CS761 Spring 2015 Homework 1

Assigned Jan. 18, due Jan. 20 before class

Instructions:

- Homeworks are to be done individually.
- Typeset your homework in latex using this file as template (e.g. use pdflatex). Show your derivations.
- Hand in the compiled pdf (not the latex file) online. Instructions will be provided. We do not accept hand-written homeworks.
- Homework will no longer be accepted once the lecture starts.
- Please email the TA if you have any questions about this solution:

Name: Xuezhou Zhang
Email: zhangxz1123@cs.wisc.edu

Each question has 5 points if answered correctly. Wrong answers or blank answers 0 points. However, an explicit "I don't know" answer is worth 1 point.

1. Define inner product on real functions

$$\langle g, h \rangle = \int_0^1 g(x)h(x)\,dx.$$

Orthogonalize the basis $1, x, x^2$.
Let $v_1 = 1$, $v_2 = x$, $v_3 = x^2$. Then we will have

$$u_1 = 1 \tag{1}$$

$$u_2 = v_2 - \frac{\langle u_1, v_2 \rangle}{\langle u_1, u_1 \rangle} u_1 = x - \frac{1}{2} \tag{2}$$

$$u_3 = v_3 - \frac{\langle u_1, v_3 \rangle}{\langle u_1, u_1 \rangle} u_1 - \frac{\langle u_2, v_3 \rangle}{\langle u_2, u_2 \rangle} u_2 = x^2 - x + \frac{1}{6}. \tag{3}$$

2. Here is a little fact: Let X be a random variable with finite mean μ and finite variance σ^2. Then $\forall t > 0$ we have

$$P(|X - \mu| \geq t) \leq \frac{\sigma^2}{t^2}.$$

Now let $X_1, X_2, \ldots$ be a sequence of independent and identically distributed random variables with mean μ and variance σ^2. Let $\bar{X}_n = \frac{1}{n}\sum_{i=1}^n X_i$. Use the little fact to prove

$$\forall \epsilon > 0, \forall \delta > 0, \exists N > 0, \forall n \geq N, P(|\bar{X}_n - \mu| \leq \epsilon) \geq 1 - \delta.$$

Well, yet another fact says that the variance of the sample mean of n i.i.d random variables equals to $\frac{\sigma^2}{n}$. Apply the little fact on $\bar{X}_n$, we have for $\forall \epsilon > 0$,

$$P(|\bar{X} - \mu| \geq \epsilon) \leq \frac{\sigma^2}{n\epsilon^2}.$$

Therefore, for any given $\epsilon > 0$, $\delta > 0$, let $N = \frac{\sigma^2}{\delta\epsilon^2}$, then the desired result follows.