CS/Math 240: Introduction to Discrete Mathematics 7/10/2007

Exam 1 - Sample Exam

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The exam will be closed-notes and no calculators will be allowed. You will be allowed a 4x6 inch index card cheat sheet. This sample exam is much longer than the actual exam will be.

1. Is the following a valid rule of inference: $(p \wedge q) \vee (\neg p \wedge \neg q)$, therefore $(p \vee q)$? Justify your answer.

2. Convert the following argument into logical form using predicates and quantifiers. (Do not worry about if it is valid or not.)

   Some baseball players take steroids.
   Barry Bonds is a baseball player.
   Therefore, Barry Bonds takes steroids.

3. Prove that for any sets A and B, $\overline{A} \cap \overline{B} = \overline{(A \cup B)}$.

4. Prove that the even positive integers have the same cardinality as the natural numbers.

5. Show that if A and B are sets with the same cardinality, then the power sets of A and B have the same cardinality.

6. "True or False" For finite sets A and B, if there is a function from A to B that is onto, then $|A| = |B|$. Justify your answer.

7. Give an exact value for the number of (not necessarily connected) directed graphs on n vertices. Justify your answer.

8. Consider the relation R over the set of cars such that $(x, y) \in R$ if and only if x and y have the same number of seat belts. Is R an equivalence relation? Justify your answer.

9. This problem deals with the problem of computing the intersection of two sets.

   (a) Give pseudo-code for an algorithm that: 1) takes input unordered lists $\{x_1, ..., x_n\}$, $\{y_1, ..., y_m\}$, and a potential element z, 2) returns "yes" iff z is in either of the lists.

   (b) Prove that your algorithm is correct.

   (c) Give a big-Θ estimate of the number of comparisons of your algorithm.

10. Consider a function $T(n)$ defined by the following recurrence relation:

$$T(1) = 1,$$

$$T(n) = n \cdot T(n/2) + 1, \text{ for all integers } n > 1.$$

Give a big-Θ estimate of the value of $T(n)$. You may assume that n is a power of 2, say $n = 2^k$.

11. Prove that for any integer $n \geq 1$, $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$.

12. Give a big-Θ estimate for the function $f(n) = \frac{n^2}{\log n} + n\log(n) - 5$.

13. Place the following functions in order from smallest to largest in terms of their asymptotic growth (no proof needed):

$$2^{\log(n^3)}, n, \log(n^2), (\log n)^2, \sqrt{n}.$$

14. Let $f(n)$ and $g(n)$ both be functions that limit to infinity as $n \to \infty$. Show that if $f(n) = O(g(n))$, then $(f(n))^2 = O((g(n))^2)$.

15. Prove that for all integers $n > 6$, $3^n < n!$.

16. Consider the following pseudo-code for an algorithm that claims to solve the *longest path* problem.

    Long-Path.
    Input: Graph G with n vertices, vertices a and z
    Output: longest path from a to z through G that does not contain any cycles

    (1) Set path to initially be empty. Set $v = a$.
    (2) **for** $i = 1$ **to** $n - 1$
    (3) Find the largest weight edge out of v that does not cause a cycle with the path already constructed. Add this edge to the path, and update v to be the vertex on the other side of this edge.
    (4) Output the path that has been constructed.

    Give a graph as a counterexample that shows that this algorithm is not correct.

17. This problem deals with the building of a maximal spanning tree - a spanning tree that has the largest total weight possible.

    (a) Give pseudo-code for an algorithm to find a maximal spanning tree.

    (b) Prove that your algorithm is correct.

    (c) Give a big-O estimate of the running time of your algorithm.

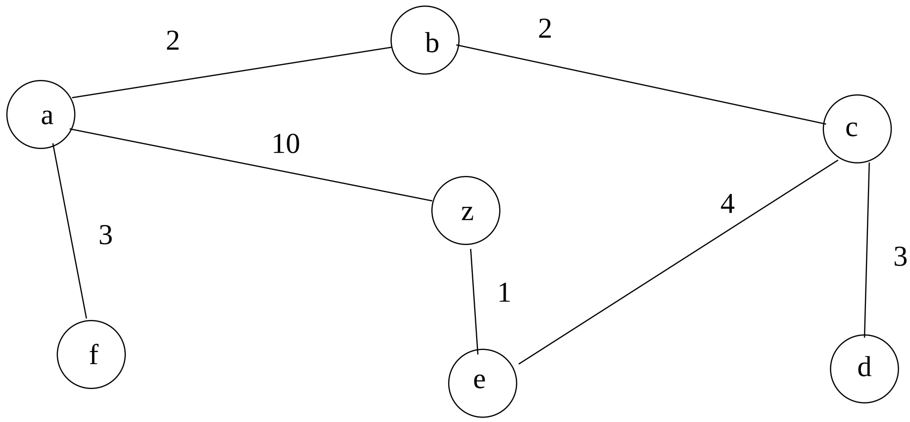
18. Name a way that a finite partially ordered set could be represented as 0's and 1's to be stored on a computer?

    For the representation you have in mind, how many comparisons are needed to determine for two elements x and y of the poset if $x \leq y$ or not?

19. For the shortest path problem, we are given two vertices and look for the shortest path between them. The shortest-any-path problem is to find the shortest path between any two pairs of vertices in a graph (find the pair of vertices with the shortest, shortest path). Give an algorithm to find the shortest-any-path on a graph that does not have any negative edge weights.

20. Prove that there are $n - 1$ edges in a spanning tree of a graph with n vertices.

21. Give the steps in the execution of Dijkstra's algorithm to find the shortest path from a to z in the following graph. That is, for each iteration of the algorithm, give the list of vertices that the algorithm has determined shortest paths to already along with their shortest path values.



22. Let $S = \{a, b, c, d, e, f\}$ be a set, and suppose we have a relation $\leq$ on S defined by the following Hasse diagram. Give a topological sorting of S according to this relation.