Game Theory

Fingers, prisoners, goats

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[based on slides from Andrew Moore http://www.cs.cmu.edu/~awm/tutorials]

Overview

- Matrix normal form
- Chance games
- Games with hidden information
- Non-zero sum games

Pure strategy

- A pure strategy for a player is the mapping between all possible states the player can see, to the move the player would make.
- Player A has 4 pure strategies:

  A's strategy I: (1→L, 4→L)

  A's strategy II: (1→L, 4→R)

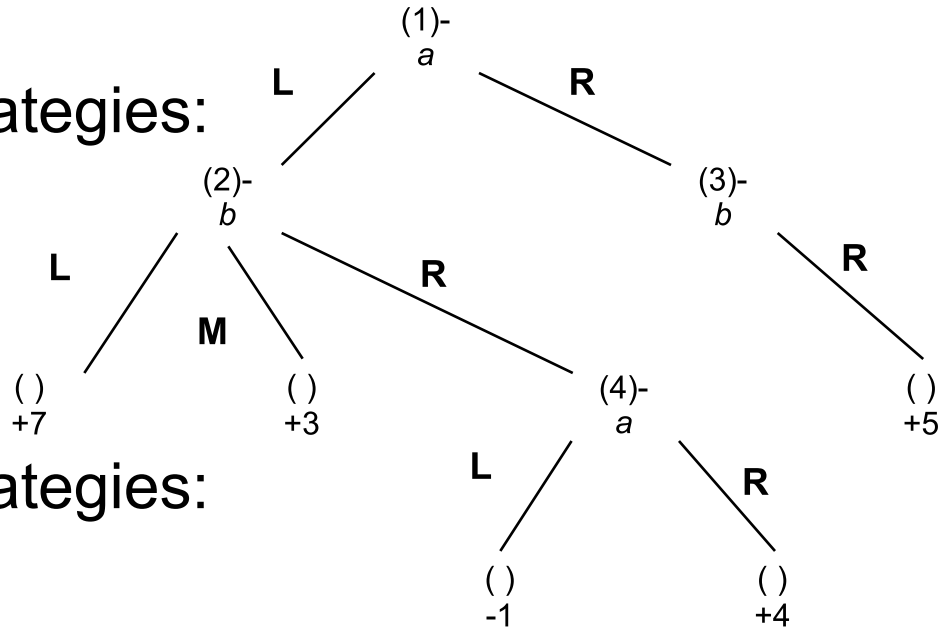
  A's strategy III: (1→R, 4→L)

  A's strategy IV: (1→R, 4→R)
- Player B has 3 pure strategies:

  B's strategy I: (2→L, 3→R)

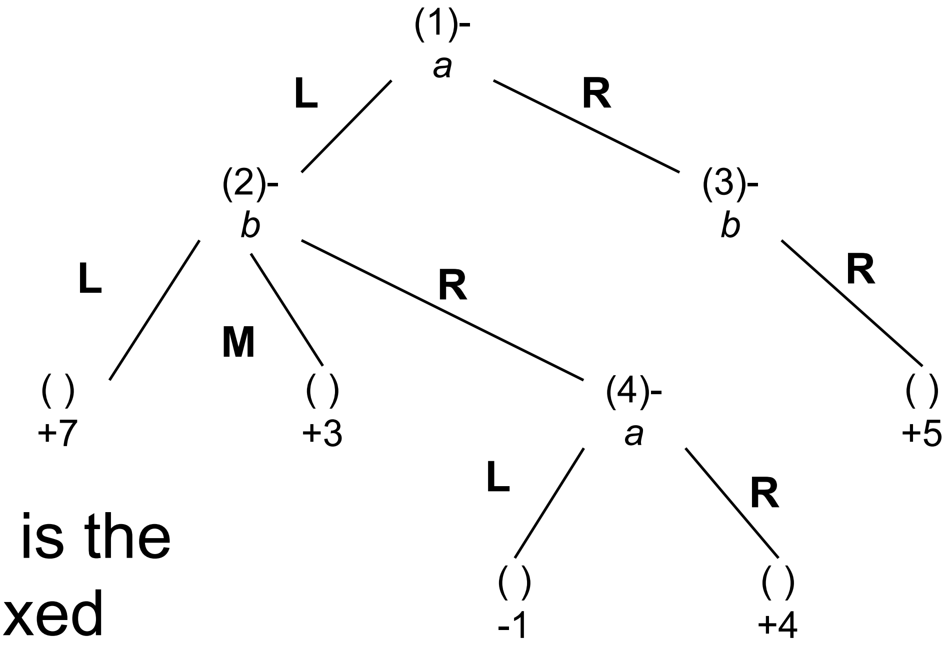
  B's strategy II: (2→M, 3→R)

  B's strategy III: (2→R, 3→R)
- How many pure strategies if each player can see *N* states, and has *b* moves at each state?



Matrix Normal Form of games

A's strategy I: (1→L, 4→L)
A's strategy II: (1→L, 4→R)
A's strategy III: (1→R, 4→L)
A's strategy IV: (1→R, 4→R)
B's strategy I: (2→L, 3→R)
B's strategy II: (2→M, 3→R)
B's strategy III: (2→R, 3→R)

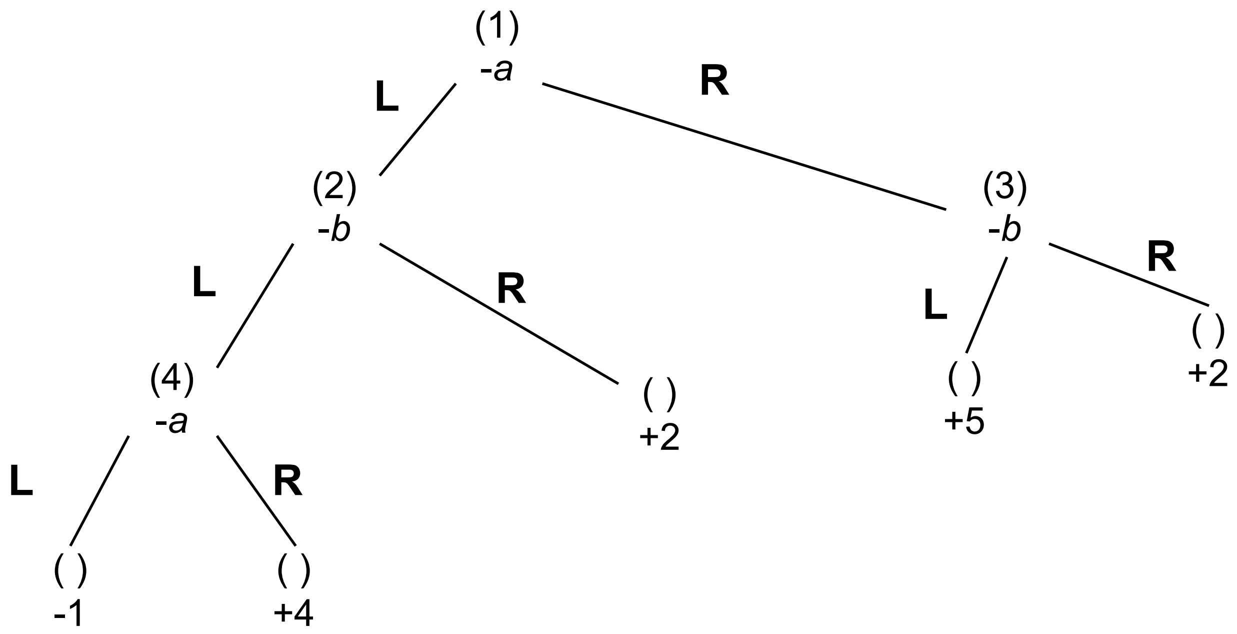


- The matrix normal form is the game value matrix indexed by each player's strategies.

	B-I	B-II	B-III
A-I	7	3	-1
A-II	7	3	4
A-III	5	5	5
A-IV	5	5	5

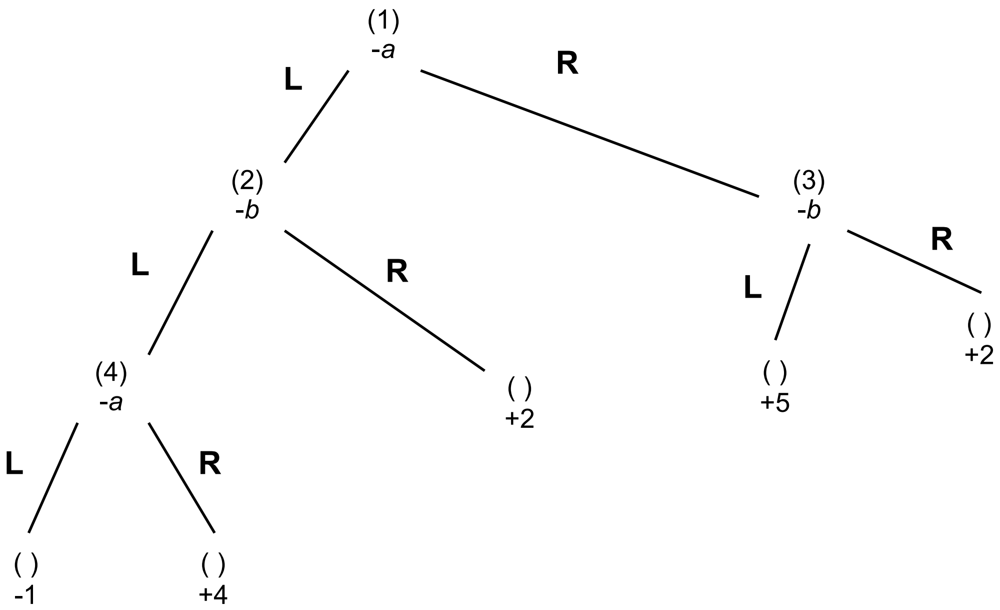
The matrix encodes every outcome of the game! The rules etc. are no longer needed.

Matrix normal form example



- How many pure strategies does A have?
- How many does B have?
- What is the matrix form of this game?

Matrix normal form example



	B-I	B-II	B-III	B-IV
A-I	-1	-1	2	2
A-II	4	4	2	2
A-III	5	2	5	2
A-IV	5	2	5	2

- How many pure strategies does A have? 4

A-I (1→L, 4→L) A-II (1→L,4→R) A-III (1→R,4→L) A-IV (1→R, 4→R)

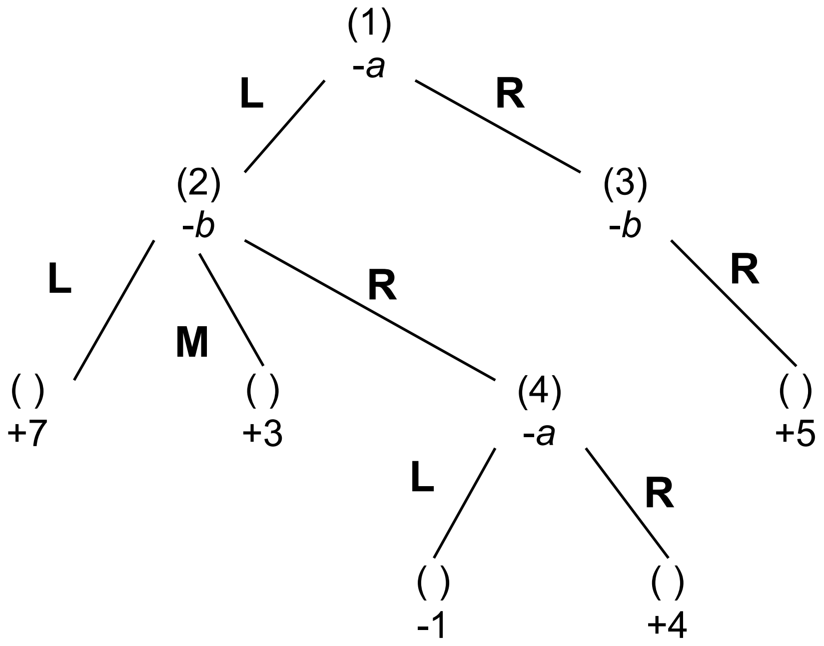
- How many does B have? 4

B-I (2→L, 3→L) B-II (2→L,3→R) B-III (2→R,3→L) B-IV (2→R, 3→R)

- What is the matrix form of this game?

Minimax in Matrix Normal Form

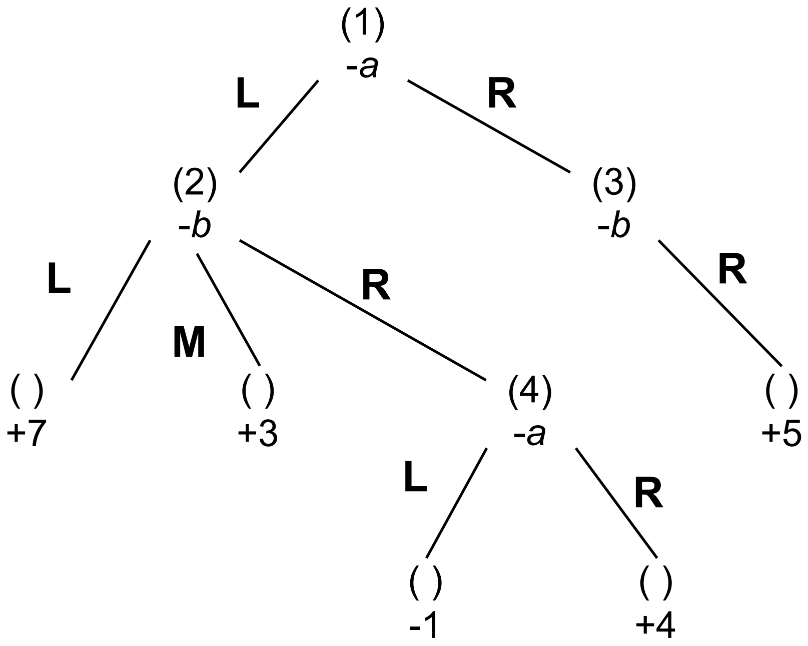
- Player A: for each strategy, consider all B’s counter strategies (a row in the matrix), find the minimum value in that row. Pick the row with the maximum minimum value.
- Here maximin=5



	B-I	B-II	B-III
A-I	7	3	-1
A-II	7	3	4
A-III	5	5	5
A-IV	5	5	5

Minimax in Matrix Normal Form

- Player B: find the maximum value in each column. Pick the column with the minimum maximum value.
- Here minimax = 5



Fundamental game theory result (proved by von Neumann):

> *In a 2-player, zero-sum game of perfect information, Minimax==Maximin. And there always exists an optimal pure strategy for each player.*

	B-I	B-II	B-III
A-I	7	3	-1
A-II	7	3	4
A-III	5	5	5
A-IV	5	5	5

Minimax in Matrix Normal Form

Interestingly, A can tell B in advance what strategy A will use (the maximin), and this information will not help B!

Similarly B can tell A what strategy B will use.

In fact A knows what B’s strategy will be.

And B knows A’s too.

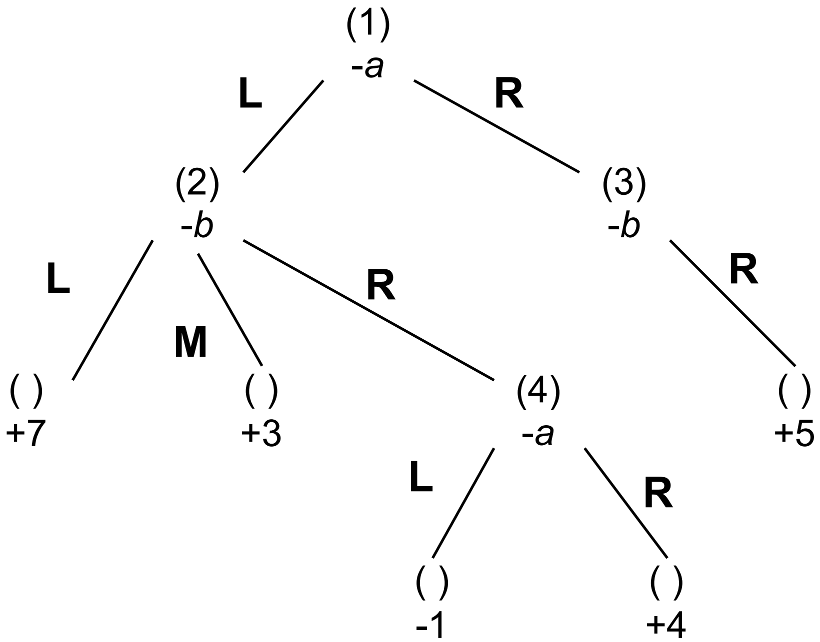
And A knows that B knows

…

The game is at an equilibrium

al

p tegy for each player.



	B-I	B-II	B-III
A-I	7	3	-1
A-II	7	3	4
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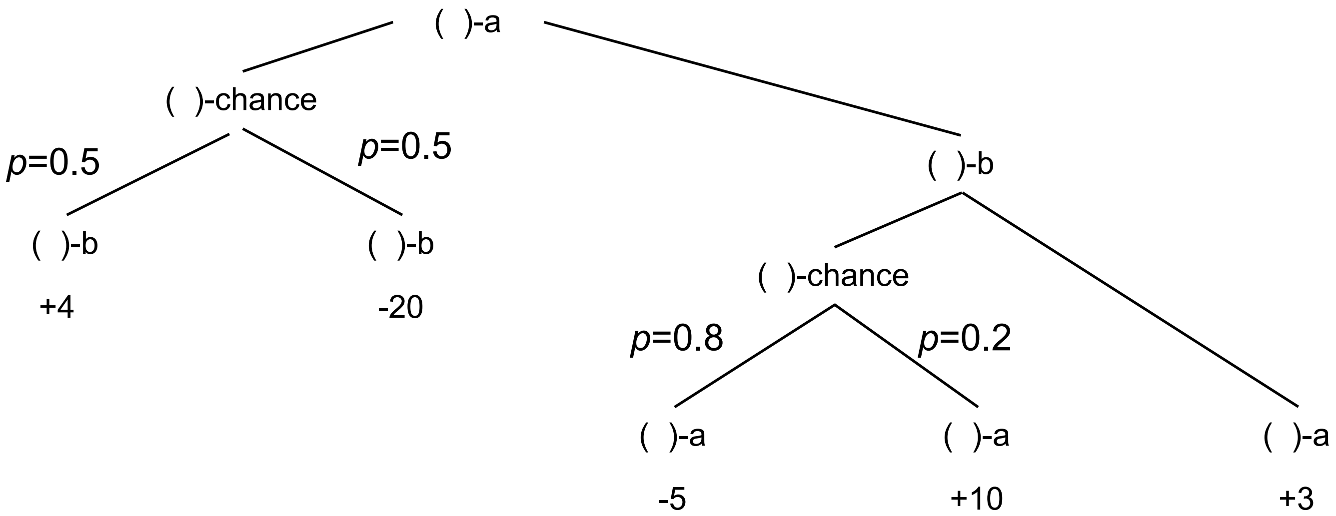
A-IV	5	5	5

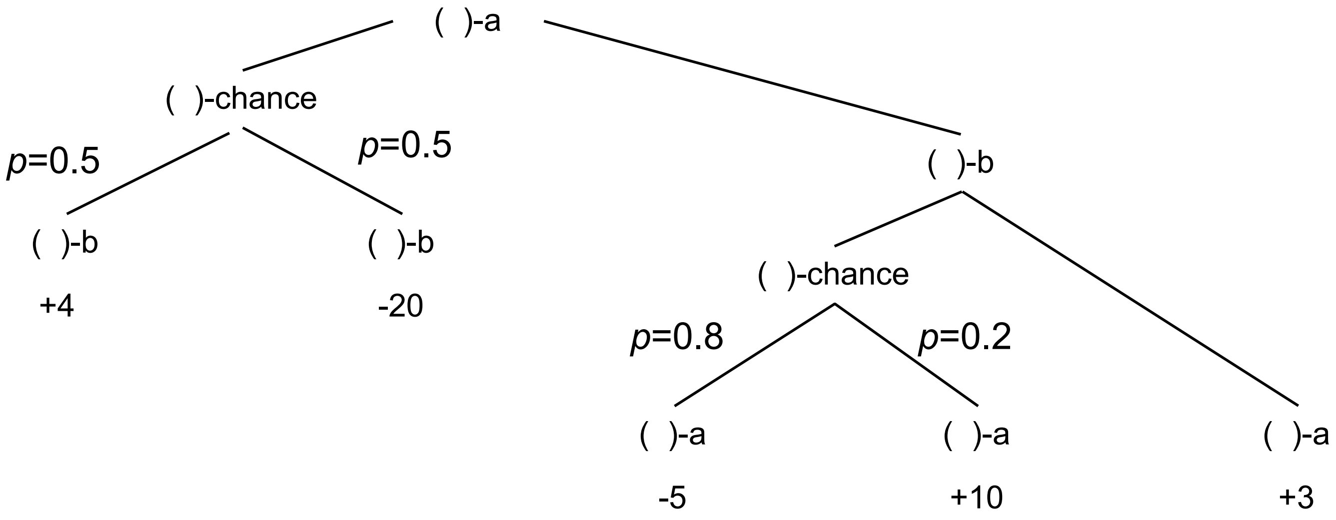
Matrix Normal Form for NONdeterministic games

- Recall the chance nodes (coin flip, die roll etc.): neither player moves, but a random move is made according to the known probability



- The game theoretic value is the expected value if both players are optimal
- What's the matrix form of this game?

Matrix Normal Form for NONdeterministic games



- A-I: L, A-II: R, B-I: L, B-II: R
- The i,jth entry is the expected value with strategies A-i,B-j
- von Neumann's result still holds
- Minimax == Maximin

	B-I	B-II
A-I	-8	-8
A-II	-2	3

Non-zero sum games

Non-zero sum games

- One player's gain is not the other's loss
- Matrix normal form: simply lists all players' gain

	B-I	B-II
A-I	-5, -5	-10, 0
A-II	0, -10	-1, -1

Convention: A's gain first, B's next

Note B now wants to maximize the blue numbers.

- Previous zero-sum games trivially represented as

	O-I	O-II
E-I	2, -2	-3, 3
E-II	-3, 3	4, -4

Prisoner’s dilemma

	B-testify	B-refuse
A-testify	-5, -5	0, -10
A-refuse	-10, 0	-1, -1

Strict domination

- A's strategy i dominates A's strategy j, if for every B's strategy, A is better off doing i than j.

	B-testify	B-refuse
A-testify	-5, -5	0, -10
A-refuse	-10, 0	-1, -1

If B-testify: A-testify (-5) is better than A-refuse (-10)

If B-refuse: A-testify (0) is better than A-refuse (-1)

A: Testify is always better than refuse.

A-testify strictly dominates (all outcomes strictly better than) A-refuse.

Strict domination

- Fundamental assumption of game theory: **get rid of strictly dominated strategies – they won't happen.**
- In some cases like prisoner's dilemma, we can use strict domination to predict the outcome, if both players are rational.

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→

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A-testify	-5, -5	0, -10

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→

	B-testify	~~B-refuse~~
A-testify	-5, -5	~~0, -10~~

↓

	B-testify
A-testify	-5, -5

Another strict domination example

- Iterated elimination of strictly dominated strategies

		Player B			
		I	II	III	IV
Player A	I	3 , 1	4 , 1	5 , 9	2 , 6
	II	5 , 3	5 , 8	9 , 7	9 , 3
	III	2 , 3	8 , 4	6 , 2	6 , 3
	IV	3 , 8	3 , 1	2 , 3	4 , 5

Strict domination?

- Strict domination doesn’t always happen…

	I	II	III
I	0 , 4	4 , 0	5 , 3
II	4 , 0	0 , 4	5 , 3
III	3 , 5	3 , 5	6 , 6

- What do you think the players will do?

Nash equilibria

- (player 1's strategy s_1^*, player 2's strategy s_2^*, … player n's strategy s_n^*) is a Nash equilibrium, iff

$$s_i^* = \arg\max_s v(s_1^*, \cdots, s_{i-1}^*, s, s_{i+1}^*, \cdots s_n^*)$$

- This says: if everybody else plays at the Nash equilibrium, player i will hurt itself unless it also plays at the Nash equilibrium.

N.E. is a local maximum in unilateral moves.

	I	II	III
I	0 , 4	4 , 0	5 , 3
II	4 , 0	0 , 4	5 , 3
III	3 , 5	3 , 5	6 , 6

Nash equilibria examples

	B-testify	B-refuse
A-testify	-5, -5	0, -10
A-refuse	-10, 0	-1, -1

1. Is there always a Nash equilibrium?
2. Can there be more than one Nash equilibrium?

Player B

Player A

	I	II	III	IV
I	3 , 1	4 , 1	5 , 9	2 , 6
II	5 , 3	5 , 8	9 , 7	9 , 3
III	2 , 3	8 , 4	6 , 2	6 , 3
IV	3 , 8	3 , 1	2 , 3	4 , 5

Example: no N.E. with pure strategies

- two-finger Morra

	O-I	O-II
E-I	2, -2	-3, 3
E-II	-3, 3	4, -4

- No pure strategy Nash equilibrium, but...

Two-player zero-sum deterministic game with hidden information

- Hidden information: something you don’t know but your opponent knows, e.g. hidden cards, or simultaneous moves
- Example: two-finger Morra
    - Each player (O and E) displays 1 or 2 fingers
    - If sum f is odd, O collects \$f from E
    - If sum f is even, E collects \$f from O
    - Strategies?
    - Matrix form?

Two-player zero-sum deterministic game with hidden information

- Hidden information: something you don't know but your opponent knows, e.g. hidden cards, or simultaneous moves
- Example: two-finger Morra
  - Each player (O and E) displays 1 or 2 fingers
  - If sum f is odd, O collects $f from E
  - If sum f is even, E collects $f from O
  - Strategies?
  - Matrix form?
  - Maximin= –3, minimax=2
  - The two are not the same!
  - What should O and E do?

	O-I	O-II
E-I	2,-2	-3,3
E-II	-3,3	4,-4

Game theoretic value when there is hidden information

- It turns out O can win a little over 8 cents on average in each game, if O does the right thing.
- Again O can tell E what O will do, and E can do nothing about it!
- The trick is to use a mixed strategy instead of a pure strategy.
  - A mixed strategy is defined by a probability distribution (p_1, p_2, … p_n). n = # of pure strategies the player has
  - At the start of each game, the player picks number i according to p_i, and uses the ith pure strategy for this round of the game
- von Neumann: every two-player zero-sum game (even with hidden information) has an optimal (mixed) strategy.

Boring math: Two-finger Morra

	O-I	O-II
E-I	2	-3
E-II	-3	4

- E’s mixed strategy: (p:I, (1-p):II)
- O’s mixed strategy: (q:I, (1-q):II)
- What is p, q?
- step 1: let’s fix p for E, and O knows that.
  - What if O always play O-I (q=1)? v_1=p*2+(1-p)*(-3)
  - What if O always play O-II (q=0)? v_0=p*(-3)+(1-p)*4
  - And if O uses some other q? q*v_1+(1-q)*v_0
  - O is going to pick q to minimize q*v_1+(1-q)*v_0
  - Since this is a linear combination, such q must be 0 or 1, not something in between!
  - The value for E is min(p*2+(1-p)*(-3), p*(-3)+(1-p)*4)
- step 2: E choose the p that maximizes the value above.

More boring math

- step 1: let's fix p for E.
  - The value for E is min($p*2+(1-p)*(-3)$, $p*(-3)+(1-p)*4$), in case O is really nasty
- step 2: E choose the p* that maximizes the value above.

$$p^* = \text{argmax}_p \min(p*2+(1-p)*(-3),\ p*(-3)+(1-p)*4)$$

- Solve it with (proof by "it's obvious")

$$p*2+(1-p)*(-3) = p*(-3)+(1-p)*4$$

- E's optimal p* = 7/12, value = -1/12 (expect to lose $! That's the best E can do!)
- Similar analysis on O shows q* = 7/12, value = 1/12

This is a zero-sum, but unfair game.

Recipe for computing A's optimal mixed strategy for a n*m game

- n*m game = A has n pure strategies and B has m. v_{ij}=(i,j)th entry in the matrix form.
- Say A uses mixed strategy (p_1, p_2, … p_n).

A's expected gain if B uses pure strategy 1: $g_1 = p_1v_{11}+p_2v_{21}+\ldots+p_nv_{n1}$

A's expected gain if B uses pure strategy 2: $g_2 = p_1v_{12}+p_2v_{22}+\ldots+p_nv_{n2}$

…

A's expected gain if B uses pure strategy m: $g_m = p_1v_{1m}+p_2v_{2m}+\ldots+p_nv_{nm}$

- Choose (p_1, p_2, … p_n) to maximize

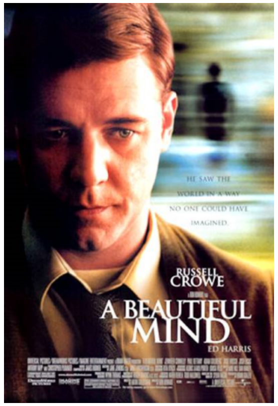
$$\min(g_1, g_2, \ldots, g_m)$$

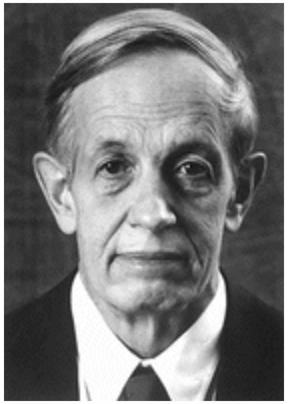
$$\text{Subject to: } p_1+p_2+ \ldots +p_n=1$$

$$0 \leq p_i \leq 1 \text{ for all } i$$

Fundamental theorems

- In a n-player pure strategy game, if iterated elimination of strictly dominated strategies leaves all but one cell (s_1^*, s_2^*, … s_n^*), then it is the unique NE of the game
- Any NE will survive iterated elimination of strictly dominated strategies
- [Nash 1950]: If n is finite, and each player has finite strategies, then there exists at least one NE (possibly involving mixed strategies)





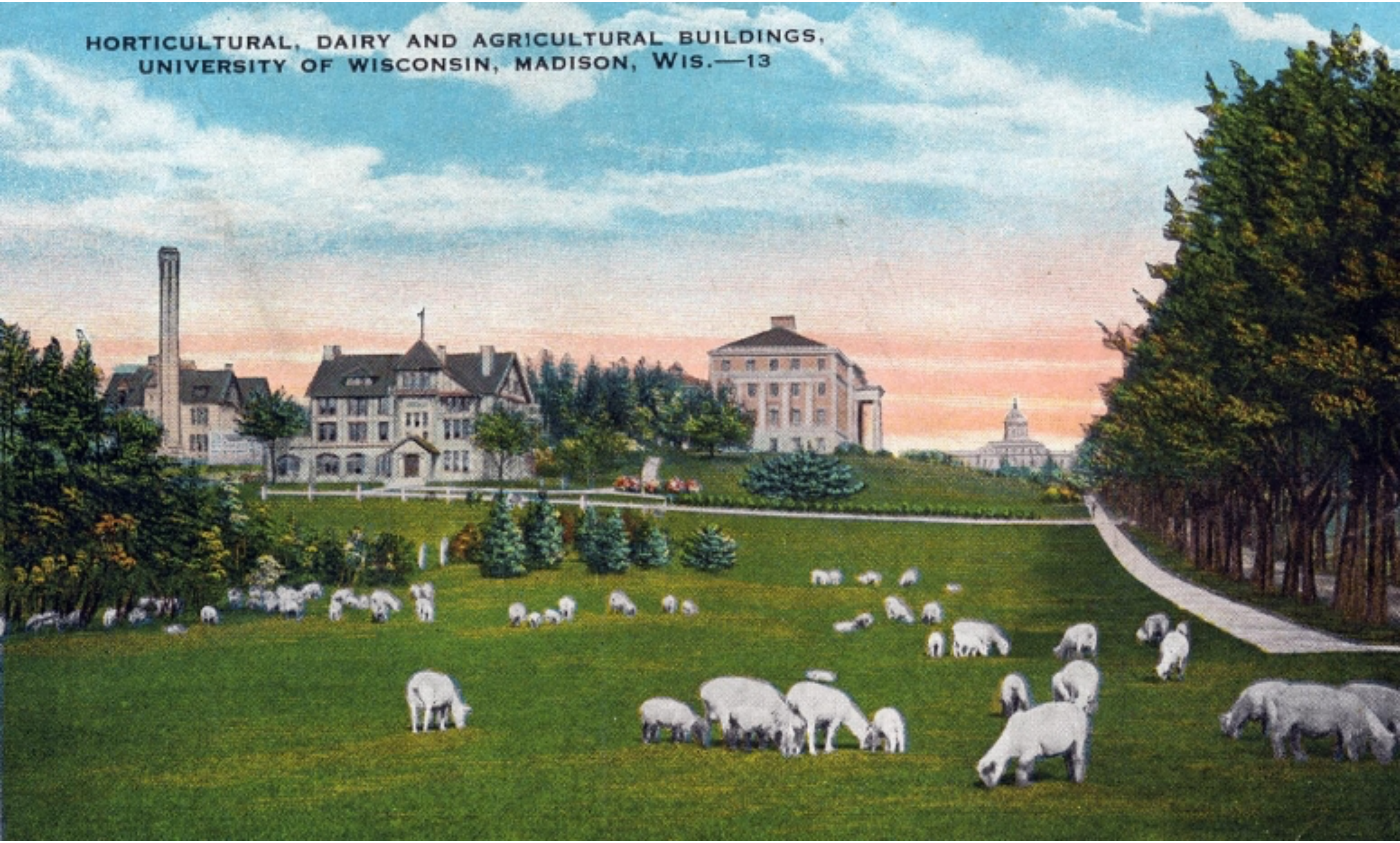
What if there are multiple, equally good NE?

- Pat enjoys football
- Chris enjoys hockey
- Pat and Chris enjoy spending time together

		Chris	
		H	F
Pat	H	1 2	0 0
	F	0 0	2 1

- 2 Nash equilibria
- Pat prefers the (F,F) equilibrium, Chris prefers (H,H)
- Could they choose (F,H) with value 0?
- Solution?

Nash equilibria in a continuous game: The Tragedy of the Commons



A 1913 postcard shows the agricultural campus, where sheep used to graze. Bascom Hall is visible in the background right. UW-MADISON ARCHIVES

The tragedy of the Commons

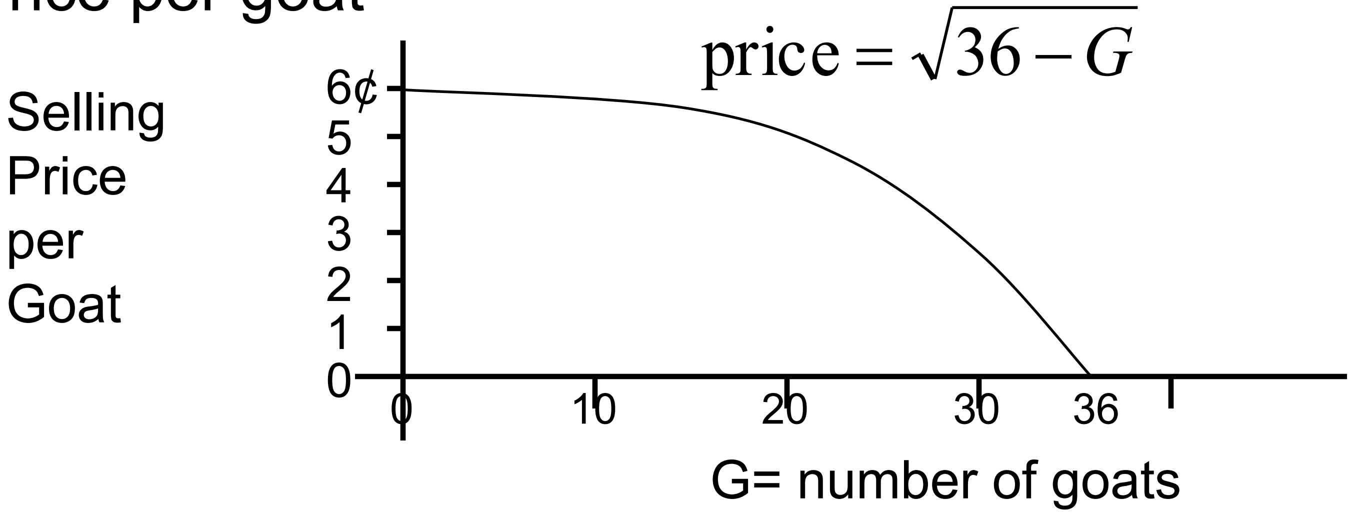
- Everybody can graze goats on the Common

The tragedy of the Commons

- The more goats, the less well fed they are.

The tragedy of the Commons

- Price per goat



$$\text{price} = \sqrt{36 - G}$$

- How many goats should one rational farmer graze?

allow real number, e.g. 1.5 goat is fine

- How much would the farmer earn?
- How many goats should *n* farmers each graze?

Continuous game

- Each farmer has infinite number of strategies $g_i \in [0,36]$
- The value for farmer *i*, when the *n* farmers play at $(g_1, g_2, \ldots, g_n)$ is

$$g_i \sqrt{36 - \sum_{j=1}^{n} g_j}$$

- Assume a Nash equilibrium exists, call it $(g_1^*, g_2^*, \ldots, g_n^*)$
- $g_i^* = \text{argmax}_{\,g_i}$ [value for farmer *i*, with $(g_1^* \ldots g_i \ldots, g_n^*)$]
- What’s the value?

The tragedy of the Commons

$$g_i^* = \arg\max_{g_i} g_i \sqrt{36 - \sum_{\substack{j=1 \\ j \neq i}}^{n} g_j^* - g_i}$$

$$\frac{\partial \text{ value}}{\partial g_i} = 0$$

$$g_i^* = 24 - \frac{2}{3} \sum_{\substack{j=1 \\ j \neq i}}^{n} g_j^*$$

We have n variables and n equations :

$$g_1^* = 24 - \frac{2}{3} \sum_{\substack{j=1 \\ j \neq 1}}^{n} g_j^*$$

$$\ldots$$

$$g_n^* = 24 - \frac{2}{3} \sum_{\substack{j=1 \\ j \neq n}}^{n} g_j^*$$

g_i^* must be the same (proof by “It’s bloody obvious”)

The tragedy of the Commons

So what?

The tragedy of the Commons

$$g_i^* = 24 - \frac{2}{3}(n-1)g_i^*$$

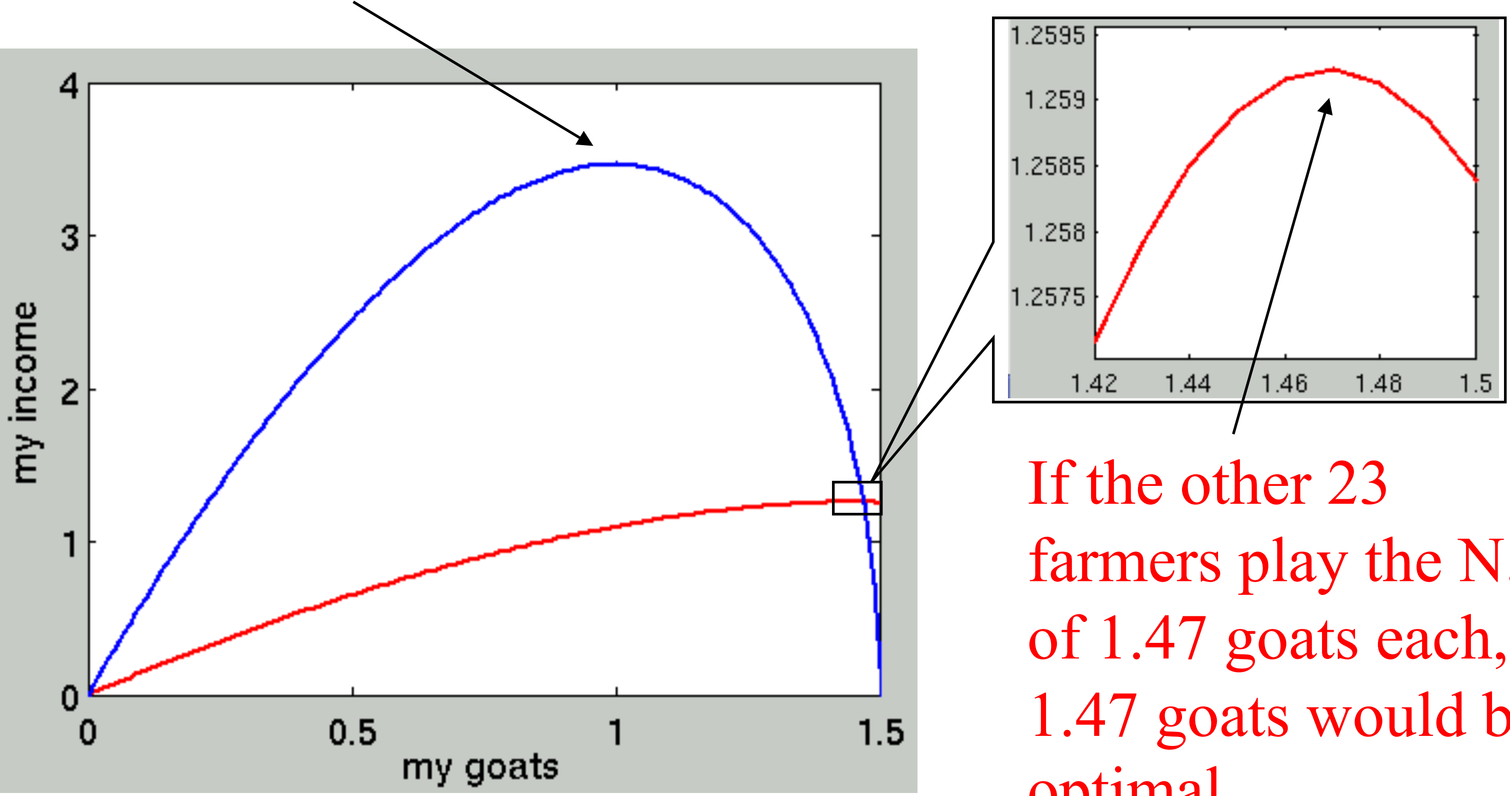
$$g_i^* = \frac{72}{2n+1}$$

So what?

- Say there are n=24 farmers. Each would rationally graze g_i^* = 72/(2*24+1) = 1.47 goats
- Each would get $g_i \sqrt{36 - \sum_{j=1}^{n} g_j}$ = 1.26¢
- But if they cooperate and each graze only 1 goat, each would get 3.46¢

The tragedy of the Commons

If all 24 farmers agree on the same number of goals to raise, 1 goat per farmer would be optimal

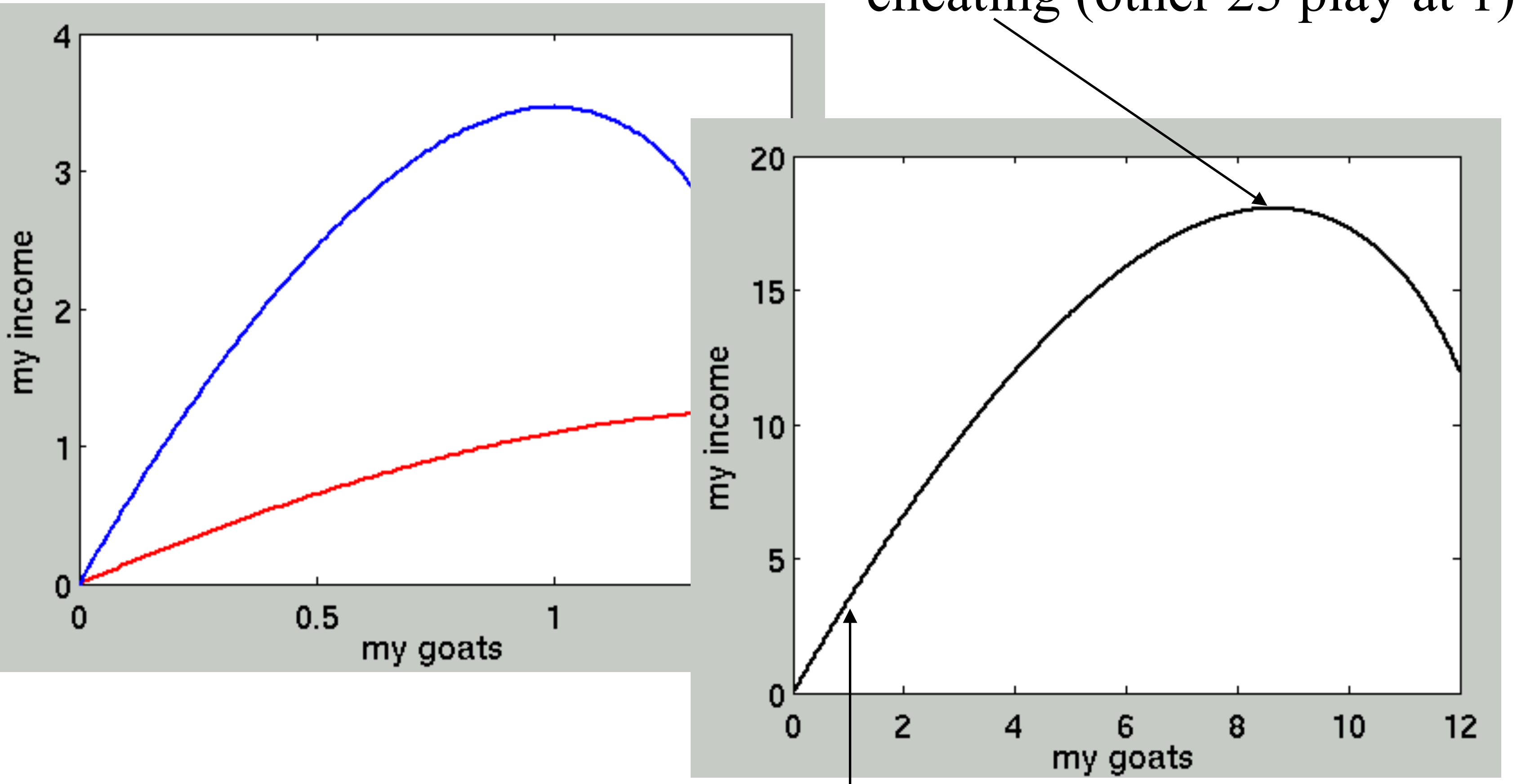


If the other 23 farmers play the N.E. of 1.47 goats each, 1.47 goats would be optimal

The tragedy of the Commons

If all 24 farmers agree on the same number of goals to raise, 1 goat per farmer would be optimal

But this is not a N.E.! A farmer can benefit from cheating (other 23 play at 1):



‘by rule’

The tragedy

- Rational behaviors lead to sub-optimal solutions!
- Maximizing individual welfare not necessarily maximizes social welfare
- What went wrong?

The tragedy

- Rational behaviors lead to sub-optimal solutions!
- Maximizing individual welfare not necessarily maximizes social welfare
- What went wrong?

Shouldn't have allowed unlimited free grazing.

It's not just the  : the use of the atmosphere and the oceans for dumping of pollutants.

Mechanism design: designing the rules of a game

Auction 1: English auction

- Auctioneer increase the price by d
- Until only 1 bidder left
- Winner pays the highest failed bid b_m, plus d

- Dominant strategy: keep bidding if price below your value v
- Simple: no need to consider other bidders' strategies
- High communication cost

Auction 2: first price sealed bid

- Each bidder makes a single, sealed bid to the auctioneer
- Winner pays the bid amount

- If you believe the maximum bid of all other bidders is b_m, you should bid min(v, $b_m+\varepsilon$)
- Have to guess other bidders' bids, hard

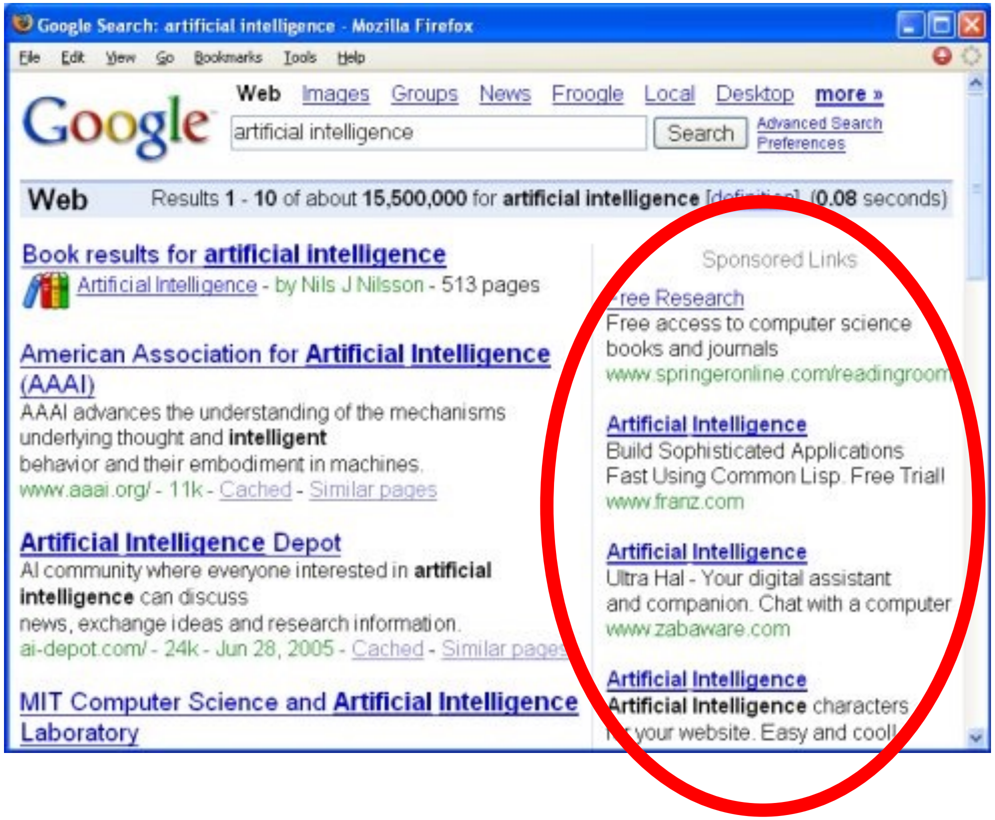
Auction 3: Vickrey (second price sealed) bid

- Each bidder makes a single, sealed bid to the auctioneer
- Winner pays the 2^{nd} highest bid

- Dominant strategy: bid your true value *v*
- Low communication cost, and simple

Why should I care?
(How does this lecture relate to AI?)

- in a world where robots interact with each other…
- in cyberspace where softbots interact with each other
- Google Adword
  - Uses a variation of Vickrey auction (2nd price auction)



Web Results **1** - **10** of about **15,500,000** for **artificial intelligence** [definition]. (**0.08** seconds)

Book results for artificial intelligence
Artificial Intelligence - by Nils J Nilsson - 513 pages

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AAAI advances the understanding of the mechanisms underlying thought and **intelligent** behavior and their embodiment in machines.
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Artificial Intelligence
Artificial Intelligence characters r your website. Easy and cool!

What you should know

- Matrix Normal Form of a game
- Strategies in game
- What do mixed strategies mean
- Strict dominance
- Nash equilibrium
- Tragedy of the Commons
- Basic concept of auctions