First Order Logic

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[Based on slides from Burr Settles]
Problems with propositional logic

• Consider the game “minesweeper” on a 10x10 field with only one landmine.

• How do you express the knowledge, with propositional logic, that the squares adjacent to the landmine will display the number 1?
Problems with propositional logic

• Consider the game “minesweeper” on a 10x10 field with only one landmine.

• How do you express the knowledge, with propositional logic, that the squares adjacent to the landmine will display the number 1?

• Intuitively with a rule like
  \[ \text{landmine}(x,y) \Rightarrow \text{number1}(\text{neighbors}(x,y)) \]

  but propositional logic cannot do this…
Problems with propositional logic

• Propositional logic has to say, e.g. for cell (3,4):
  - Landmine_3_4 ⇒ number1_2_3
  - Landmine_3_4 ⇒ number1_2_4
  - Landmine_3_4 ⇒ number1_2_5
  - Landmine_3_4 ⇒ number1_3_3
  - Landmine_3_4 ⇒ number1_3_5
  - Landmine_3_4 ⇒ number1_4_3
  - Landmine_3_4 ⇒ number1_4_4
  - Landmine_3_4 ⇒ number1_4_5

  And similarly for each of Landmine_1_1, Landmine_1_2, Landmine_1_3, …, Landmine_10_10!

• Difficult to express large domains concisely
• Don’t have objects and relations
• First Order Logic is a powerful upgrade
Ontological commitment

- Logics are characterized by what they consider to be ‘primitives’

<table>
<thead>
<tr>
<th>Logic</th>
<th>Primitives</th>
<th>Available Knowledge</th>
</tr>
</thead>
<tbody>
<tr>
<td>Propositional</td>
<td>facts</td>
<td>true/false/unknown</td>
</tr>
<tr>
<td>First-Order</td>
<td>facts, objects, relations</td>
<td>true/false/unknown</td>
</tr>
<tr>
<td>Temporal</td>
<td>facts, objects, relations, times</td>
<td>true/false/unknown</td>
</tr>
<tr>
<td>Probability Theory</td>
<td>facts</td>
<td>degree of belief 0…1</td>
</tr>
<tr>
<td>Fuzzy</td>
<td>degree of truth</td>
<td>degree of belief 0…1</td>
</tr>
</tbody>
</table>
First Order Logic syntax

- **Term**: an object in the world
  - **Constant**: Jerry, 2, Madison, Green, …
  - **Variables**: x, y, a, b, c, …
  - **Function**(term$_1$, …, term$_n$)
    - Sqrt(9), Distance(Madison, Chicago)
    - Maps one or more objects to another object
    - Can refer to an unnamed object: LeftLeg(John)
    - Represents a user defined functional relation

- A **ground term** is a term without variables.
FOL syntax

- **Atom**: smallest T/F expression
  - **Predicate**(term$_1$, …, term$_n$)
    - Teacher(Jerry, you), Bigger(sqrt(2), x)
    - Convention: read “Jerry (is)Teacher(of) you”
    - Maps one or more objects to a truth value
    - Represents a user defined relation
  
- **term$_1$ = term$_2$**
  - Radius(Earth)=6400km, 1=2
  - Represents the equality relation when two terms refer to the same object
FOL syntax

• **Sentence**: T/F expression
  - Atom
  - Complex sentence using connectives: $\land \lor \neg \Rightarrow \Leftarrow$
    - Spouse(Jerry, Jing) $\Rightarrow$ Spouse(Jing, Jerry)
    - Less(11,22) $\land$ Less(22,33)
  - Complex sentence using quantifiers $\forall, \exists$
• Sentences are evaluated under an interpretation
  - Which objects are referred to by constant symbols
  - Which objects are referred to by function symbols
  - What subsets defines the predicates
FOL quantifiers

• Universal quantifier: ∀

• Sentence is true **for all** values of x in the domain of variable x.

• Main connective typically is ⇒
  ▪ Forms if-then rules
  ▪ “all humans are mammals”
    \[ \forall x \text{ human}(x) \Rightarrow \text{mammal}(x) \]
  ▪ Means if x is a human, then x is a mammal
FOL quantifiers

\( \forall x \ \text{human}(x) \implies \text{mammal}(x) \)

- It’s a big AND: Equivalent to the conjunction of all the instantiations of variable \( x \):
  
  \[
  (\text{human}(Jerry) \implies \text{mammal}(Jerry)) \land \\
  (\text{human}(Jing) \implies \text{mammal}(Jing)) \land \\
  (\text{human}(laptop) \implies \text{mammal}(laptop)) \land \ldots
  \]

- Common mistake is to use \( \land \) as main connective

  \( \forall x \ \text{human}(x) \land \text{mammal}(x) \)

- This means everything is human and a mammal!

  \[
  (\text{human}(Jerry) \land \text{mammal}(Jerry)) \land \\
  (\text{human}(Jing) \land \text{mammal}(Jing)) \land \\
  (\text{human}(laptop) \land \text{mammal}(laptop)) \land \ldots
  \]
FOL quantifiers

- Existential quantifier: $\exists$
- Sentence is true for some value of $x$ in the domain of variable $x$.

- Main connective typically is $\land$
  - “some humans are male”
    $\exists x \ human(x) \land male(x)$
  - Means there is an $x$ who is a human and is a male
FOL quantifiers

\( \exists x \human(x) \land \male(x) \)

- It’s a big OR: Equivalent to the disjunction of all the instantiations of variable \( x \):
  
  \[(\human(Jerry) \land \male(Jerry)) \lor \]
  
  \[(\human(Jing) \land \male(Jing)) \lor \]
  
  \[(\human(laptop) \land \male(laptop)) \lor \ldots \]

- Common mistake is to use \( \Rightarrow \) as main connective
  - “Some pig can fly”
    
    \( \exists x \pig(x) \Rightarrow \fly(x) \) (wrong)
FOL quantifiers

\[ \exists x \text{ human}(x) \land \text{male}(x) \]

- It’s a big OR: Equivalent to the disjunction of all the instantiations of variable x:

\[ \text{human}(\text{Jerry}) \land \text{male}(\text{Jerry}) \lor \text{human}(\text{Jing}) \land \text{male}(\text{Jing}) \lor \text{human}(\text{laptop}) \land \text{male}(\text{laptop}) \lor \ldots \]

- Common mistake is to use \[ \Rightarrow \] as main connective
  - “Some pig can fly”
    \[ \exists x \, \text{pig}(x) \Rightarrow \text{fly}(x) \] (wrong)
- This is true if there is something not a pig!
  \[ (\text{pig}(\text{Jerry}) \Rightarrow \text{fly}(\text{Jerry})) \lor (\text{pig}(\text{laptop}) \Rightarrow \text{fly}(\text{laptop})) \lor \ldots \]
FOL quantifiers

• Properties of quantifiers:
  - $\forall x \forall y$ is the same as $\forall y \forall x$
  - $\exists x \exists y$ is the same as $\exists y \exists x$

• Example:
  - $\forall x \forall y \text{likes}(x,y)$
    Everyone likes everyone.
  - $\forall y \forall x \text{likes}(x,y)$
    Everyone is liked by everyone.
FOL quantifiers

- Properties of quantifiers:
  - $\forall x \exists y$ is not the same as $\exists y \forall x$
  - $\exists x \forall y$ is not the same as $\forall y \exists x$

- Example:
  - $\forall x \exists y \text{likes}(x,y)$
    Everyone likes someone (can be different).
  - $\exists y \forall x \text{likes}(x,y)$
    There is someone who is liked by everyone.
FOL quantifiers

• Properties of quantifiers:
  - $\forall x \; P(x)$ when negated becomes $\exists x \; \neg P(x)$
  - $\exists x \; P(x)$ when negated becomes $\forall x \; \neg P(x)$

• Example:
  - $\forall x \; \text{sleep}(x)$
    Everybody sleeps.
  - $\exists x \; \neg \text{sleep}(x)$
    Somebody does not sleep.
FOL quantifiers

- Properties of quantifiers:
  - $\forall x \ P(x)$ is the same as $\neg \exists x \ \neg P(x)$
  - $\exists x \ P(x)$ is the same as $\neg \forall x \ \neg P(x)$

- Example:
  - $\forall x \ sleep(x)$
    Everybody sleeps.
  - $\neg \exists x \ \neg sleep(x)$
    There does not exist someone who does not sleep.
FOL syntax

• A free variable is a variable that is not bound by an quantifier, e.g. $\exists y \text{ Likes}(x,y)$: $x$ is free, $y$ is bound

• A well-formed formula (wff) is a sentence in which all variables are quantified (no free variable)

• Short summary so far:
  - Constants: Bob, 2, Madison, …
  - Variables: $x$, $y$, $a$, $b$, $c$, …
  - Functions: Income, Address, Sqrt, …
  - Predicates: Teacher, Sisters, Even, Prime…
  - Connectives: $\land \lor \neg \Rightarrow \Leftrightarrow$
  - Equality: $=$
  - Quantifiers: $\forall \exists$
More summary

- **Term**: constant, variable, function. Denotes an object. (A ground term has no variables)

- **Atom**: the smallest expression assigned a truth value. Predicate and =

- **Sentence**: an atom, sentence with connectives, sentence with quantifiers. Assigned a truth value

- **Well-formed formula** (wff): a sentence in which all variables are quantified
Thinking in logical sentences

Convert the following sentences into FOL:

• “Elmo is a monster.”
  - What is the constant? Elmo
  - What is the predicate? Is a monster
  - Answer: monster(Elmo)

• “Tinky Winky and Dipsy are teletubbies”

• “Tom, Jerry or Mickey is not a mouse.”
Thinking in logical sentences

We can also do this with relations:

• “America bought Alaska from Russia.”
  ▪ What are the constants?
    • America, Alaska, Russia
  ▪ What are the relations?
    • Bought
  ▪ Answer: bought(America, Alaska, Russia)

• “Warm is between cold and hot.”

• “Jerry and Jing are married.”
Thinking in logical sentences

Now let’s think about quantifiers:

- “Jerry likes everything.”
  - What’s the constant?
    - Jerry
  - Thing?
    - Just use a variable x
  - Everything?
    - Universal quantifier
  - Answer: \( \forall x \text{ likes}(\text{Jerry, } x) \)
    - i.e. \( \text{likes}(\text{Jerry, IceCream}) \land \text{likes}(\text{Jerry, Jing}) \land \text{likes}(\text{Jerry, Armadillos}) \land \ldots \)

- “Jerry likes something.”
- “Somebody likes Jerry.”
Thinking in logical sentences

We can also have multiple quantifiers:

• “somebody heard something.”
  ▪ What are the variables?
    • Somebody, something
  ▪ How are they quantified?
    • Both are existential
  ▪ Answer: $\exists x, y \text{ heard}(x, y)$

• “Everybody heard everything.”

• “Somebody did not hear everything.”
Thinking in logical sentences

Let’s allow more complex quantified relations:

- “All stinky shoes are allowed.”
  - How are ideas connected?
    - Being a shoe and being stinky implies it’s allowed
  - Answer: \( \forall x \text{ shoe}(x) \land \text{stinky}(x) \Rightarrow \text{allowed}(x) \)

- “No stinky shoes are allowed.”
  - Answers:
    - \( \forall x \text{ shoe}(x) \land \text{stinky}(x) \Rightarrow \neg \text{allowed}(x) \)
    - \( \neg \exists x \text{ shoe}(x) \land \text{stinky}(x) \land \text{allowed}(x) \)
    - \( \neg \exists x \text{ shoe}(x) \land \text{stinky}(x) \Rightarrow \text{allowed}(x) \) (?)
Thinking in logical sentences

• “No stinky shoes are allowed.”
  ▪ \(\neg \exists x \ shoe(x) \land stinky(x) \Rightarrow allowed(x)\) (?)
  ▪ \(\neg \exists x \neg (shoe(x) \land stinky(x)) \lor allowed(x)\)
  ▪ \(\forall x \neg (\neg (shoe(x) \land stinky(x)) \lor allowed(x))\)
  ▪ \(\forall x (shoe(x) \land stinky(x)) \land \neg allowed(x)\)

• But this says “Jerry is a stinky shoe and Jerry is not allowed.”

• How about
  \(\forall x allowed(x) \Rightarrow \neg (shoe(x) \land stinky(x))\)
Thinking in logical sentences

And some more complex relations:

• “No one sees everything.”
  • Answer: \( \neg \exists x \ \forall y \ \text{sees}(x, y) \)

• Equivalently: “Everyone doesn’t see something.”
  • Answer: \( \forall x \ \exists y \ \neg \text{sees}(x, y) \)

• “Everyone sees nothing.”
  • Answer: \( \forall x \ \neg \exists y \ \text{sees}(x, y) \)
Thinking in logical sentences

And some *really* complex relations:

- “Any good amateur can beat some professional.”
  - Ingredients: x, amateur(x), good(x), y, professional(y), beat(x,y)
  - Answer:
    \[
    \forall x \ [\{\text{amateur}(x) \land \text{good}(x)\} \implies \exists y \ \{\text{professional}(y) \land \text{beat}(x,y)\}] 
    \]

- “Some professionals can beat all amateurs.”
  - Answer:
    \[
    \exists x \ [\text{professional}(x) \land \forall y \ \{\text{amateur}(y) \implies \text{beat}(x,y)\}] 
    \]
Thinking in logical sentences

We can throw in functions and equalities, too:

- “Jerry and Jing are the same age.”
  - Are functional relations specified?
  - Are equalities specified?
  - Answer: \( \text{age(Jerry)} = \text{age(Jing)} \)

- “There are exactly two shoes.”
  - ?
Thinking in logical sentences

• “There are exactly two shoes.”
  ▪ First try:
  \[ \exists x \exists y \text{ shoe}(x) \land \text{ shoe}(y) \]
Thinking in logical sentences

• “There are exactly two shoes.”
  - First try:
    $$\exists x \ \exists y \ \text{shoe}(x) \land \text{shoe}(y)$$
  - Second try:
    $$\exists x \ \exists y \ \text{shoe}(x) \land \text{shoe}(y) \land \neg(x=y)$$
Thinking in logical sentences

• “There are exactly two shoes.”
  - First try:
    \[ \exists x \, \exists y \, \text{shoe}(x) \land \text{shoe}(y) \]
  - Second try:
    \[ \exists x \, \exists y \, \text{shoe}(x) \land \text{shoe}(y) \land \neg(x=y) \]
  - Third try:
    \[ \exists x \, \exists y \, \text{shoe}(x) \land \text{shoe}(y) \land \neg(x=y) \land \neg(x=y) \land \forall z \, (\text{shoe}(z) \implies (x=z) \lor (y=z)) \]
Thinking in logical sentences

- Interesting words: always, sometimes, never
  - “Good people *always* have friends.”
Thinking in logical sentences

• Interesting words: always, sometimes, never
  ▪ “Good people *always* have friends.”
    \[ \forall x \, \text{person}(x) \land \text{good}(x) \Rightarrow \exists y (\text{friend}(x,y)) \]
  ▪ “Busy people *sometimes* have friends.”
Thinking in logical sentences

• Interesting words: always, sometimes, never
  ▪ “Good people always have friends.”
    \( \forall x \text{ person}(x) \land \text{good}(x) \Rightarrow \exists y(\text{friend}(x,y)) \)
  ▪ “Busy people sometimes have friends.”
    \( \exists x \text{ person}(x) \land \text{busy}(x) \land \exists y(\text{friend}(x,y)) \)
  ▪ “Bad people never have friends.”
Thinking in logical sentences

- Interesting words: always, sometimes, never
  - “Good people *always* have friends.”
    \[
    \forall x \text{ person}(x) \land \text{good}(x) \Rightarrow \exists y (\text{friend}(x,y))
    \]
  - “Busy people *sometimes* have friends.”
    \[
    \exists x \text{ person}(x) \land \text{busy}(x) \land \exists y (\text{friend}(x,y))
    \]
  - “Bad people *never* have friends.”
    \[
    \forall x \text{ person}(x) \land \text{bad}(x) \Rightarrow \neg \exists y (\text{friend}(x,y))
    \]
Thinking in logical sentences

Tricky sentences

• “x is above y if and only if x is directly on the top of y, or else there is a pile of one or more other objects directly on top of one another, starting with x and ending with y.”
Thinking in logical sentences

Tricky sentences

• “x is above y if and only if x is directly on the top of y, or else there is a pile of one or more other objects directly on top of one another, starting with x and ending with y.”

\[ \forall x \ \forall y \ \text{above}(x,y) \iff \]
\[ \left[ \text{onTop}(x,y) \lor \exists z \{ \text{onTop}(x,z) \land \text{above}(z,y) \} \right] \]
Professor Snape’s Puzzle

Danger lies before you, while safety lies behind,
Two of us will help you, whichever you would find,
One among us seven will let you move ahead,
Another will transport the drinker back instead,
Two among our number hold only nettle-wine,
Three of us are killers, waiting hidden in line
Choose, unless you wish to stay here forevermore
To help you in your choice, we give you these clues four:
First, however slyly the poison tries to hide
You will always find some on nettle wine's left side
Second, different are those who stand at either end
But if you would move onward, neither is your friend;
Third as you see clearly, all are different size
Neither dwarf nor giant hold death in their insides;
Fourth, the second left and the second on the right
Are twins once you taste them, though different at first sight.
1. $\exists x \ A(x) \land (\forall y \ A(y) \Rightarrow x=y)$
2. $\exists x \ B(x) \land (\forall y \ B(y) \Rightarrow x=y)$
3. $\exists x \exists y \ W(x) \land W(y) \land \neg(x=y) \land (\forall z \ W(z) \Rightarrow z=x \lor z=y)$
4. $\forall x \ \neg(A(x) \lor B(x) \lor W(x)) \Rightarrow P(x)$
5. $\forall x \forall y \ W(x) \land L(y,x) \Rightarrow P(y)$
6. $\neg(P(b1) \land P(b7))$
7. $\neg(W(b1) \land W(b7))$
8. $\neg A(b1)$
9. $\neg A(b7)$
10. $\neg P(b3)$
11. $\neg P(b6)$
12. $(P(b2) \land P(b6)) \lor (W(b2) \land W(b6))$
Next: Inference for FOL

• Recall that in propositional logic, inference is easy
  ▪ Enumerate all possibilities (truth tables)
  ▪ Apply sound inference rules on facts

• But in FOL, we have the concepts of variables, relations, and quantification
  ▪ This complicates things quite a bit!

• We will discuss inference in FOL next time.