Please Remember to Write Your Name

CS731 Exam, 10 points each

April 11, 2011

1. Let $X_1, \ldots, X_n \overset{i.i.d.}{\sim} \text{Cauchy}(x; x_0, \gamma)$. What would the Central Limit Theorem look like on these random variables? Briefly justify your answer.

   CLT does not apply because Cauchy does not have a finite mean.

2. Find the maximum likelihood estimator for $\theta > 0$ from $X_1, \ldots, X_n \overset{i.i.d.}{\sim} \text{uniform}(-\theta, \theta)$.

   $\hat{\theta} = \max(|X_1|, \ldots, |X_n|)$.

3. Let $X \sim \text{Bernoulli}(p)$. Let $\hat{p}_1 = X$ and $\hat{p}_2 = 0.5$. Assume squared loss. Compute the risk of $\hat{p}_1$ and $\hat{p}_2$, respectively. Then compute the maximum risk of $\hat{p}_1$ and $\hat{p}_2$.

   $R(p, \hat{p}_1) = \mathbb{E}[(X-p)^2] = \mathbb{V}[X] = p(1-p)$. $R(p, \hat{p}_2) = \mathbb{E}[(0.5-p)^2] = (0.5-p)^2$. $R^{\max}(\hat{p}_1) = \max_p p(1-p) = 1/4$. $R^{\max}(\hat{p}_2) = \max_p (0.5-p)^2 = 1/4$.

4. (For this question you do not need to give the exact solution) Draw by hand the line $x_1 + 2x_2 = 1$ for $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$. Draw the ℓ_1 ball that corresponds to the solution

$$\begin{aligned} \min \quad & \|\mathbf{x}\|_1 && (1) \\ \text{s.t.} \quad & x_1 + 2x_2 = 1. && (2) \end{aligned}$$

   Clearly mark the solution. On a separate plot, draw the ℓ_2 ball that corresponds to the solution

$$\begin{aligned} \min \quad & \|\mathbf{x}\|_2 && (3) \\ \text{s.t.} \quad & x_1 + 2x_2 = 1. && (4) \end{aligned}$$

   Clearly mark the solution too.

   A unit diamond and a unit circle.

5. Consider the Bernoulli distribution in exponential family form: $p(x) = \exp(x\theta - \log(1 + \exp(\theta)))$. Give its marginal polytope.

   $\phi(x) = x$. $\mathcal{M} = [0, 1]$.

6. Let $A^*(\mu) = \mu \log \mu + (1-\mu)\log(1-\mu)$ for $\mu \in (0, 1)$. Solve

$$\sup_{\mu \in (0,1)} \theta\mu - A^*(\mu). \quad (5)$$

   $\mu = \frac{1}{1+\exp(-\theta)}$. The objective is $\log(\exp(\theta) + 1)$.

7. Consider a Markov chain with two states x_1, x_2 and transition matrix $T(x_1 \mid x_1) = 0, T(x_2 \mid x_1) = 1, T(x_1 \mid x_2) = 1, T(x_2 \mid x_2) = 0$. What is the stationary distribution of this Markov chain?

   There is none, the chain is periodic.

8. Consider the exponential distribution

$$p(x) = \begin{cases} e^{-x} & x \geq 0 \\ 0 & x < 0 \end{cases} \tag{6}$$

whose mean is at 1, the variance is 1, and the mode is at 0. Can we use it as a smoothing kernel for kernel density estimation? Justify your answer.

Let $K(x) = p(x+1)$.

9. Consider the Chinese Restaurant Process with concentration parameter α. Recall that the i-th customer goes to a new table with probability $\alpha/(\alpha+i-1)$. After serving n customers, on average how many tables are occupied?

$\sum_{i=1}^{n} \alpha/(\alpha+i-1)$

10. Consider a random vector $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$ chosen uniformly on the unit circle within the first quadrant: $\|\mathbf{x}\|^2 = 1$, $x_1 \geq 0$, $x_2 \geq 0$. Compute the expected squared norm of its projection to the first axis: $\mathbb{E}[\|x_1\|^2]$. Show your steps. (Hint: $\int \cos^2(\theta)d\theta = \theta/2 + 1/4\sin(2\theta) + C$.)

1/2.