CS540
Uninformed Search

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Main messages

• Many AI problems can be formulated as search.

• Iterative deepening is good when you don’t know much.
Problem: The boat only holds two, but you can’t leave the goat with the cabbage or the wolf with the goat.

Solution:
1. Take the goat across.
2. Return alone.
3. Take the cabbage across.
4. Leave the wolf.

Why did you have a wolf?
The search problem

- **State space** $S$: all valid configurations
- **Initial states (nodes)** $I=\{(\text{CSDF,})\} \subseteq S$
  - Where’s the boat?
- **Goal states** $G=\{(),\text{CSDF}\} \subseteq S$
- **Successor function** $\text{succs}(s) \subseteq S$: states reachable in one step (one arc) from $s$
  - $\text{succs}((\text{CSDF,})) = \{\text{(CD, SF)}\}$
  - $\text{succs}((\text{CDF,S})) = \{\text{(CD,FS), (D,CFS), (C, DFS)}\}$
- **Cost** $(s, s') = 1$ for all arcs. (weighted later)
- The search problem: find a solution path from a state in $I$ to a state in $G$.
  - Optionally minimize the cost of the solution.
Search examples

• 8-puzzle

- States = configurations
- successor function = up to 4 kinds of movement
- Cost = 1 for each move
Search examples

• Water jugs: how to get 1?

• Goal? (How many goal states?)

• Successor function: fill up (from tap or other jug), empty (to ground or other jug)
Search examples

- Route finding (state? Successors? Cost weighted)
8-queens

• State: complete configuration vs. column-by-column

• Tree instead of graph
A directed graph in state space

In general there will be many generated, but un-expanded states at any given time.

One has to choose which one to expand next.
Different search strategies

- The generated, but not yet expanded states form the fringe (OPEN).
- The essential difference is which one to expand first.
- Deep or shallow?
Uninformed search on trees

- **Uninformed** means we only know:
  - The goal test
  - The `succs()` function

- But **not** which non-goal states are better: that would be informed search (next lecture).

- For now, we also assume `succs()` graph is a **tree**.
  - Won’t encounter repeated states.
  - We will relax it later.

- Search strategies: BFS, UCS, DFS, IDS, BIBFS
- Differ by what un-expanded nodes to expand
Breadth-first search (BFS)

Expand the shallowest node first

• Examine states **one** step away from the initial states
• Examine states **two** steps away from the initial states
• and so on…

ripple
Breadth-first search (BFS)

Use a queue (First-in First-out)

1. en_queue(Initial states)
2. While (queue not empty)
3. s = de_queue()
4. if (s==goal) success!
5. T = succs(s)
6. en_queue(T)
7. endWhile
Breadth-first search (BFS)

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Use a **queue** (First-in First-out)

1. `en_queue(Initial states)`
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6. `en_queue(T)`
7. `endWhile`

queue (fringe, OPEN) → [CB] → A
Breadth-first search (BFS)

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queue (fringe, OPEN)  \( \rightarrow [EDC] \rightarrow B \)
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queue (fringe, OPEN) → [GFED] → C

If G is a goal, we've seen it, but we don't stop!
Breadth-first search (BFS)

Use a queue (First-in First-out)
- en_queue(Initial states)
- While (queue not empty)
  - s = de_queue()
  - if (s==goal) success!
  - T = succs(s)
  - for t in T: t.prev=s
- en_queue(T)
- endWhile

Looking stupid?
Indeed. But let’s be consistent…

... until much later we pop G.

We need back pointers to recover the solution path.
Performance of BFS

- Assume:
  - the graph may be infinite.
  - Goal(s) exists and is only finite steps away.

- Will BFS find at least one goal?
- Will BFS find the least cost goal?

- Time complexity?
  - # states generated
  - Goal $d$ edges away
  - Branching factor $b$

- Space complexity?
  - # states stored
Performance of BFS

Four measures of search algorithms:

• **Completeness** (not finding all goals): yes, BFS will find a goal.

• **Optimality**: yes if edges cost 1 (more generally positive non-decreasing in depth), no otherwise.

• **Time complexity** (worst case): goal is the last node at radius $d$.
  - Have to generate all nodes at radius $d$.
  - $b + b^2 + \ldots + b^d \sim O(b^d)$

• **Space complexity** (bad)
  - Back pointers for all generated nodes $O(b^d)$
  - The queue / fringe (smaller, but still $O(b^d)$)
What’s in the fringe (queue) for BFS?

- Convince yourself this is $O(b^d)$
### Performance of search algorithms on trees

*b*: branching factor (assume finite)  \quad  *d*: goal depth

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1. Edge cost constant, or positive non-decreasing in depth
Performance of BFS

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  - $b + b^2 + \ldots + b^d \sim O(b^d)$

- **Space complexity** (bad, Figure 3.11)
  - Back points for all generated nodes $O(b^d)$
  - The queue (smaller, but still $O(b^d)$)

Solution: Uniform-cost search
Uniform-cost search

• Find the least-cost goal
• Each node has a path cost from start (= sum of edge costs along the path). Expand the least cost node first.
• Use a priority queue instead of a normal queue
  ▪ Always take out the least cost item
  ▪ Remember heap? time $O(\log(\#\text{items in heap}))$

That’s it*

* Complications on graphs (instead of trees). Later.
Uniform-cost search (UCS)

- Complete and optimal (if edge costs $\geq \varepsilon > 0$)
- Time and space: can be much worse than BFS
  - Let $C^*$ be the cost of the least-cost goal
  - $O(b^{C*/\varepsilon})$, possibly $C^*/\varepsilon >> d$
## Performance of search algorithms on trees

**b**: branching factor (assume finite)  
**d**: goal depth

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<td>Uniform-cost search²</td>
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<td>Y</td>
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1. edge cost constant, or positive non-decreasing in depth
   - edge costs $\geq \varepsilon > 0$. $C^*$ is the best goal path cost.
General State-Space Search Algorithm

function general-search(problem, QUEUEING-FUNCTION)
    ;; problem describes the start state, operators, goal test, and
    ;; operator costs
    ;; queueing-function is a comparator function that ranks two states
    ;; general-search returns either a goal node or "failure"

    nodes = MAKE-QUEUE(MAKE-NODE(problem.INITIAL-STATE))
    loop
        if EMPTY(nodes) then return "failure"
        node = REMOVE-FRONT(nodes)
        if problem.GOAL-TEST(node.STATE) succeeds
            then return node
        nodes = QUEUEING-FUNCTION(nodes, EXPAND(node, problem.OPERATORS))
            ;; succ(s)=EXPAND(s, OPERATORS)
            ;; Note: The goal test is NOT done when nodes are generated
            ;; Note: This algorithm does not detect loops
    end
Recall the bad space complexity of BFS

Four measures of search algorithms:

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- **Space complexity** (bad, Figure 3.11)
  - Back points for all generated nodes $O(b^d)$
  - The queue (smaller, but still $O(b^d)$)

Solution: Uniform-cost search

Solution: Depth-first search
Depth-first search

Expand the deepest node first

1. Select a direction, go deep to the end
2. Slightly change the end
3. Slightly change the end some more…

fan
Depth-first search (DFS)

Use a stack (First-in Last-out)

1. push(Initial states)
2. While (stack not empty)
3. \[ s = \text{pop()} \]
4. if (s==goal) success!
5. \[ T = \text{succs}(s) \]
6. push(T)
7. endWhile

stack (fringe)
\[
[] \iff
\]
What’s in the fringe for DFS?

- $m = \text{maximum depth of graph from start}$
- $m(b-1) \sim O(mb)$
  (Space complexity)

- “backtracking search” even less space
  - generate siblings (if applicable)

\[ O(b^d) \quad \text{c.f. BFS} \]
What’s wrong with DFS?

• Infinite tree: may not find goal (incomplete)
• May not be optimal
• Finite tree: may visit almost all nodes, time complexity $O(b^m)$

c.f. BFS $O(b^d)$
### Performance of search algorithms on trees

* b: branching factor (assume finite)  
* d: goal depth  
* m: graph depth

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1. edge cost constant, or positive non-decreasing in depth
   - edge costs \( \geq \varepsilon > 0 \). \( C^* \) is the best goal path cost.
How about this?

1. DFS, but stop if path length > 1.
2. If goal not found, repeat DFS, stop if path length > 2.
3. And so on…

fan within ripple
Iterative deepening

• Search proceeds like BFS, but fringe is like DFS
  ▪ Complete, optimal like BFS
  ▪ Small space complexity like DFS

• A huge waste?
  ▪ Each deepening repeats DFS from the beginning
  ▪ No! \( db+(d-1)b^2+(d-2)b^3+\ldots+b^d \sim O(b^d) \)
  ▪ Time complexity like BFS

• Preferred uninformed search method
# Performance of search algorithms on trees

<table>
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1. edge cost constant, or positive non-decreasing in depth
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b: branching factor (assume finite)  
d: goal depth  
m: graph depth
### Performance of search algorithms on trees

- **b**: branching factor (assume finite)
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1. edge cost constant, or positive non-decreasing in depth
   - edge costs $\geq \varepsilon > 0$. $C^*$ is the best goal path cost.

How to reduce the number of states we have to generate?
Bidirectional search

- Breadth-first search from both start and goal
- Fringes meet
- Generates $O(b^{d/2})$ instead of $O(b^d)$ nodes
Bidirectional search

• But
  ▪ The fringes are $O(b^{d/2})$
  ▪ How do you start from the 8-queens goals?

slide 40
# Performance of search algorithms on trees

- **b**: branching factor (assume finite)
- **d**: goal depth
- **m**: graph depth

<table>
<thead>
<tr>
<th>Search Method</th>
<th>Complete</th>
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<th>Space</th>
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<td>Y, if 1</td>
<td>(O(b^d))</td>
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</tr>
<tr>
<td>Uniform-cost search(^2)</td>
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<td>Y</td>
<td>(O(b^{C*/\varepsilon}))</td>
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</tr>
<tr>
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<td>(O(bd))</td>
</tr>
<tr>
<td>Bidirectional search(^3)</td>
<td>Y</td>
<td>Y, if 1</td>
<td>(O(b^{d/2}))</td>
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1. edge cost constant, or positive non-decreasing in depth
- edge costs \(\geq \varepsilon > 0\). \(C^*\) is the best goal path cost.
- both directions BFS; not always feasible.
If state space graph is not a tree

- The problem: repeated states

- Ignore the danger of repeated states: wasteful (BFS) or impossible (DFS). Can you see why?

- How to prevent it?
If state space graph is not a tree

• We have to remember already-expanded states (CLOSED).
• When we take out a state from the fringe (OPEN), check whether it is in CLOSED (already expanded).
  ▪ If yes, throw it away.
  ▪ If no, expand it (add successors to OPEN), and move it to CLOSED.
If state space graph is not a tree

- BFS:
  - Still $O(b^d)$ space complexity, not worse
- DFS:
  - Known as Memorizing DFS (MEMDFS)
    - Space and time now $O(\min(N, b^M))$ – much worse!
    - $N$: number of states in problem
    - $M$: length of longest cycle-free path from start to anywhere
  - Alternative: Path Check DFS (PCDFS): remember only expanded states on current path (from start to the current node)
    - Space $O(M)$
    - Time $O(b^M)$
Example

(All edges are directed, pointing downwards)
Nodes expanded by:

• Depth-First Search: S A D E G
  Solution found: S A G

• Breadth-First Search: S A B C D E G
  Solution found: S A G

• Uniform-Cost Search: S A D B C E G
  Solution found: S B G (This is the only uninformed search that worries about costs.)

• Iterative-Deepening Search: S A B C S A D E G
  Solution found: S A G
Depth-First Search

<table>
<thead>
<tr>
<th>expanded node</th>
<th>nodes list</th>
</tr>
</thead>
<tbody>
<tr>
<td>S</td>
<td>{ A B C }</td>
</tr>
<tr>
<td>A</td>
<td>{ D E G B C }</td>
</tr>
<tr>
<td>D</td>
<td>{ E G B C }</td>
</tr>
<tr>
<td>E</td>
<td>{ G B C }</td>
</tr>
<tr>
<td>G</td>
<td>{ B C }</td>
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Solution path found is S A G  <-- this G has cost 10
Number of nodes expanded (including goal node) = 5
Breadth-First Search

Expanded nodes with nodes list:

<table>
<thead>
<tr>
<th>node</th>
<th>nodes list</th>
</tr>
</thead>
<tbody>
<tr>
<td>S</td>
<td>{ S }</td>
</tr>
<tr>
<td>A</td>
<td>{ B C D E G }</td>
</tr>
<tr>
<td>B</td>
<td>{ C D E G G' }</td>
</tr>
<tr>
<td>C</td>
<td>{ D E G G' G'' }</td>
</tr>
<tr>
<td>D</td>
<td>{ E G G' G'' }</td>
</tr>
<tr>
<td>E</td>
<td>{ G G' G'' }</td>
</tr>
<tr>
<td>G</td>
<td>{ G' G'' }</td>
</tr>
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Solution path found is S A G  <-- this G also has cost 10
Number of nodes expanded (including goal node) = 7
Uniform-Cost Search

expanded
node        nodes list
 ----        -----------
  S          { S }
  A          { A(1) B(5) C(8) }
  D          { D(4) B(5) C(8) E(8) G(10) }  (note, we don’t return G)
  B          { B(5) C(8) E(8) G(10) }
  C          { C(8) E(8) G(9) G(10) }
  E          { E(8) G(9) G(10) G(13) }
  G          { G(9) G(10) G(13) }

Solution path found is S B G  <-- this G has cost 9, not 10
Number of nodes expanded (including goal node) = 7
What you should know

• Problem solving as search: state, successors, goal test
• Uninformed search
  ▪ Breadth-first search
    • Uniform-cost search
  ▪ Depth-first search
  ▪ Iterative deepening ✭
  ▪ Bidirectional search
• Can you unify them (except bidirectional) using the same algorithm, with different priority functions?
• Performance measures
  ▪ Completeness, optimality, time complexity, space complexity