoundRobin}(S_s, \{T_i(t-1)\}_{i \in S_s}, b, r)$
7: Pull by $a$ (pull arm $i$ $a_i$ times, $\forall i$).
8: $T_i(t) \leftarrow T_i(t-1) + a_i$, $\forall i$
9: $t \leftarrow t + 1$
10: end for
11: $S_{s+1}$ be the set of max${\{\left\lceil S_s \right\rceil /2 \}, [b/r]}$ arms in $S_s$ with the largest empirical average.
12: end for
13: Output $S_{s+1}$

Methods We use two algorithms as the baseline: RoundRobin and Halving. RoundRobin repeatedly pull arms according to RoundRobin (Algorithm 1) with no eliminations until the batch budget runs out. Halving adapts Sequential Halving algorithm 10, a top-1 identification algorithm, to perform top-$k$ identification under the batch setting 3 though it lacks theoretical guarantee. We describe Halving in Algorithm 4. Both Halving and BatchSAR could waste the budget if run as is due to the integer operators. Thus, in Halving we randomly spread out the remaining budget over stages, and in BatchSAR we use $m_i'$ defined in 10 in place of $m_s$.

Result We run the algorithms for $k \in \{100,200,400,800\}$ and $B \in \{100,200,400,800\}$. We run each experiment 20 times and measure false negative rate (FNR). FNR is the proportion of the top-$k$ arms that do not appear in the output of the algorithm: $|\hat{k}\setminus \{\text{the output of the algorithm}\}|$.

High FNR was pointed out by 9 as the main issue in adaptive experiments. The result is shown in Figure 1(b-e). We plot the FNR vs. batch budget $B$ for each $k$, where the error bar is the 95% confidence interval. In all experiments, BatchSAR and Halving significantly outperform RoundRobin. RoundRobin spends the vast majority of arm pulls on arms with small expected rewards and fails to distinguish the top-$k$ arms from those near top-$k$. BatchSAR outperforms Halving in most cases, especially when the budget is large.

5.3 Application 2: Twitter User Tracking

We are interested in identifying the top-$k$ users in Twitter who talk about a given topic most frequently. For example, in the study of bullying researchers are interested in a longitudinal study on the long term effects of bullying experience 21. Since there exist twelve 5-minute intervals in an hour, we either follow a user or not, we discuss a procedure below where $r$ can be larger.

We observe that the tweet rate of users changes significantly over time of day. Figure 2(a) shows the daily trend. We plot the number of bullying tweets generated in each 5-minute interval where the count is accumulated over 31 days. To get around this diurnal issue, we define a pull of an arm to be following a user for 5 minutes every hour in a day, totaling two hours. Since there exist twelve 5-minute intervals in an hour, one can follow $b = 12 \cdot 5000 = 60,000$ users in a day.

3 We change the number of stages from $\lfloor \log_2 n \rfloor$ to $\lfloor \log_2(n/k) \rfloor$ following 15 and divide stages in terms of the number of batches rather than samples.

4 statuses/filter API with follow parameter.
Note those 60,000 users need not be unique; one can follow the same user up to 12 times, each in a different 5-minute interval. Thus, \( r=12 \). Formally, we assume that the bullying tweet rate of a user \( i \) changes every hour of day \( h \), but remains the same within. Let \( Y_{i,d,h,c} \) indicate whether or not (1 or 0) the user \( i \) generates a bullying tweet at day \( d \) hour \( h \) interval \( c \in [12] \). \( Y_{i,d,h,c} \) follows a Bernoulli(\( p_{i,h} \)) distribution, where \( p_{i,h} \) is the tweet probability of user \( i \) during a 5-minute interval in hour \( h \). We define the reward of an arm \( i \) on day \( d \) as \((1/24) \sum_{h=1}^{24} Y_{i,d,h,c(h)} \) where \( c(h) \) indicates which interval to pick at each hour \( h \). Then, the expected reward \( \mu_i \) of an arm \( i \) is the average Bernoulli probability over a day: \( \mu_i = (1/24) \sum_{h=1}^{24} p_{i,h} \).

**Dataset** The dataset consists of 707,776 bullying tweets collected over 31 days in January 2013. The number of distinct users \( n \) is 522,062. We compute the average reward for each user at each hour and take it as the true tweet probability \( p_{i,h} := 1/n \sum_{i=1}^{31} 1/24 \sum_{c=1}^{12} Y_{i,d,h,c} \) from which we compute the expected reward \( \mu_i \) as previously shown. We plot the expected reward in Figure 2(b), where both axes are in log scale. The plot nicely follows a power law.

**Result** We use the same methods from Section 5.2. We set \( k=100 \) and run each algorithm 10 times due to randomness in breaking ties. We measure the gain \( \sum_{i \in J} \mu_i \), where \( J \) is the set of \( k \) arms output by an algorithm. The gain measures the quality of the selected arms by the magnitudes of their expected rewards. The result is summarized in Figure 2(c). The number in parentheses is the 95% confidence interval, and Oracle is the gain on the true top-\( k \) arms (\( J = \{1, \ldots, k\} \)), which is the best achievable. Both Halving and BatchSAR significantly outperforms RoundRobin. Halving, though without theoretical guarantee, performs slightly better than BatchSAR. We suspect that the reason is that Halving works well in low budget situations as it was in Section 5.2. A thorough empirical comparison is left as a future work.

**6 Future Work**

There are numerous opportunities for future works in the \((b,r)\)-batch setting. First, adapting existing pure exploration algorithms to the batch setting and analyzing their batch complexity. Not only Halving presented in this work without guarantee but also LUCB would be interesting to analyze since it is known to perform best in practice [12]. Second, investigating the lower bound on the batch complexity. Specifically, we have observed that the contribution of the repeated pull limit \( r \) to the batch complexity is nontrivial. It would be interesting to see if the same form of contribution happens in the lower bound as well. Finally, we would like to extend the batch setting to accommodate different batch sizes as in [22], but for \( n > 2 \) arms (22 only pertains to \( n = 2 \)) and in the pure exploration setting rather than the regret setting. Varying batch size MAB problems can arise in clinical trials and in crowdsourcing systems where the crowd size dynamically changes over time.

**Acknowledgments**

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References


Top Arm Identification in Multi-Armed Bandits with Batch Arm Pulls


Supplementary Material

A Proof of Lemma 1

Let $S_\tau = \sum_{s=1}^{\tau} X_s$. Note that the upper and lower bound around $S_\tau$ is symmetric. Thus, it suffices to show that the upper bound on $S_\tau$ fails with probability at most $3\omega^2$. Show that $\{e^{S_\tau}\}$ is sub-Martingale, apply Doob’s maximal inequality to it, then use the definition of the sub-Gaussian and Markov’s inequality to have

$$
P\left(\max_{\tau=1,...,m} S_\tau \geq x\right) \leq e^{-\frac{x^2}{2m\sigma^2}}.
$$

Define

$$
\mathcal{E}_1 = \bigcap_{k \geq 0} \left\{ S_{2^k} \leq 2\sigma \sqrt{2^k \log(\log_2(2 \cdot 2^k)/\omega)} \right\}
$$

and

$$
\mathcal{E}_2 = \bigcap_{k \geq 1} \bigcap_{\tau = 2^k} \left\{ S_\tau - S_{2^k} \leq 2\sigma \sqrt{2^k \log(\log_2(2 \cdot 2^k)/\omega)} \right\}.
$$

One can verify that $\mathcal{E}_1 \cap \mathcal{E}_2$ implies the upper bound stated in the Lemma since (i) for $\tau = 1$ and 2, $\mathcal{E}_1$ implies the bound and (ii) for all $\tau \geq 3$ such that $2^k < \tau \leq 2^{k+1}$ for some $k$,

$$
S_\tau \leq S_{2^k} + 2\sigma \sqrt{2^k \log(\log_2(2 \cdot 2^k)/\omega)} \leq 4\sigma \sqrt{2^k \log(\log_2(2 \cdot 2^k)/\omega)} < 4\sigma \sqrt{\tau \log(\log_2(2\tau)/\omega)}.
$$

The failure probability of the bound is at most $P(\mathcal{E}_1) + P(\mathcal{E}_2)$, which we bound with $3\omega^2$ as follows:

$$
P(\mathcal{E}_1) \leq \sum_{k \geq 0} e^{-2\log(\log_2(2^k+1))/\omega} \leq \frac{\pi^2}{6}\omega^2 < 2\omega^2
$$

$$
P(\mathcal{E}_2) \leq \sum_{k \geq 1} \sum_{\tau = 2^k+1} P\left( \bigcup_{\tau = 2^k+1} \left\{ S_\tau - S_{2^k} \geq 2\sigma \sqrt{2^k \log(\log_2(2 \cdot 2^k)/\omega)} \right\} \right)
$$

$$
= \sum_{k \geq 1} \sum_{\tau = 2^k+1} P\left( \bigcup_{\tau = 2^k+1} \left\{ \sum_{i=2^k+1}^{\tau} X_i \geq 2\sigma \sqrt{2^k \log(\log_2(2 \cdot 2^k)/\omega)} \right\} \right)
$$

$$
= \sum_{k \geq 1} \sum_{j=1}^{2^k} P\left( \bigcup_{j=1}^{2^k} \left\{ S_j \geq 2\sigma \sqrt{2^k \log(\log_2(2 \cdot 2^k)/\omega)} \right\} \right)
$$

$$
\leq \sum_{k \geq 1} \left( \frac{\omega}{k+1} \right)^2 = (\pi^2/6 - 1)\omega^2 < \omega^2.
$$

B Proof of Theorem 1

Lemma 3 states that the true mean of arm $i$ can be bounded by the empirical top-$i$ arm.

**Lemma 3.** Denote by $\hat{i}$ the index of the arm whose empirical mean is i-th largest: i.e., $\hat{\mu}_i \geq \ldots \geq \hat{\mu}_n$. Assume that the empirical means of the arms are controlled by $\epsilon$: i.e., $\forall i, |\hat{\mu}_i - \mu_i| \leq \epsilon$. Then,

$$
\forall i, \mu_i - \epsilon \leq \hat{\mu}_i \leq \mu_i + \epsilon.
$$

**Proof.** To show $\mu_i - \epsilon \leq \hat{\mu}_i$, consider two cases: (i) empirically i-th largest arm is true top-$i$, i.e. $\hat{i} \leq i$, and (ii) is not true top-$i$, i.e. $\hat{i} > i$. Case (i) implies that $\hat{\mu}_i \geq \mu_i$. Then,

$$
\hat{\mu}_i \geq \mu_i - \epsilon \geq \mu_i - \epsilon.
$$

Case (ii) implies that there exists an arm $\ell$ that is true top-$i$, but not empirical top-$i$, i.e. $\mu_\ell \geq \mu_i$ and $\hat{\mu}_i \leq \hat{\mu}_\ell$, since otherwise $\hat{i}$ (the empirical top-i) must have been squeezed out of the empirical top-$i$, which is a contradiction. Then,

$$
\hat{\mu}_i \geq \hat{\mu}_\ell \geq \mu_\ell - \epsilon \geq \mu_i - \epsilon.
$$

One can show $\hat{\mu}_i \leq \mu_i + \epsilon$ by symmetry. \qed
The key insight in proving the batch complexity of BatchRacing is that once there are only \( \lfloor b/r \rfloor \) or less surviving arms, there is no choice but to pull each arm \( r \) times at every batch, possibly wasting some arm pulls. Before then, the algorithm fully exploits pulling \( b \) times in a round and enjoys a full factor of \( b \) reduction.

**Proof.** Note a random variable \( X \) such that \( X \in [0,1] \) w.p. 1 is sub-Gaussian with scale \( \sigma = 1/2 \). By Lemma \[3\] \( \mathbb{P}(\bigcap_{i=1}^{n} \mathcal{E}_i(\omega)) \geq 1 - \delta \). Thus, it is safe to assume that \( \bigcap_{i=1}^{n} \mathcal{E}_i(\omega) \); the true mean of each arm is trapped by the LCB and UCB.

**Correctness** We show that if an arm \( i \) is accepted (rejected) then \( i \) is (not) top-\( k \). It suffices to prove for round \( t \) such that \( A_i \) and \( R_i \) are empty for the same reason stated in the proof of Lemma \[2\]. An arm \( i \) being accepted implies that there are \( (n - k) \) arms whose true means are smaller than \( \mu_i \) since the LCB and UCB of each arm trap its true mean. Thus, \( i \) is a top-\( k \) arm. Similarly, if an arm \( j \) is rejected then \( j \) is not top-\( k \).

**Batch Complexity** Let \( T'(t) = \min_{i \in S_t} T_i(t) \). When \( T'(t) \) goes above \( T_i \) for some arm \( i \), then we know by Lemma \[2\] that arm \( i \) is accepted or rejected and is removed from \( S_t \). We consider the worst case where an arm \( i \) is removed only after satisfying \( T'(t) \geq T_i \), and the order of the removal is arm \( \sigma(n), \sigma(n-1), \ldots, \sigma(3), \) and \( \sigma(2) \) or \( \sigma(1) \). Any early removal of an arm and switching the order of the removal necessarily make the rest of the problem easier: i.e., required number of arm pulls are reduced.

By Lemma \[2\] when \( T'(t) \) reaches \( T_i \), arm \( \sigma(i) \) is removed. If \( b = r = 1 \), the batch complexity can be trivially obtained by summing up \( T_i \) for \( i \in [n] \). Thus, we consider \( b \geq 2 \) for the rest of the proof. Let \( t_i \) be the round that \( T'(t_i) \) just passed \( T_\sigma(i) \); i.e., \( T'(t_i - 1) < T_i \) and \( T'(t_i) \geq T_\sigma(i) \). Note that \( T'(t_i) \) can be greater than \( T_\sigma(i) \) due to the “batch effect” where the increment of the number of samples of an arm can be greater than 1. Such discrepancy can be large when \( b \) is much larger than \( |S_t| \). It is easy to see that \( T'(t_i) \leq T_i + \lfloor b/|S_t| \rfloor - 1 \). Then,

\[
T\sigma(i)(t_i) \leq T'(t_i) + 1 \leq T\sigma(i) + \left\lfloor \frac{b}{|S_t|} \right\rfloor = T\sigma(i) + \left\lfloor \frac{b}{n - i + 1} \right\rfloor. \tag{11}
\]

Consider the round \( t' = t_i+1 \) where arm \( \sigma(\lfloor b/r' \rfloor + 1) \) is just removed. The total number of samples used up to \( t' \) is

\[
b't' = \sum_{i=1}^{n} T\sigma(i)(t') \leq \left\lfloor \frac{b}{r'} \right\rfloor (T'(t') + 1) + \sum_{i=\lfloor b/r' \rfloor + 1}^{n} T\sigma(i) + \left\lfloor \frac{b}{n - i + 1} \right\rfloor.
\]

At round \( t' + 1 \), there are only \( \lfloor b/r' \rfloor \) surviving arms. The algorithm then has no choice but to pull every surviving arm \( r' \) times in each round until we remove the hardest two arms \( \sigma(1) \) and \( \sigma(2) \) during which arm \( \sigma(3), \ldots, \sigma(\lfloor b/r' \rfloor) \) are removed as a byproduct. The extra number of batches that must be performed until termination is then \((1/2)(T\sigma(2) - T'(t'))\). Therefore, the total number of batches is

\[
t' + \left\lfloor \frac{1}{r^2}(T\sigma(2) - T'(t')) \right\rfloor \leq \frac{1}{r^2}T\sigma(2) + \frac{1}{b} \sum_{i=\lfloor b/r' \rfloor + 1}^{n} (T\sigma(i) + \left\lfloor \frac{b}{n - i + 1} \right\rfloor) + \frac{1}{r^2} + 1
\]

\[
\leq \frac{1}{r^2}T\sigma(2) + \frac{1}{b} \left( \sum_{i=\lfloor b/r' \rfloor + 1}^{n} T\sigma(i) \right) + 1 + \log n + \frac{n}{b} + \frac{1}{r^2} + 1
\]

\[
= O \left( \frac{1}{r^2}T\sigma(2) + \frac{1}{b} \sum_{i=\lfloor b/r' \rfloor + 1}^{n} T\sigma(i) + \log n \right).
\]

\[\square\]

### C  Proof of Theorem \[2\]

We define an event \( \xi_s \) where every arm’s empirical mean is controlled as follows:

\[
\xi_s = \begin{cases}
\forall i \in [n], |\hat{\mu}_{i,T_i(t_s)} - \mu_i| \leq \frac{\Delta_{s(\lfloor n - i \rfloor + 1)}}{\sqrt{n}} & \text{if } s \leq n - \tilde{n} \\
\forall i \in [n], |\hat{\mu}_{i,T_{n-s+1}(t_n-\tilde{n}+1)} - \mu_i| \leq \frac{\Delta_{s(2)}}{\sqrt{n}} & \text{if } s = n - \tilde{n} + 1
\end{cases}
\]
The proof consists of two parts. We first upper bound the probability that the empirical means are not controlled, which is \( \mathbb{P}(\cup_{s=1}^{n-\tilde{n}+1} \xi_s) \). Next, we show that when the empirical means are controlled, i.e., \( \cap_{s=1}^{n-\tilde{n}+1} \xi_s \), Algorithm 3 terminates with the correct top-\( k \) arms.

**Upperbound the Probability** \( \mathbb{P}(\cup_{s=1}^{n-\tilde{n}+1} \xi_s) \) Define \( Z = \tilde{n}/2 + \sum_{i=\tilde{n}+1}^{n} 1/i \). By Hoeffding’s bound [10], for \( s \leq n - \tilde{n} \),

\[
P(\xi_s) \leq 2 \sum_{i=1}^{n} 2 \exp \left( -2T_i(t_s) \frac{\Delta^2_{\sigma(n-s+1)}}{16} \right) \leq 2n \exp \left( -2m_s \frac{\Delta^2_{\sigma(n-s+1)}}{16} \right)
\]

\[
\leq 2n \exp \left( -\frac{1}{8} Z \frac{bB - (\sum_{i=\tilde{n}+1}^{n} [b/i]) - c_1}{(n - s + 1) \Delta_{\sigma(n-s+1)}^2} \right),
\]

and similarly \( \mathbb{P}(\xi_{n-\tilde{n}+1}) \leq 2n \exp \left( -\frac{1}{8} Z \frac{bB - (\sum_{i=\tilde{n}+1}^{n} [b/i]) - c_1}{\max_{i=2,\tilde{n}+1,\ldots,n} \Delta_{\sigma(i)}^2} \right) \). By union bound, \( \mathbb{P}(\cup_{s=1}^{n-\tilde{n}+1} \xi_s) \leq (n - \tilde{n} + 1)2n \exp \left( -\frac{1}{8} Z \frac{bB - (\sum_{i=\tilde{n}+1}^{n} [b/i]) - c_1}{H_{\frac{b}{4},c/8}} \right).

**Correctness of the first stage** We show that at every stage \( s \) the algorithm makes a correct decision given that \( \xi_s \) is true: the selected arm \( j_s \) is accepted if it is top-\( k \) and not accepted if it is not top-\( k \). We prove it by induction. We first show that the decision at the first stage is correct. Then, assuming that the decision at the round \( s \) is correct, we show that the stage \( s + 1 \) is also correct.

Consider the first stage. Assume that \( \xi_s \) is true. Suppose the decision on the selected arm \( j_1 \) at the first stage is incorrect. There are two cases: (a) \( j_1 \) is not top-\( k \) but is accepted and (b) \( j_1 \) is top-\( k \) but not accepted.

We here prove that (a) leads to a contradiction; one can show the same for (b) symmetrically. Define the empirical gap

\[
\hat{\Delta}_{\rho(i)} = \begin{cases} 
\hat{\mu}_{\rho(i),T_{\rho(i)}(t_s)} - \hat{\mu}_{\rho(k+1),T_{\rho(k+1)}(t_s)} & \text{if } i \leq k \\
\hat{\mu}_{\rho(k),T_{\rho(k)}(t_s)} - \hat{\mu}_{\rho(i),T_{\rho(i)}(t_s)} & \text{if } i > k 
\end{cases}.
\]

Since \( j_1 \) is not top-\( k \), i.e., \( j_1 > k \), and satisfies (a), \( j_1 \) has the largest empirical gap and the largest empirical mean:

\[
\hat{\Delta}_{j_1} \geq \hat{\Delta}_{\ell}, \forall \ell \in S_1.
\]

\[
\hat{\mu}_{j_1} = \hat{\mu}_{\rho(1)}.
\]

Note that \( \Delta_{\sigma(n)} \) is either \( \mu_1 - \mu_{k+1} \) or \( \mu_k - \mu_n \).

**Case (a1):** \( \Delta_{\sigma(n)} = \mu_1 - \mu_{k+1} \). By (13) and Lemma 3,

\[
\begin{align*}
\mu_{j_1} + \Delta_{\sigma(n)}/4 &\geq \hat{\mu}_{j_1} = \hat{\mu}_{\rho(1)} \geq \mu_1 - \Delta_{\sigma(n)/4} \\
\mu_{k+1} \geq \mu_{j_1} &\geq \mu_1 - \Delta_{\sigma(n)}/2 = (\mu_1 + \mu_{k+1})/2 \\
\mu_{k+1} &\geq \mu_1,
\end{align*}
\]

which contradicts \( \mu_1 \geq \mu_k > \mu_{k+1} \).

**Case (a2):** \( \Delta_{\sigma(n)} = \mu_k - \mu_n \)

By (12),

\[
\hat{\mu}_{j_1} - \hat{\mu}_{\rho(k+1)} \geq \hat{\mu}_{\rho(k)}(1) - \hat{\mu}_{\rho(n)} \\
\geq \mu_k - \Delta_{\sigma(n)/4} - (\mu_n + \Delta_{\sigma(n)/4}) = \Delta_{\sigma(n)/2}
\]

Note that arm 1, \ldots, \( k \) have the following lower bound: \( \forall i \leq k, \hat{\mu}_i \geq \mu_i - \Delta_{\sigma(n)}/4 \geq \mu_k - \Delta_{\sigma(n)}/4 \). Thus, there are \( k + 1 \) arms whose empirical mean is at least \( \mu_k - \Delta_{\sigma(n)}/4 \). This gives us a lower bound for the \((k + 1)\)-th largest empirical mean: \( \hat{\mu}_{\rho(k+1)} \geq \mu_k - \Delta_{\sigma(n)}/4 \). Then,

\[
\mu_{j_1} + \Delta_{\sigma(n)/4} - (\mu_k - \Delta_{\sigma(n)/4}) \geq \Delta_{\sigma(n)/2}
\]

\[
\mu_{j_1} \geq \mu_k,
\]
which contradicts $j_1 > k$.

**Correctness of the $s$-th stage** ($s \leq n - \tilde{n}$) First, consider $s = 2$. After the first round, the arm $j_1$ is removed from the surviving set $S_2$. Let $k' = k - |A_2|$, the remaining number of top arms to identify at the second stage. The problem we are facing at the second stage is exactly the first stage with $[n] \setminus \{\sigma(n)\}$ as the surviving set $S_1$ and $k$ replaced with $k'$. If the arm $j_1$ was $\sigma(n)$, the event $\xi_2$ where the empirical mean of each arm is controlled by $\Delta_{\sigma(n-1)}/4$ implies that the decision at the end of the second stage is correct. What happens if the arm $j_1$ was not $\sigma(n)$? Denote by $\sigma'(i)$ the arm with $i$-th smallest gap in $S_2$ and define the gap accordingly: $\Delta'_{\sigma'(1)} \leq \ldots \leq \Delta'_{\sigma'(n-1)}$. Assuming that the decision at the first stage was correct, it is easy to verify that the largest gap $\Delta'_{\sigma'(n-1)}$ in $S_2$ is no less than $\Delta_{\sigma(n-1)}$. Therefore, the event $\xi_2$ implies that the empirical means of arms are controlled by $\Delta'_{\sigma'(n-1)}/4$, which ensures that the decision at the second stage must be correct, too. In other words, removing arm $\sigma(n)$ at the first stage is the worst case; removing other arms can only make the gaps larger and make the problem “easier”.

**Correctness of the final stage** It is easy to show that if arms are controlled by $\Delta_{\sigma(2)}/4$ then the empirical means of the top-$k'$ will be greater than any non-top-$k'$ in the surviving arms $S_{n-\tilde{n}+1}$. Thus, the output $k'$ arms at the last stage is correct.