CS 540-1: Introduction to Artificial Intelligence
Homework Assignment # 6

Assigned: 11/3
Due: 11/10 before class

Hand in your homework:

This homework includes only a written portion. Please type the written portion and hand in a pdf file named hw6.pdf. Go to Learn@UW My Course Dashboard, choose the 540 course, choose Assignments/Dropbox, click on hw6: this is where you submit your file.

Late Policy:

All assignments are due at the beginning of class on the due date. One (1) day late, defined as a 24-hour period from the deadline (weekday or weekend), will result in 10% of the total points for the assignment deducted. So, for example, if a 100-point assignment is due on a Wednesday 9:30 a.m., and it is handed in between Wednesday 9:30 a.m. and Thursday 9:30 a.m., 10 points will be deducted. Two (2) days late, 25% off; three (3) days late, 50% off. No homework can be turned in more than three (3) days late. Written questions and program submission have the same deadline. A total of two (2) free late days may be used throughout the semester without penalty.

Assignment grading questions must be raised with the instructor within one week after the assignment is returned.

Collaboration Policy:

You are to complete this assignment individually. However, you are encouraged to discuss the general algorithms and ideas with classmates, TAs, and instructor in order to help you answer the questions. You are also welcome to give each other examples that are not on the assignment in order to demonstrate how to solve problems. But we require you to:

- not explicitly tell each other the answers
- not to copy answers or code fragments from anyone or anywhere
- not to allow your answers to be copied
- not to get any code on the Web

In those cases where you work with one or more other people on the general discussion of the assignment and surrounding topics, we suggest that you specifically record on the assignment the names of the people you were in discussion with.
Problem 1. Where is the center? [30]

Consider the following \( n = 7 \) points: 5.4, 9.5, 9.6, 1.5, 9.7, 9.5, 4.8. Find the center \( c \) separately for the three cases below:

1. A 2-norm center that minimizes
   \[
   \sum_{i=1}^{n} |x_i - c|^2.
   \]

2. A 1-norm center that minimizes
   \[
   \sum_{i=1}^{n} |x_i - c|.
   \]

3. An infinity-norm center that minimizes
   \[
   \max_{i=1}^{n} |x_i - c|.
   \]

4. Explain, in words, what interpretation the three centers have, respectively.

Hint: you do not need to prove why your \( c \) is the correct center. Use numerical computation to help you guess if need to.

**SOLUTION:**

1. \[
f(c) = \sum_{i=1}^{n} |x_i - c|^2 = \sum_{i=1}^{n} (x_i^2 - 2x_ic + c^2) = nc^2 - 2c\left(\sum_{i=1}^{n} x_i\right) + \sum_{i=1}^{n} x_i^2
\]

   We have \( n > 0 \) so the function gets the minimum optima when:

   \[
f'(c) = 0 \iff 2nc - 2\left(\sum_{i=1}^{n} x_i\right) = 0 \iff c = \frac{1}{n} \sum_{i=1}^{n} x_i = 7.14
\]

   This center is the mean

2. A 1-norm center that minimizes
   \[
   \sum_{i=1}^{n} |x_i - c|.
   \]

   Sort the \( n \) points from small to large, namely \( x_1, x_2, ..., x_7 \)

   We have
   \[
   |x_7 - c| + |x_1 - c| \geq x_7 - x_1
   |x_6 - c| + |x_2 - c| \geq x_6 - x_2
   |x_5 - c| + |x_3 - c| \geq x_5 - x_3
   |x_4 - c| \geq 0
   \]

   \[
   \sum_{i=1}^{n} |x_i - c| \geq x_7 - x_1 + x_6 - x_2 + x_5 - x_3 + 0
   \]

   The equality happens when \( c = x_4 = 9.5 \)

   This is the median. In our case \( n \) is odd, pick the value in the middle.
3. 

\[ f(c) = \max_{i=1}^{n} |x_i - c|. \]

Sort the \( n \) points from small to large, namely \( x_1, x_2, \ldots, x_7 \)

- \( c > x_7 \) or \( c < x_1 \)

\[ f(c) = \max_{i=1}^{n} |x_i - c| > |x_7 - x_1| \]

- \( x_1 \leq c \leq x_7 \)

We have \( |x_i - c| \leq \max(|x_7 - c|, |c - x_1|) \), so \( \max_{i=1}^{n} |x_i - c| \) is either \( |x_7 - c| \) or \( |c - x_1| \)

\[ f(c) \geq \frac{|x_7 - c| + |c - x_1|}{2} = \frac{x_7 - x_1}{2} \]

From two cases, \( \min(f(c)) = \frac{x_7 - x_1}{2} = 4.1 \)

The equality happens when \( \max_{i=1}^{n} |x_i - c| = |x_7 - c| = |c - x_1| \), hence \( c \) is the midpoint of \((x_1, x_7)\), or 5.6

This is \( (\max + \min)/2 \).

4. Explain, in words, what interpretation the three centers have, respectively.

See above.

**Problem 2. Minima, maxima [20]**

Find all the local minima and local maxima of the function \((1 - x)^4x^3\). Hint: critical point.

Recall critical points are where the first derivative is zero. That is,

\[-4(1 - x)^3x^3 + 3(1 - x)^4x^2 = 0\]

\[(1 - x)^3x^2(-4x + 3(1 - x)) = 0\]

\[(1 - x)^3x^2(-7x + 3) = 0\]

The critical points are 0, 3/7, 1. In the interval \((-\infty, 0)\) the first derivative is positive, meaning the function is increasing. In the interval \((0, 3/7)\) the first derivative is positive, too, meaning the function is still increasing. So 0 is neither a minimum nor maximum. In the interval \((3/7, 1)\) the first derivative is negative, meaning the function is decreasing. Thus 3/7 is a local maximum. In the interval \((1, \infty)\) the first derivative is positive, meaning the function is increasing. Thus 1 is a local minimum.

**Problem 3. kNN [20]**

There are three classes: A, B and C. The number of feature dimensions is \( d = 2 \). The following is the training set:

<table>
<thead>
<tr>
<th>Class</th>
<th>Training items</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>(2, 2), (3, 3), (2, 3)</td>
</tr>
<tr>
<td>B</td>
<td>(4, 3.7), (3.7, 3.8), (3.8, 4.1), (3.5, 3.6)</td>
</tr>
<tr>
<td>C</td>
<td>(2, 5), (4, 5)</td>
</tr>
</tbody>
</table>
Now use 3-NN with infinity-norm as the distance measure to solve the following problems. The infinity-norm distance between two points \( x_i \) and \( x_j \) is defined as:

\[
\|x_i - x_j\|_\infty = \max_{k=1}^d |x_{ik} - x_{jk}|
\]

where \( d \) is the number of features.

Classify the following points: (2.5, 2.5), (4, 3), (2, 4). Show the steps of your calculation. If a tie between classes arises then prefer A, B, C in that order.

Here we list the training points in that order as the rows, and the three test points as columns; in the table we show the distance. The three nearest neighbors are italicized.

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>0.5</th>
<th>2.0</th>
<th>2.0</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>A</td>
<td>0.5</td>
<td>1.0</td>
<td>1.0</td>
</tr>
<tr>
<td></td>
<td>A</td>
<td>0.5</td>
<td>2.0</td>
<td>1.0</td>
</tr>
<tr>
<td></td>
<td>B</td>
<td>1.5</td>
<td>0.7</td>
<td>2.0</td>
</tr>
<tr>
<td></td>
<td>B</td>
<td>1.3</td>
<td>0.8</td>
<td>1.7</td>
</tr>
<tr>
<td></td>
<td>B</td>
<td>1.6</td>
<td>1.1</td>
<td>1.8</td>
</tr>
<tr>
<td></td>
<td>B</td>
<td>1.1</td>
<td>0.6</td>
<td>1.5</td>
</tr>
<tr>
<td></td>
<td>C</td>
<td>2.5</td>
<td>2.0</td>
<td>1.0</td>
</tr>
<tr>
<td></td>
<td>C</td>
<td>2.5</td>
<td>2.0</td>
<td>2.0</td>
</tr>
</tbody>
</table>

Thus we see that 3NN will classify the three test points as class A, B, A, respectively.

**Problem 4. Must-links and cannot-links [30]**

Consider the following major cities in the US: Madison, Seattle, Boston, Honolulu, Anchorage; and in Canada: Vancouver, Winnipeg, Montreal.

1. Consult the Internet and create a table with the distances between the eight cities. USE KM INSTEAD OF MILES (you are a scientist). Round to the nearest hundred km.

2. Use hierarchical clustering with single linkage to produce TWO clusters. Show your steps.

3. Now repeat the above question, but with the following constraints:

   - must-link(Honolulu, Madison)
   - must-link(Anchorage, Madison)
   - cannot-link(Seattle, Vancouver)

You need to modify the hierarchical clustering with single linkage clustering algorithm as follows:

- In the very beginning before distance-based merges, put cities linked by must-links in the same cluster.
- Throughout clustering, never allow two clusters with cities in cannot-link to be merged.

Again cluster the cities into two clusters. Show your steps.

**SOLUTION:**

4-1:
Madison  | Seattle  | Boston  | Honolulu | Anchorag | Vancouver | Winnipeg | Montreal
---|---|---|---|---|---|---|---
Madison  | 2600  | 1500  | 6700  | 4400  | 2600  | 1000  | 1300
Seattle   | 4000  | 4300  | 2300  | 200   | 1900  | 3700
Boston    | 8200  | 5400  | 4100  | 400   | 2200  | 400
Honolulu  | 4500  | 4200  | 4200  | 6100  | 7900
Anchorag  |        |        | 2500  | 3400  | 5000
Vancouver |        |        | 2000  | 3800
Winnipeg  |        |        |        | 1800
Montreal  |        |        |        |        |        |        |