Persistent Homology Tutorial

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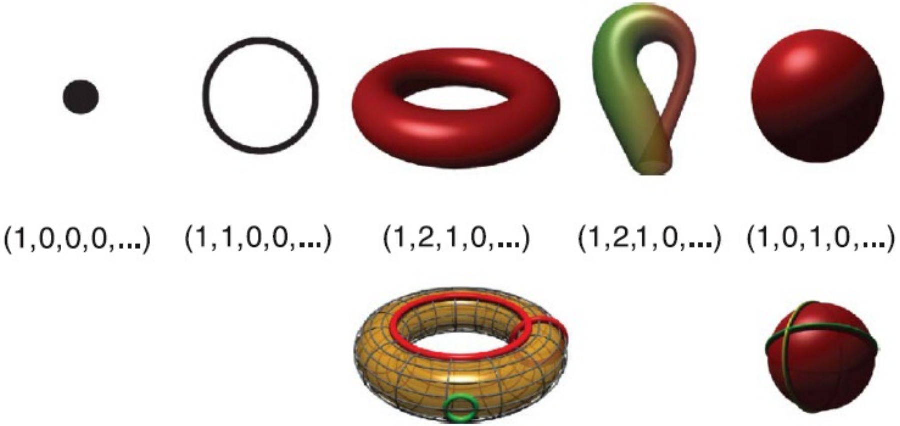
  - higher order holes, too
- Betti numbers: the number of k^{th} order holes

Betti number examples



[Reproduced from Singh *et al.* J. Vision 2008]

Plan of this talk

- Persistent homology tutorial

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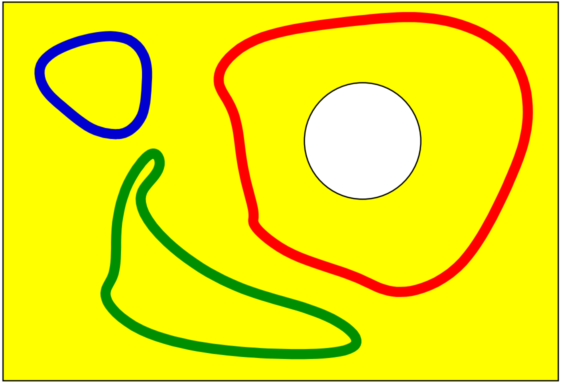
- An application in natural language processing

Holes and equivalent rubber bands



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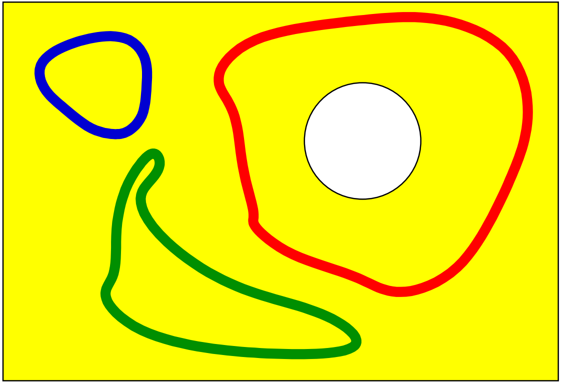

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- two equivalent classes $\Leftrightarrow$ one hole.

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- All our groups G are abelian: $\forall a, b \in G, a * b = b * a$.

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- not $\langle \mathbb{R}_-, \times \rangle$

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- cosets have equal sizes and partition G.

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- negation in natural language: G_N

$*$	$\sqcup$	not
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homomorphism (isomorphism) from G_N to $\mathbb{Z}_2$: $\phi(\sqcup) = 0, \phi(\text{not}) = 1$.

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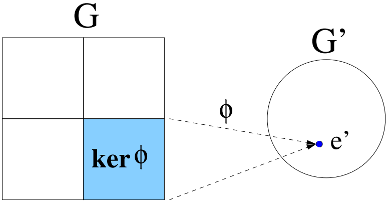

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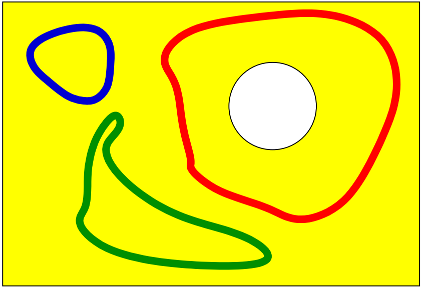
- $\mathrm{rank}(\mathbb{Z}) = 1$ since $\mathbb{Z} = \langle \{1\} \rangle$.
- $\mathrm{rank}(\mathbb{Z} \times \mathbb{Z}) = 2$ since $\mathbb{Z} \times \mathbb{Z} = \langle \{(0,1),(1,0)\} \rangle$.

The group of rubber bands



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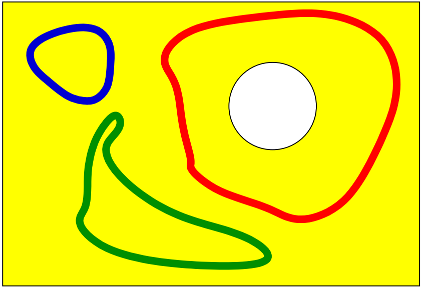

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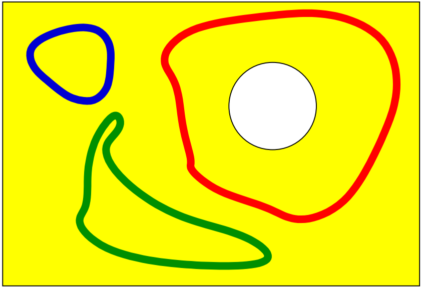

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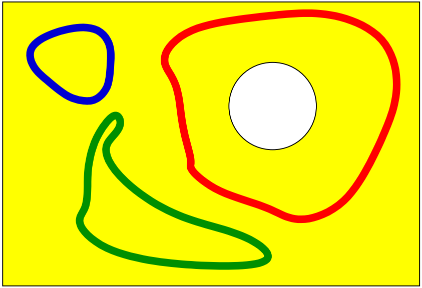

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- Computation: need discrete rubber bands $\Rightarrow$ simplicial complex

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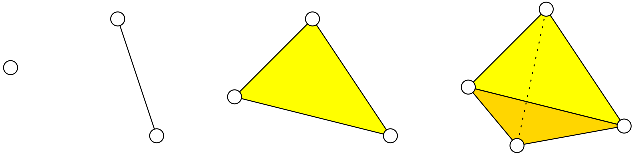
- $p = 0, 1, 2, 3$



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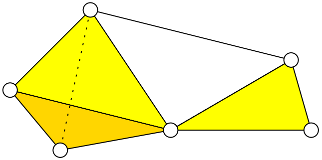

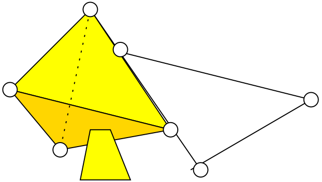

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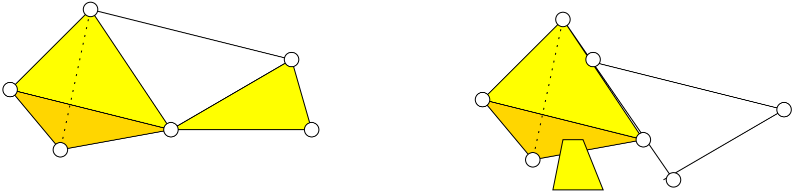

- Simplicial complex = the yellow space in the rubber band picture

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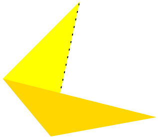

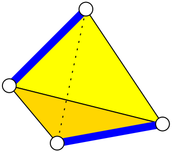

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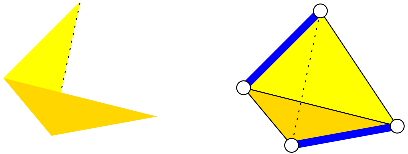

- A p-chain does not have to be connected.

Chain group

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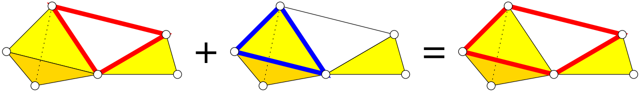
- Mod-2 addition



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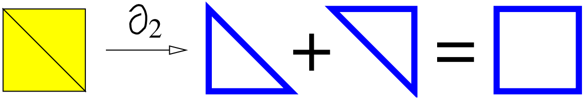
- Faces shared by an even number of p-simplices in the chain will cancel out:



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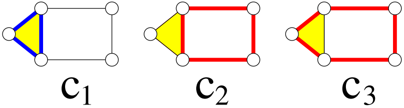
Boundary-Cycle

- The boundary of any $(p+1)$-chain is always a p-cycles



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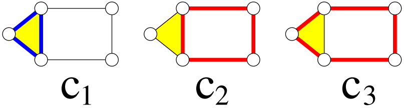

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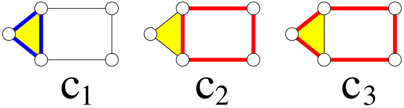

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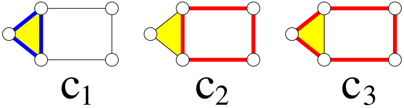

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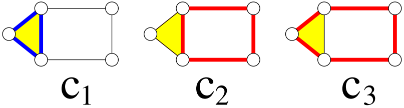

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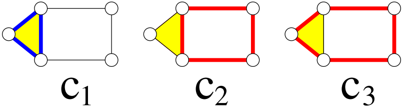

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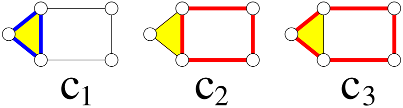

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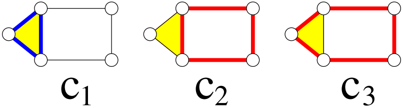

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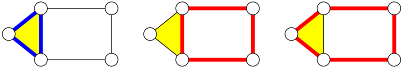

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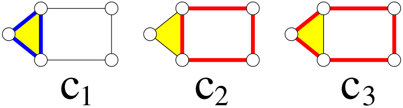

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- The equivalence class: $c + B_p$

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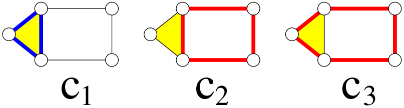

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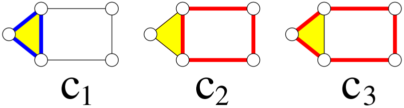

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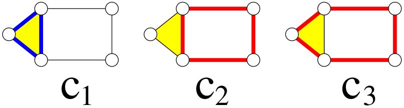

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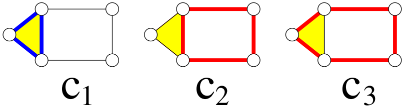

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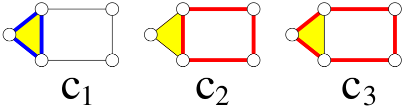

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- The homology group $H_1 = Z_1/B_1$ isomorphic to $\mathbb{Z}_2$

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From data to simplicial complex

- Given data $x_1, \ldots, x_n \in \mathbb{R}^d$.

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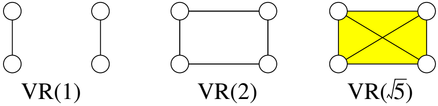

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An increasing sequence of ϵ produces a filtration, i.e., a sequence of increasing simplicial complexes $VR(\epsilon_1) \subseteq VR(\epsilon_2) \subseteq \ldots$, with the property that a simplex enters the sequence no earlier than all its faces.

Persistent homology

- In a filtration, at what value of ϵ does a hole appear, and how long does it persist till it is filled in?

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- In a filtration, at what value of ϵ does a hole appear, and how long does it persist till it is filled in?
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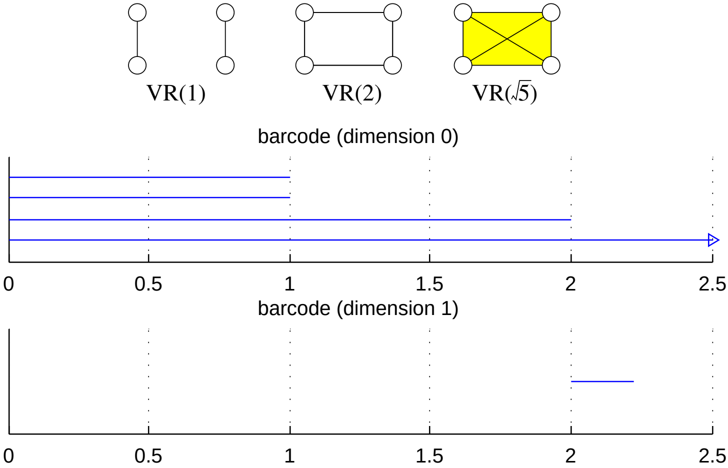
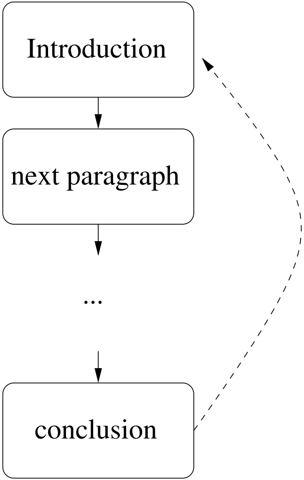


Applications to natural language processing

Good articles "tie back."



How can we capture such loopy structure in text documents?

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- We will focus on the 0-th (clusters) and 1st (holes) order homology classes.

Example: Itsy bitsy spider

The Itsy Bitsy Spider climbed up the water spout
Down came the rain and washed the spider out
Out came the sun and dried up all the rain

And the Itsy Bitsy Spider climbed up the spout again

- bag-of-words

again	all	and	bitsy	came	climbed	down	dried	itsy	out	rain	spider	spout	sun	the	up	washed	water
0	0	0	1	0	1	0	0	1	0	0	1	1	0	2	1	0	1
0	0	1	0	1	0	1	0	0	1	1	1	0	0	2	0	1	0
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- vertices

Similarity Filtration (SIF)

$D_{max} = \max D(x_i, x_j), \forall i, j = 1 \ldots n$
FOR $m = 0, 1, \ldots M$
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Similarity Filtration with Time Skeleton (SIFTS)

$D(x_i, x_{i+1}) = 0$ for $i = 1, \ldots, n-1$
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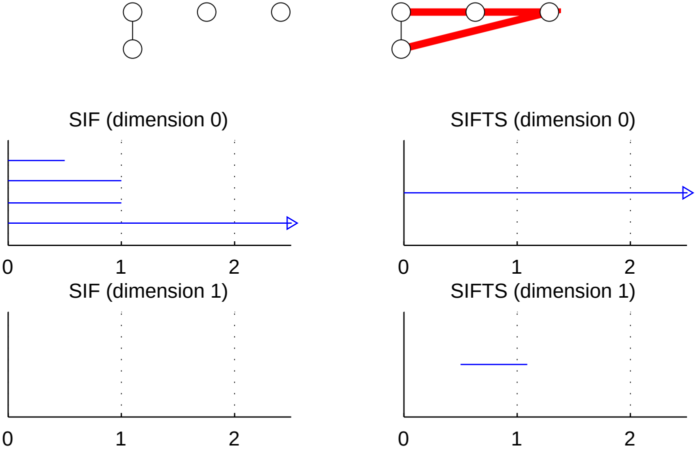
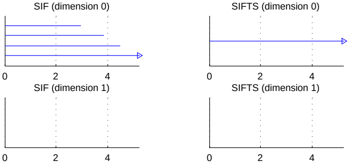
- time edges allow tie-back in time

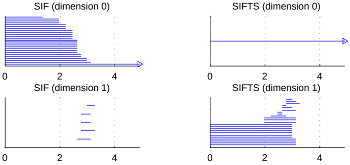
SIF vs. SIFTS on Itsy bitsy spider



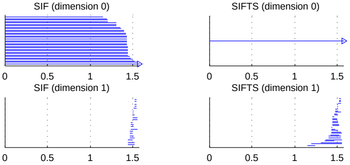
On Nursery Rhymes and Other Stories



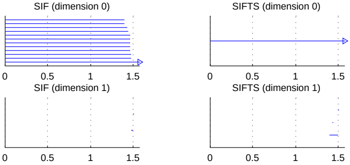
Row Row Row Your Boat



London Bridge



Little Red-Cap



Alice in Wonderland

- London Bridge: "My fair Lady" repeats 12 times.

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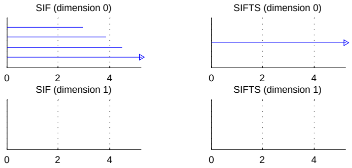

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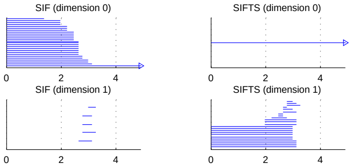

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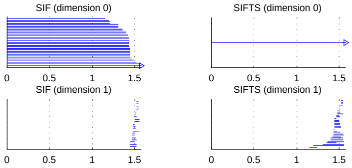

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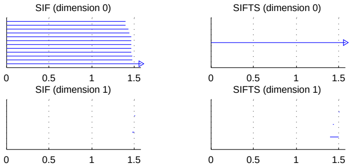

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On Child and Adolescent Writing

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  - ϵ^*: the smallest ϵ when the first hole in H_1 forms.

	child	adolescent	adol. trunc.
holes?	87%	100%*	98%*
$\|H_1\|$	3.0 ($\pm$0.2)	17.6 ($\pm$0.9)*	3.9 ($\pm$0.2)*
ϵ^*	1.35 ($\pm$.02)	1.27 ($\pm$.02)*	1.38 ($\pm$.01)

*: statistically significantly different from "child"

Is Homology Merely Counting Repeats?

- On $x_1 \leadsto x_2 \leadsto x_3$ where x_1, x_2, x_3 SIFTS will find two holes: $x_1 \rightleftharpoons x_2$, $x_2 \rightleftharpoons x_3$

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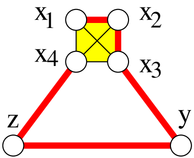
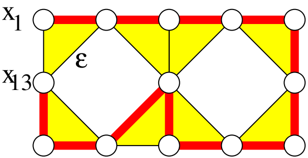
- No.





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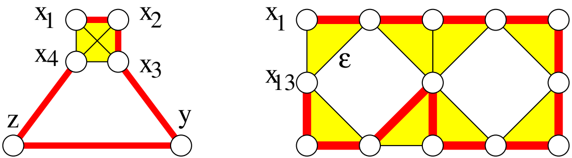

  - Left: $k-1=3$, SIFTS correctly finds $\beta_1 = 1$

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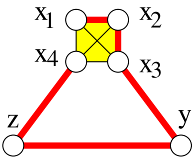

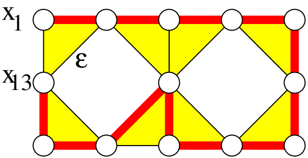

  - Left: $k-1=3$, SIFTS correctly finds $\beta_1 = 1$
  - Right: $k-1=12$, merging x 0 holes, SIFTS correctly finds $\beta_1 = 2$

Summary

- Persistent homology may offer new representations for machine learning

To read more, see the references in
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Summary

- Persistent homology may offer new representations for machine learning
- How to best use it?

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