Deterministic Simulations and Hierarchy Theorems For Randomized Algorithms

Jeff Kinne

University of Wisconsin-Madison, Department of Computer Sciences

Thesis Defense, April 27, 2010

Does Randomness Add Significant Power?

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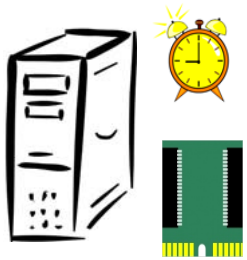
Running Time /
Memory Space



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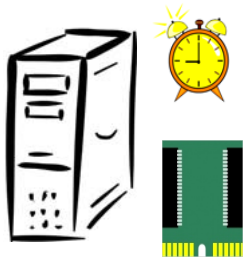

primality testing

undirected connectivity

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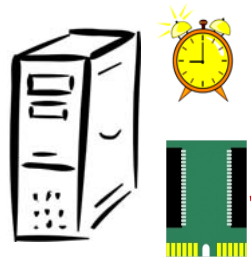

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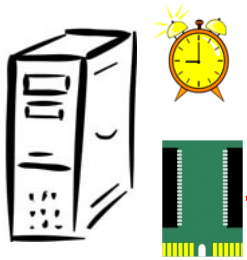

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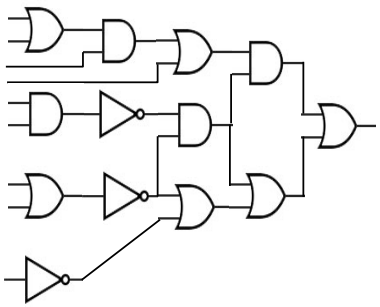
Circuits



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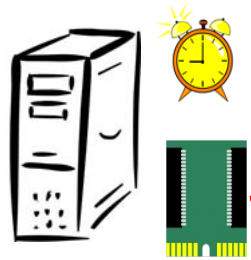

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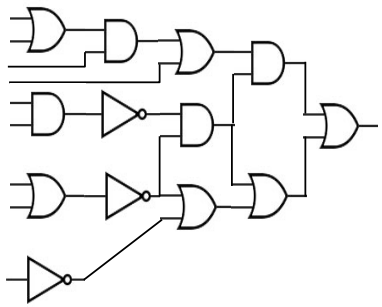

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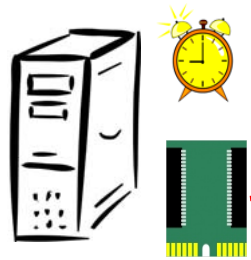

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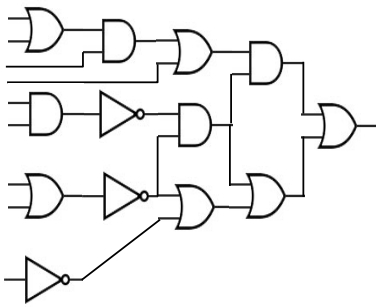
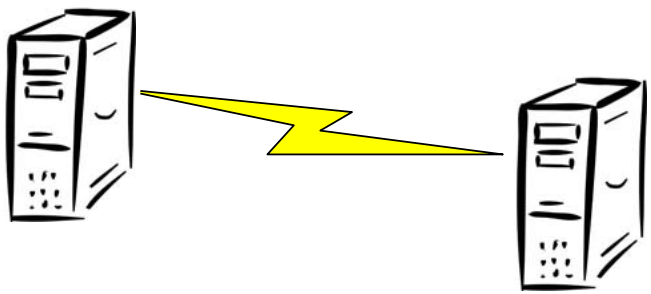


Communication complexity



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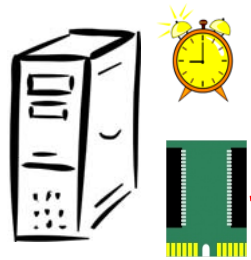

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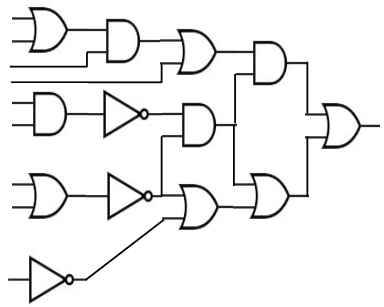

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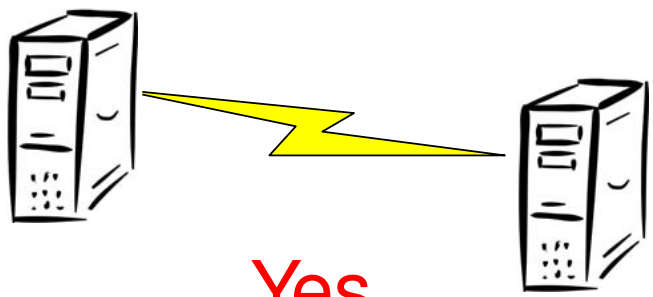

Yes

equality

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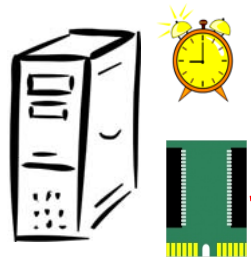

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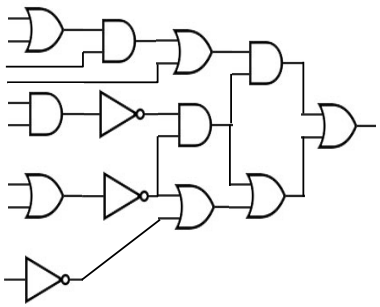
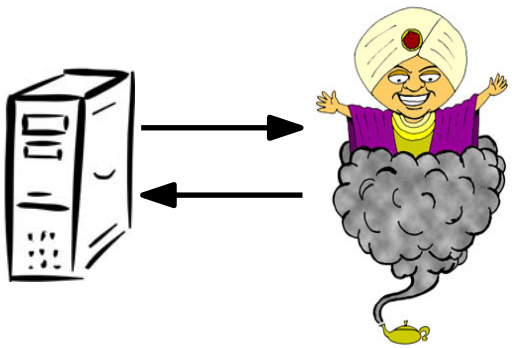


Interactive proofs



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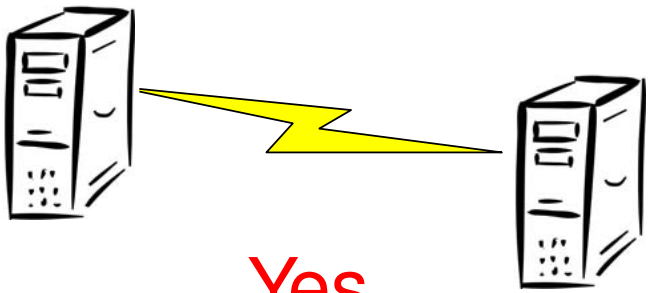

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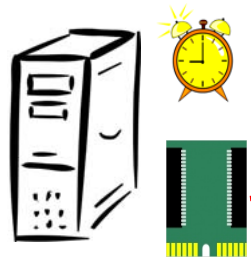

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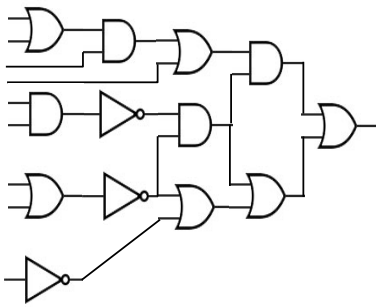

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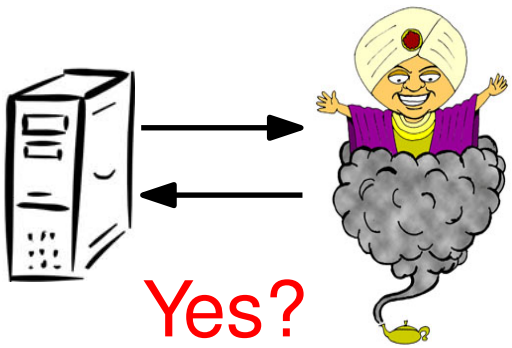
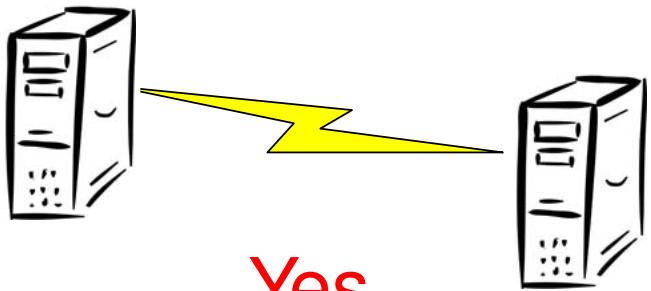


Yes?

quantified Boolean formula

Communication complexity



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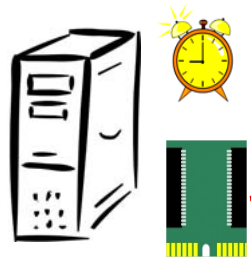

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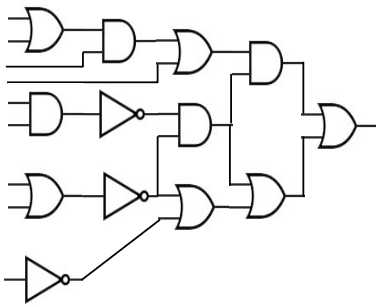

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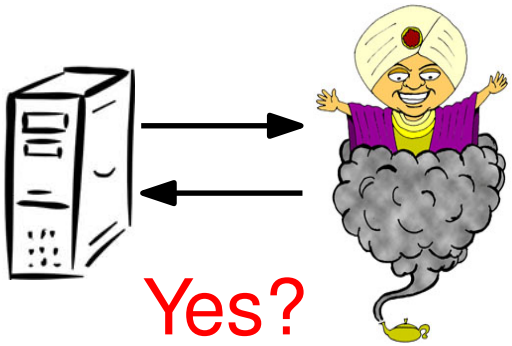

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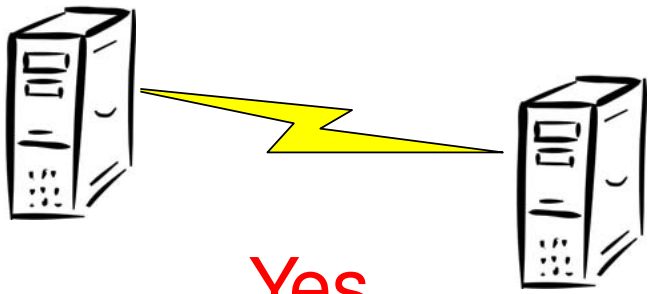

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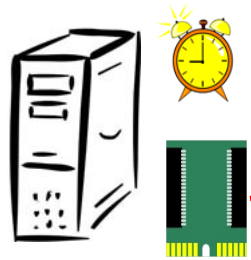

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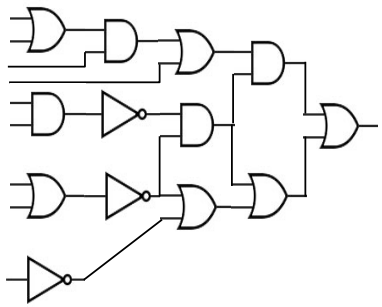

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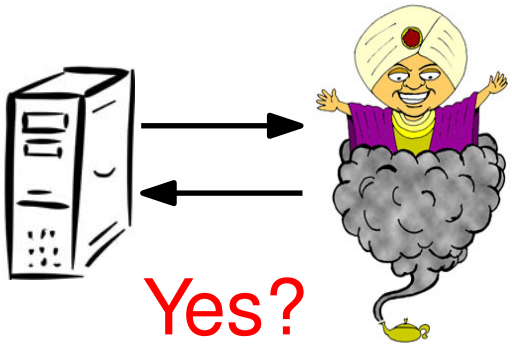

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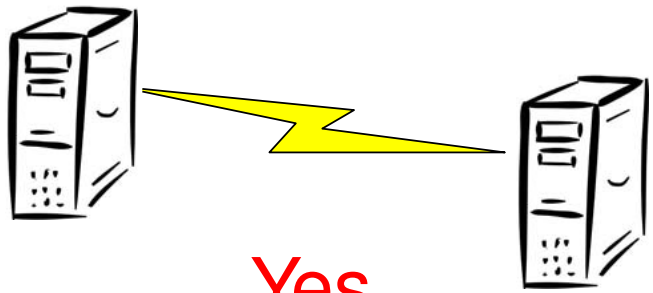

Yes

equality

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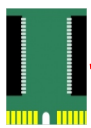
Running Time / Memory Space

No?

poly identity testing



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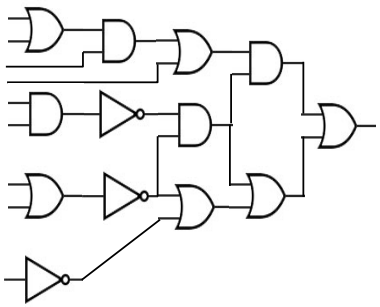

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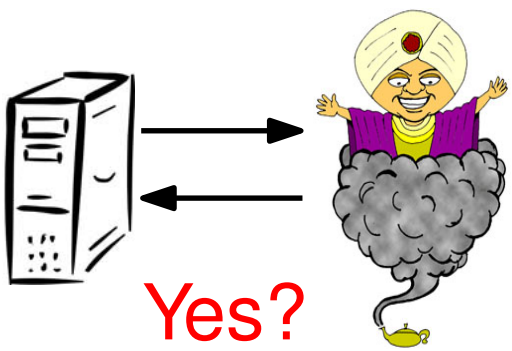

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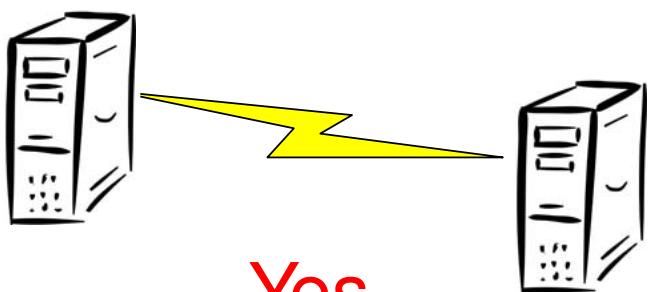

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Outline

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How Powerful is Randomness?

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- "Typically-correct" Derandomization

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- Hierarchy Theorems for Randomized Algorithms

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- "Typically-correct" Derandomization
- Hierarchy Theorems for Randomized Algorithms
- Derandomizing Monotone Computations

Randomized Algorithm

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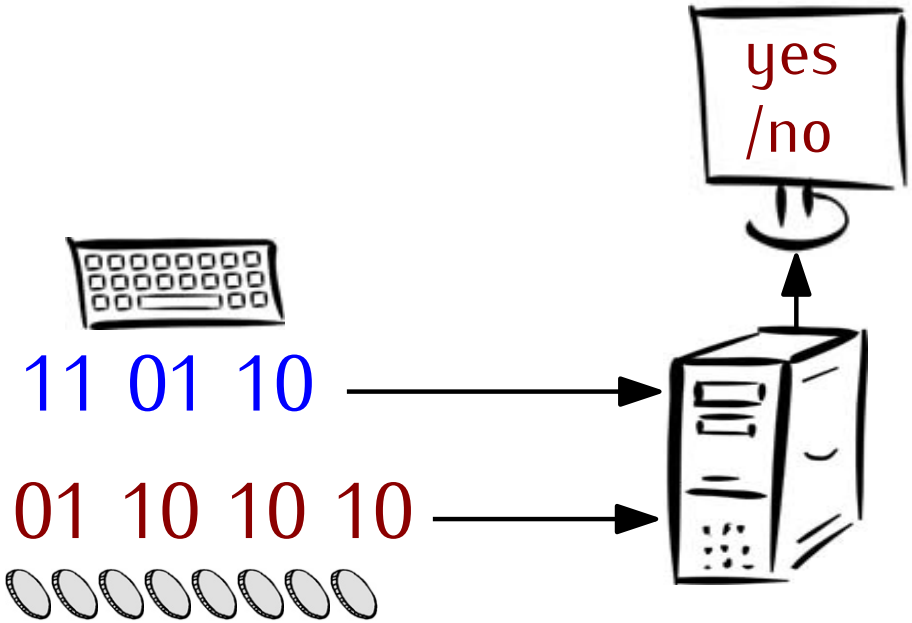

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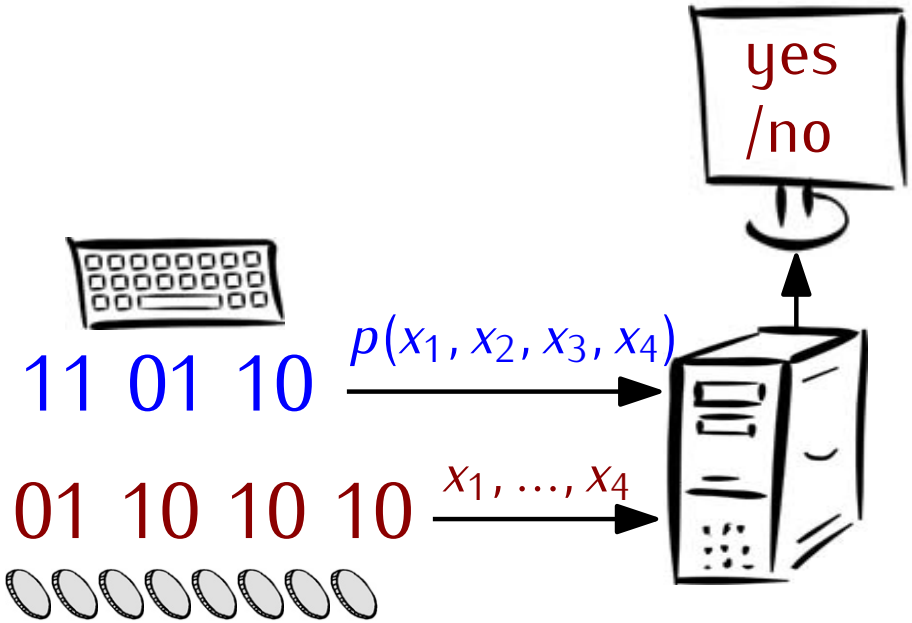

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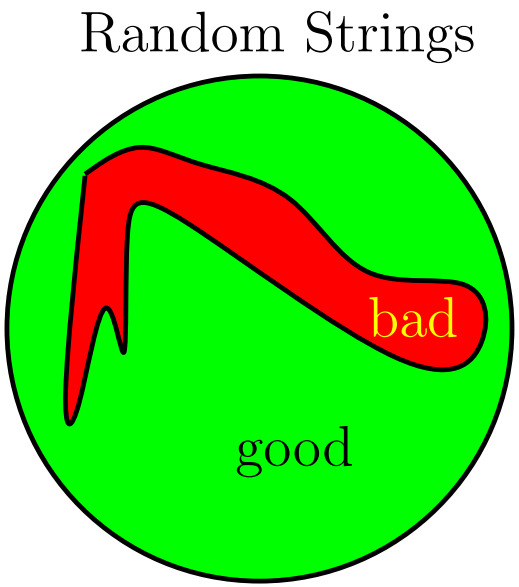

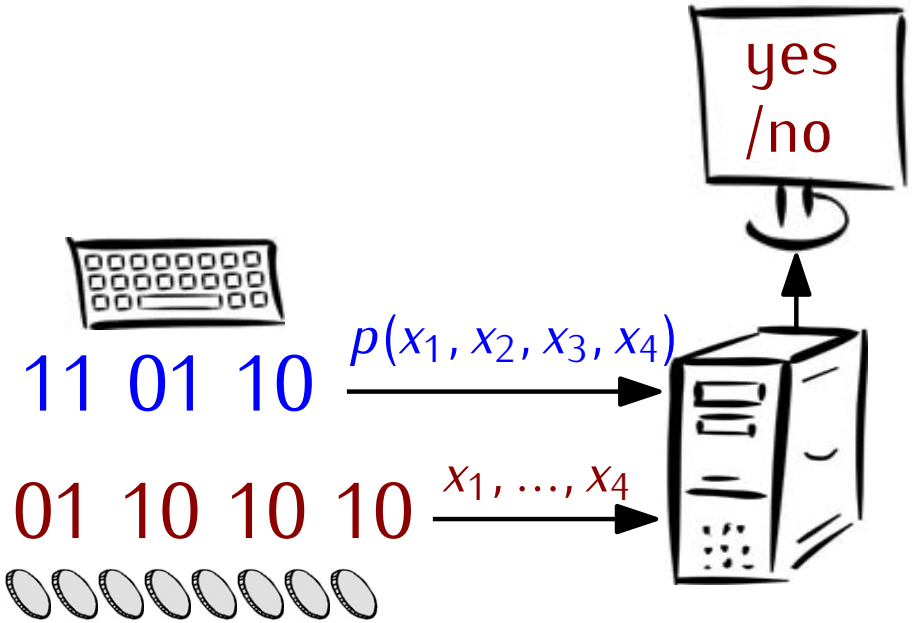

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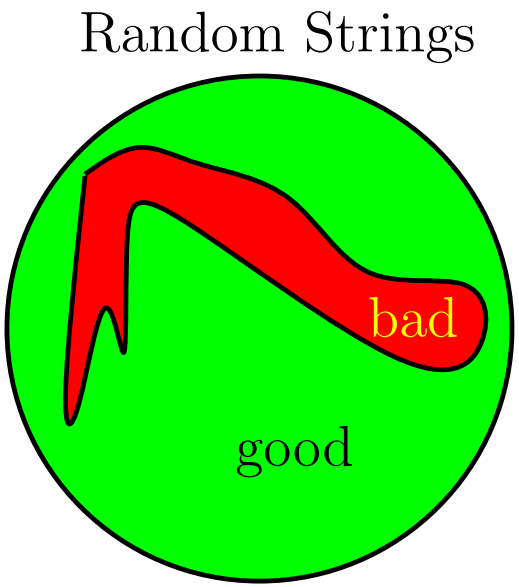
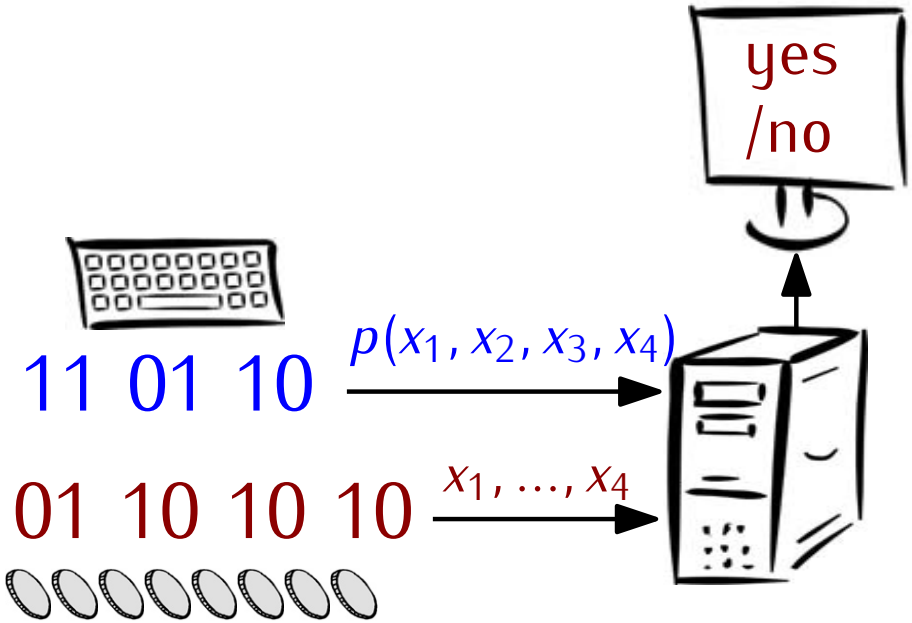
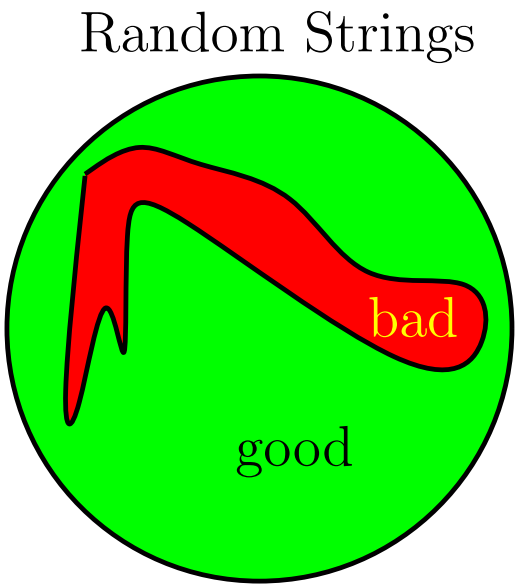
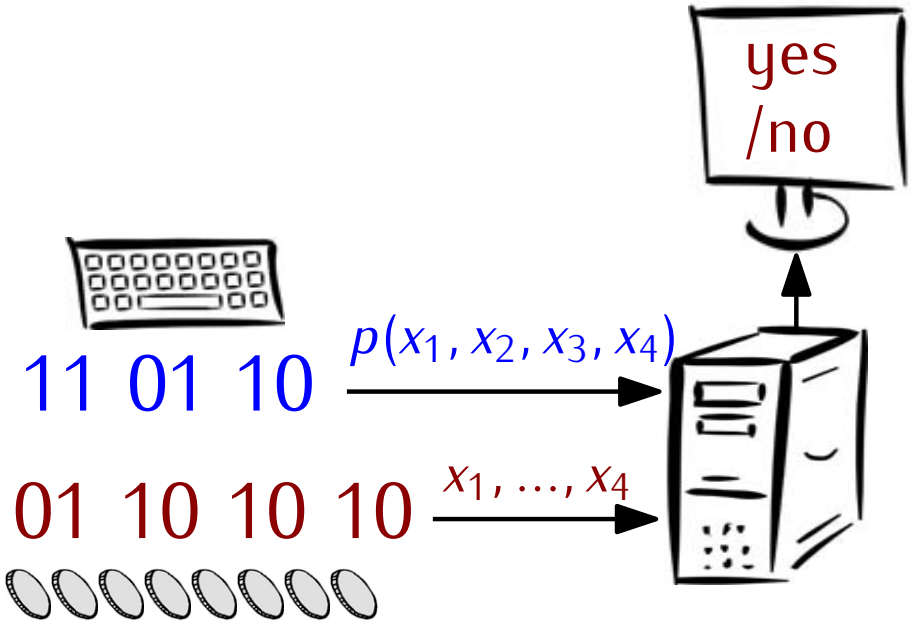




- Bounded error: For every fixed input, correct with Pr $>$ 99%

Randomized Algorithm





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- **BPP**: Bounded-error Probabilistic Poly time

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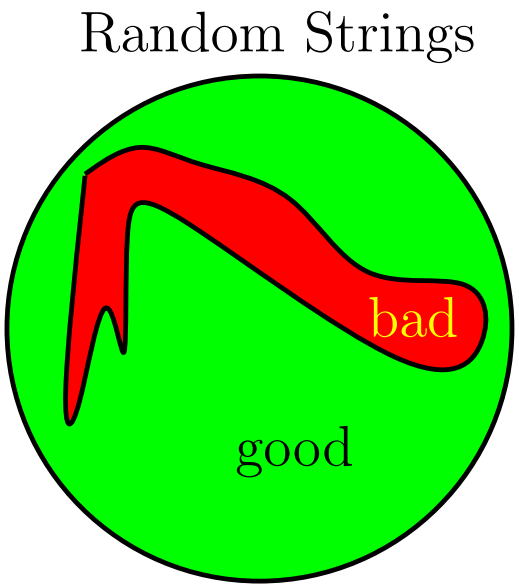
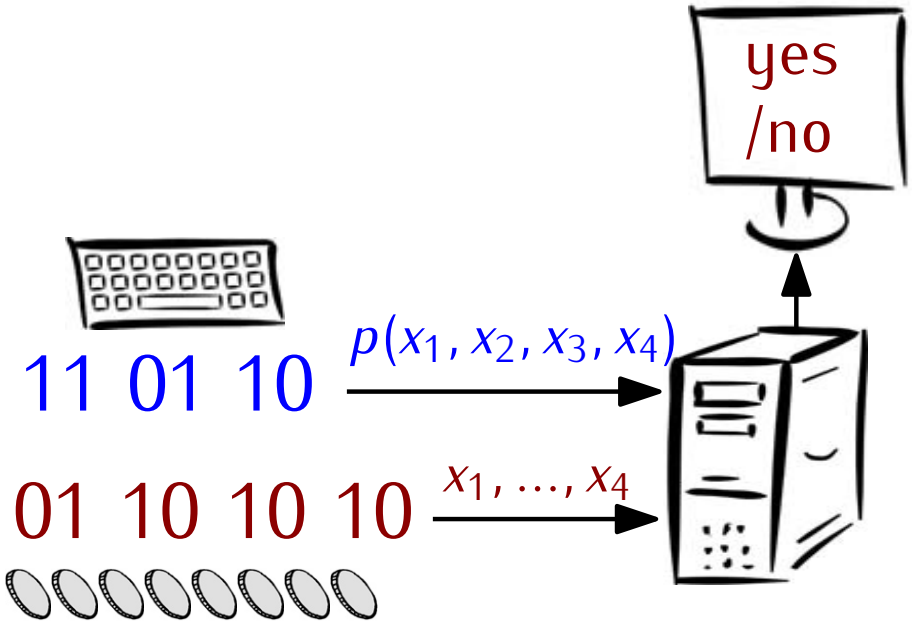


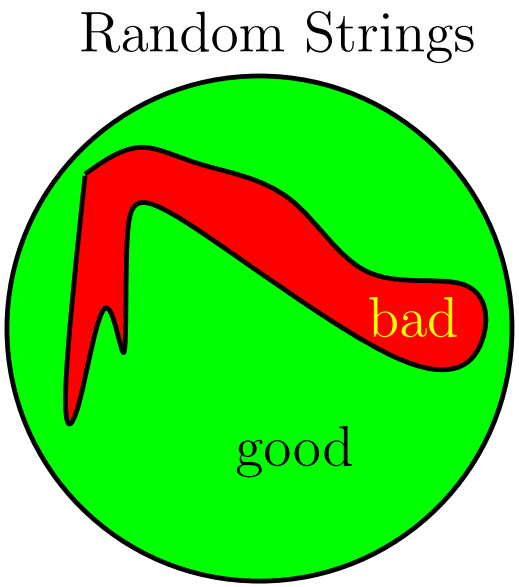


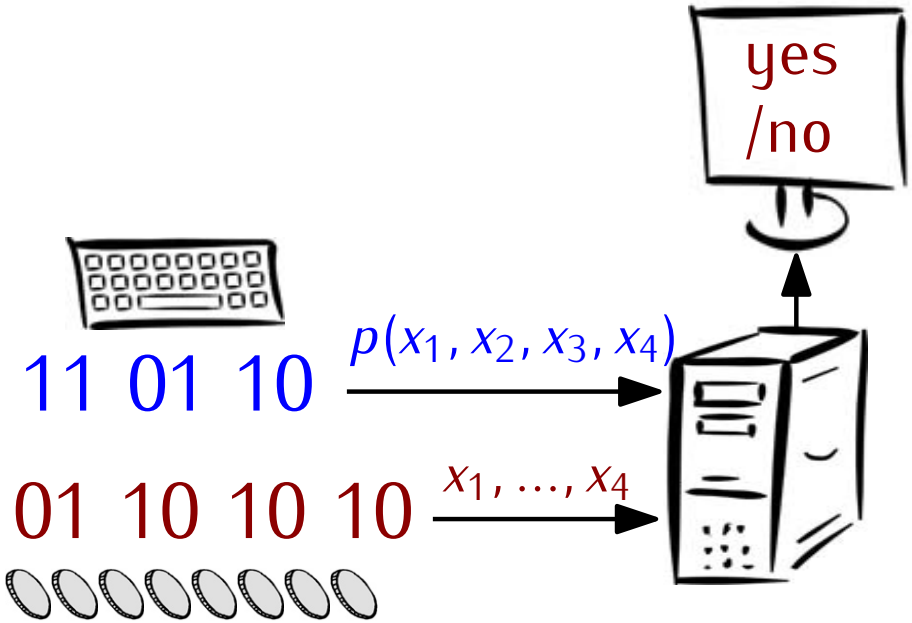
- Bounded error: For every fixed input, correct with Pr $> 99\%$
- **BPP**: Bounded-error Probabilistic Poly time **BPL**: Log space

Typically-Correct Derandomization

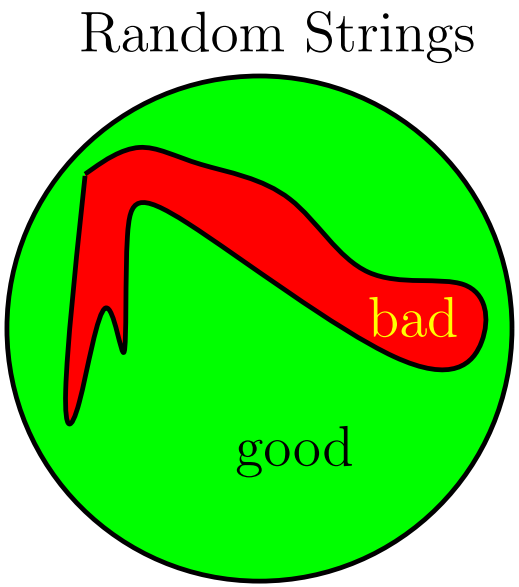
Naive Derandomization

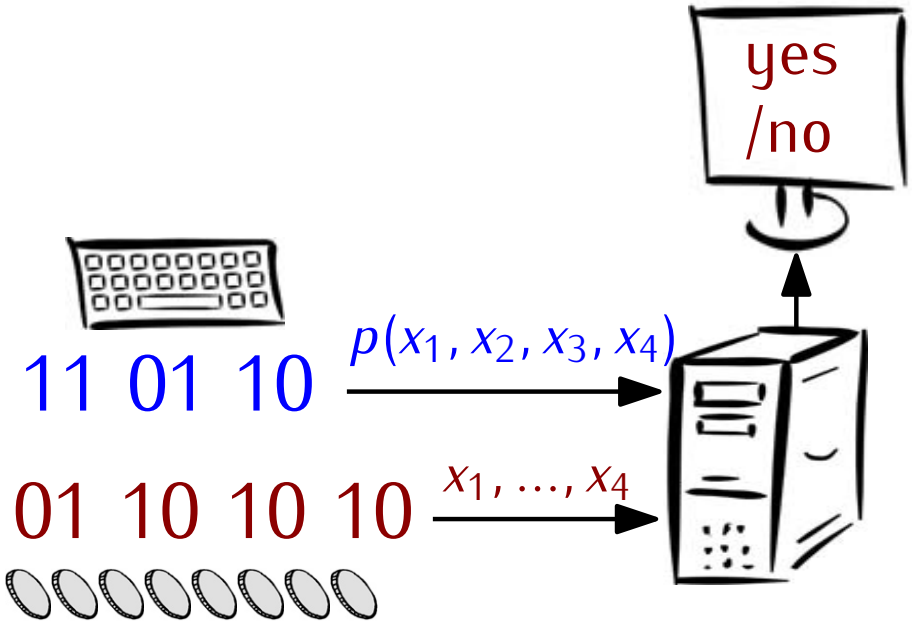
Naive Derandomization





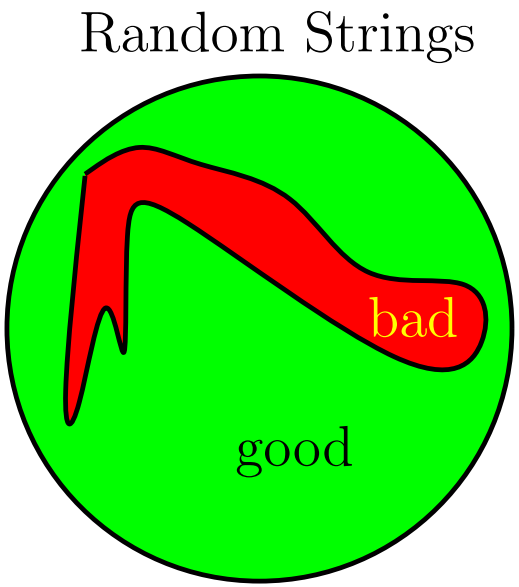
Naive Derandomization

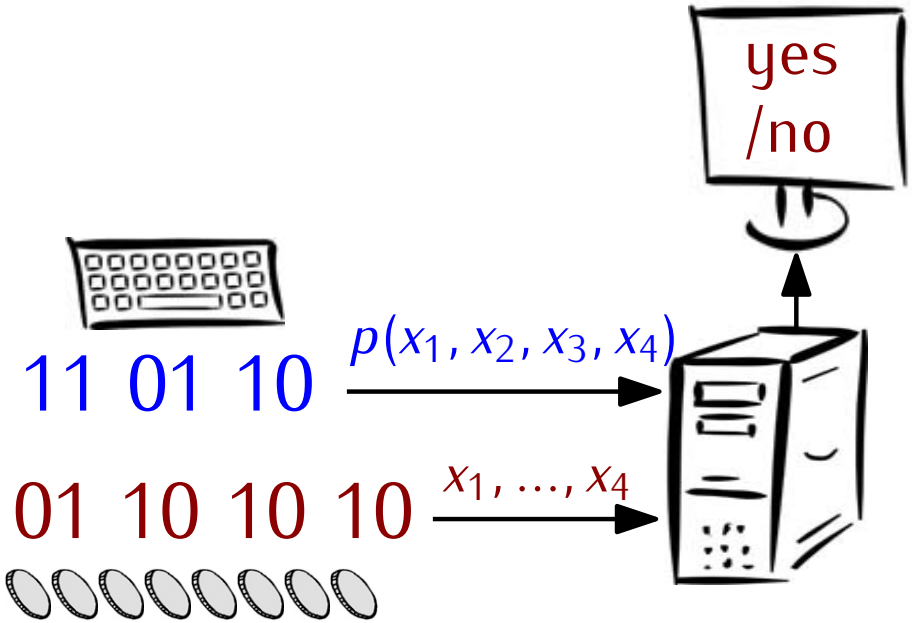




Try all possible random bit strings

Naive Derandomization





Try all possible random bit strings – exponentially many

Derandomization – the Standard PRG Approach

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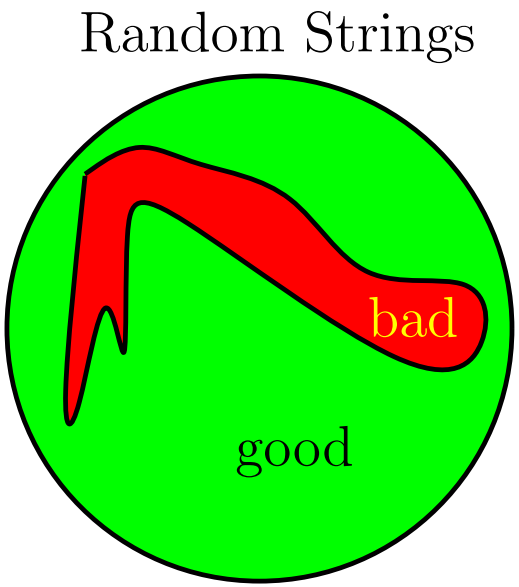

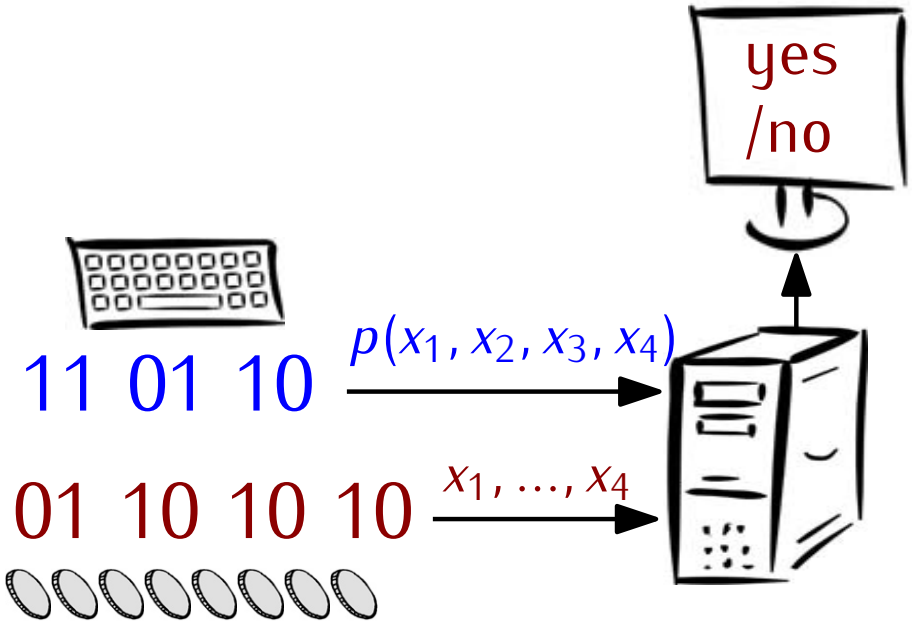

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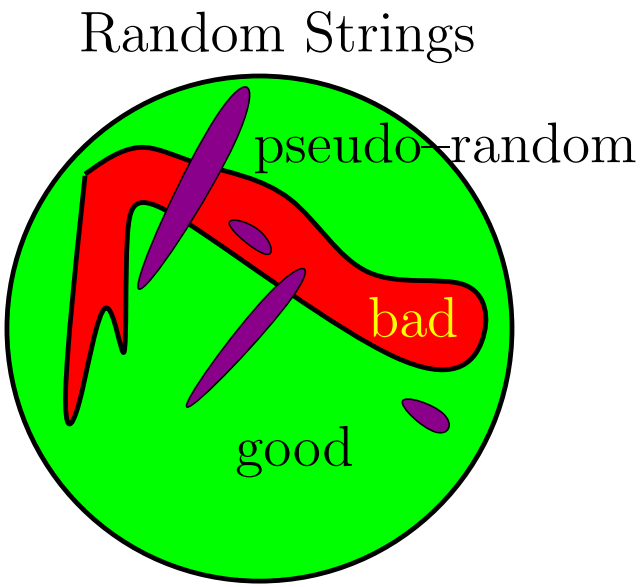

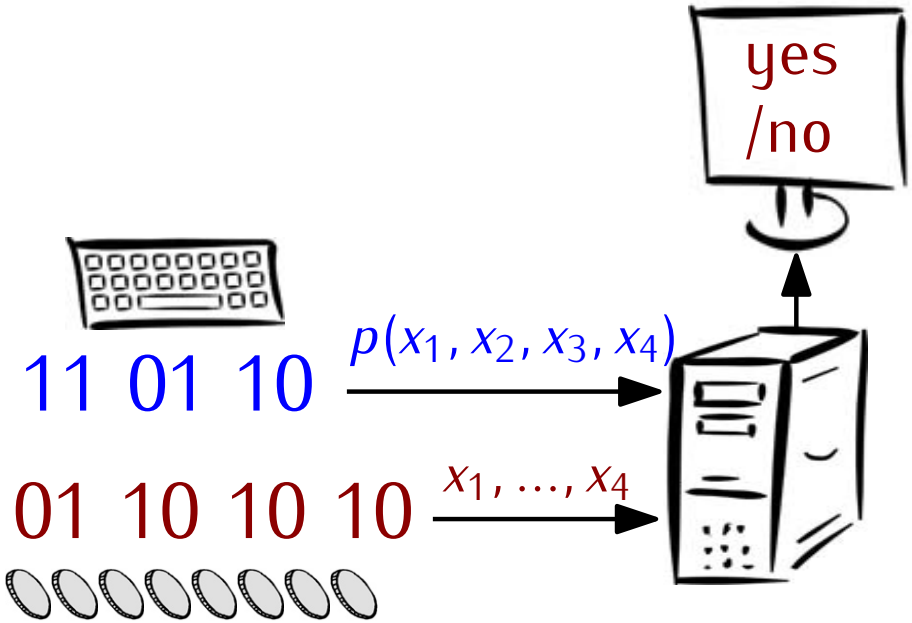

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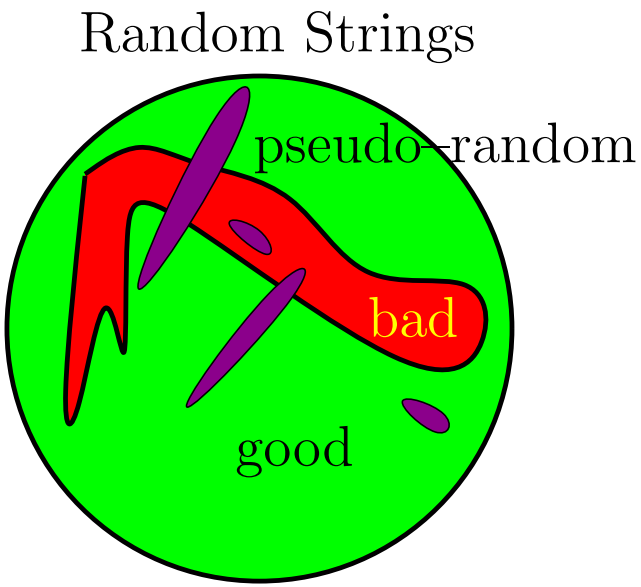
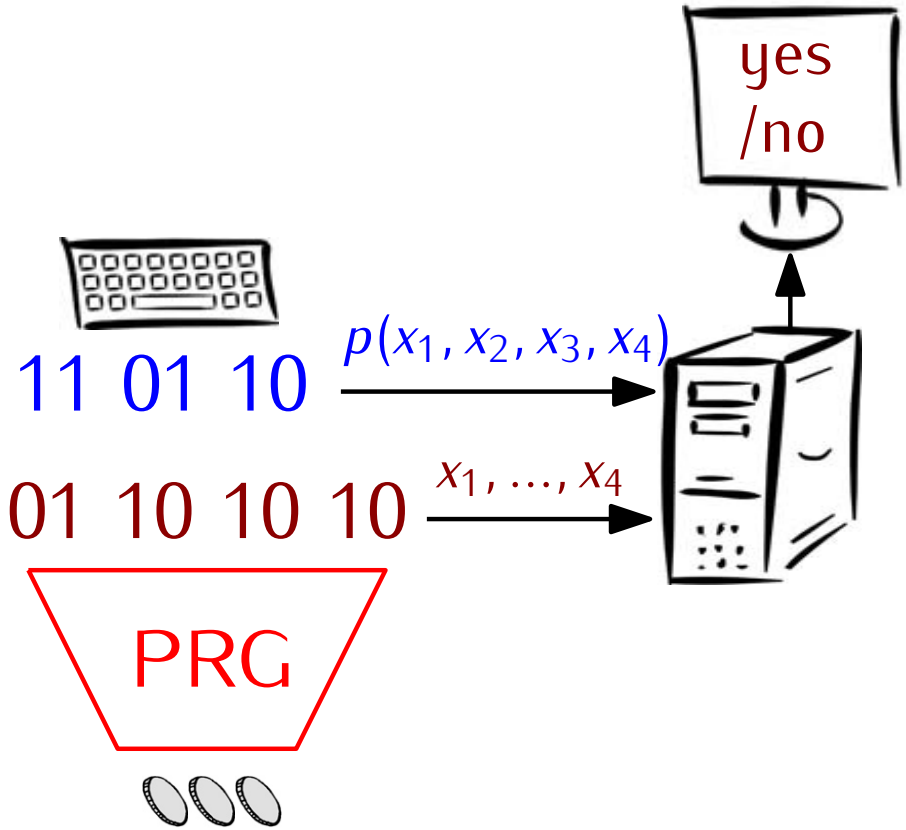
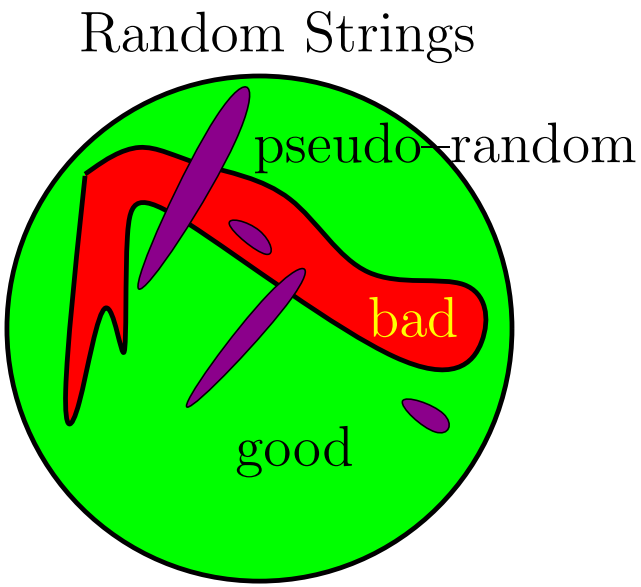
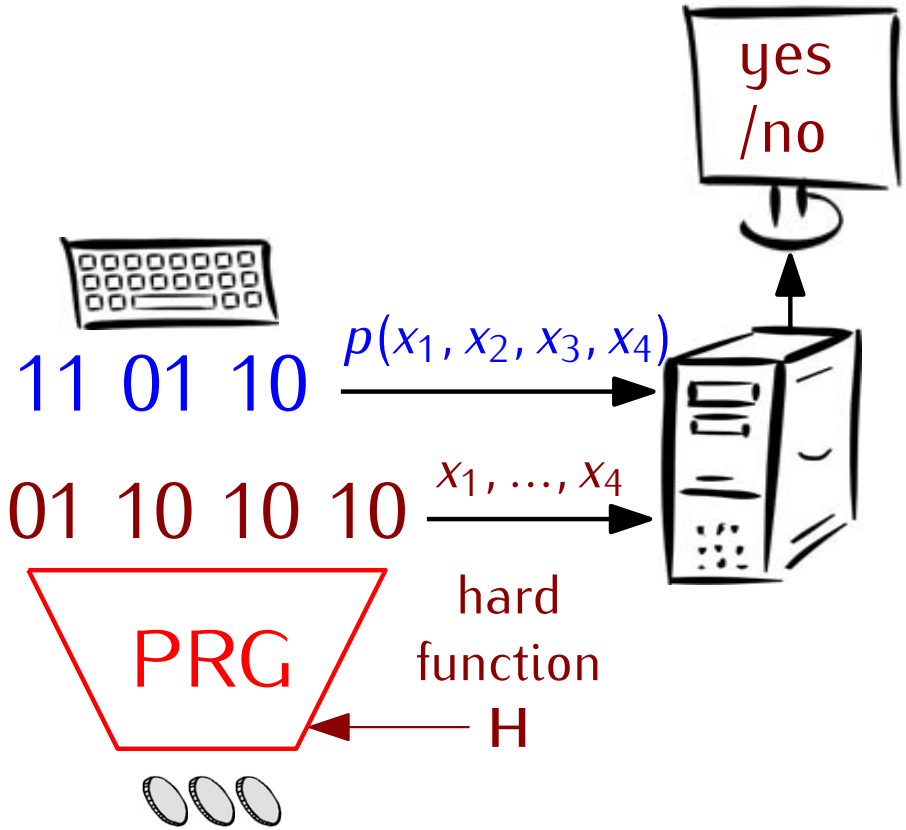




Derandomization – the Standard PRG Approach





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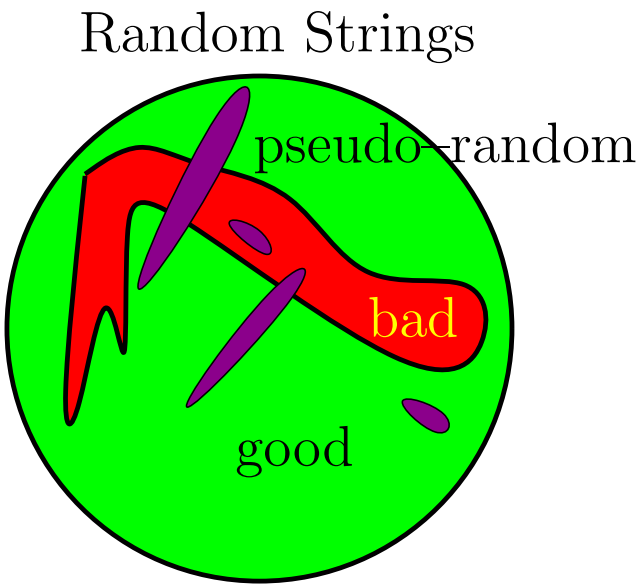

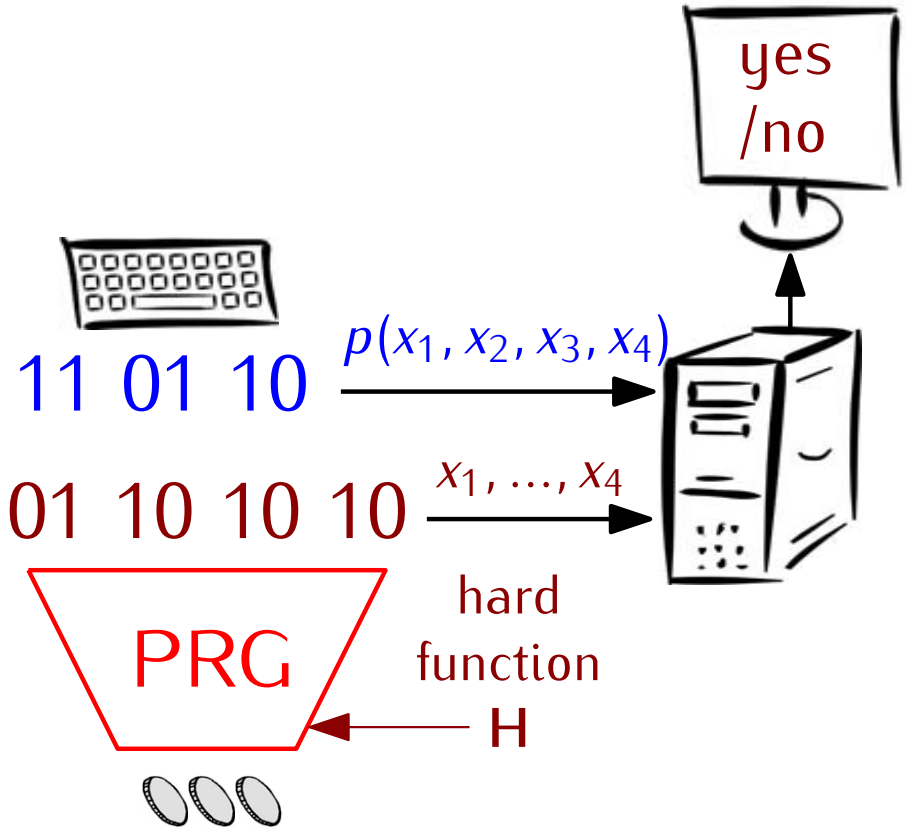

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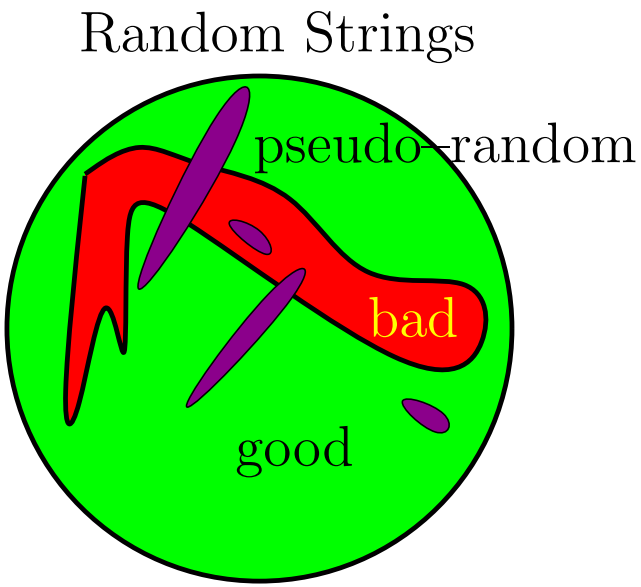

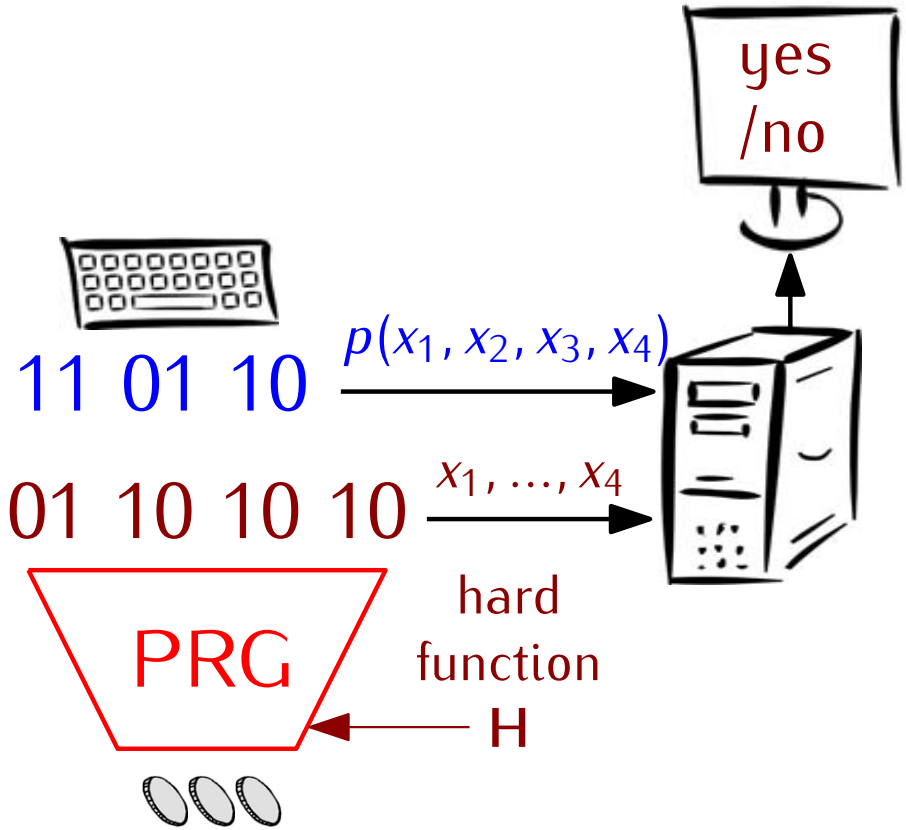

Poly many strings to try $\Rightarrow$ O(log n) seed, exp stretch

A New Approach – Typically-Correct Derandomization

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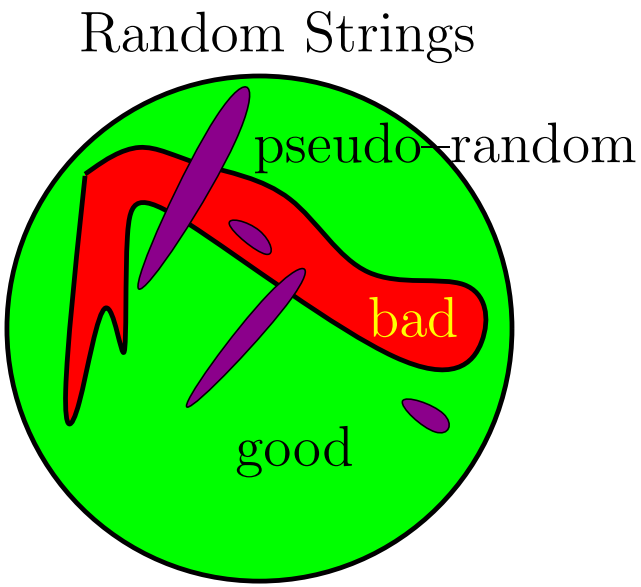

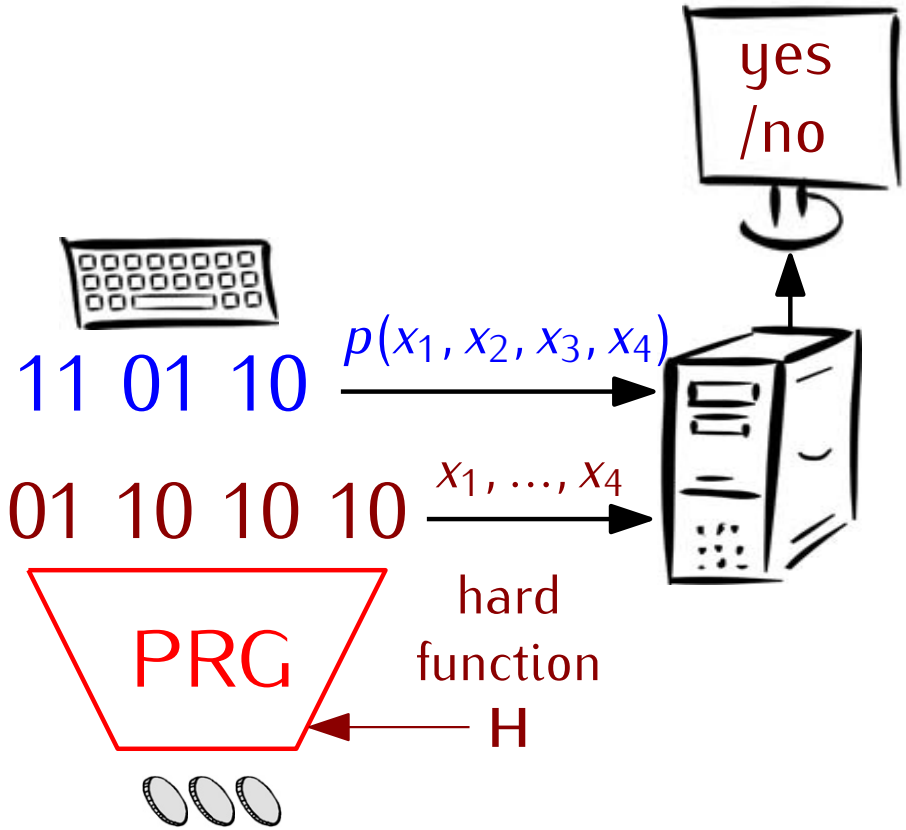

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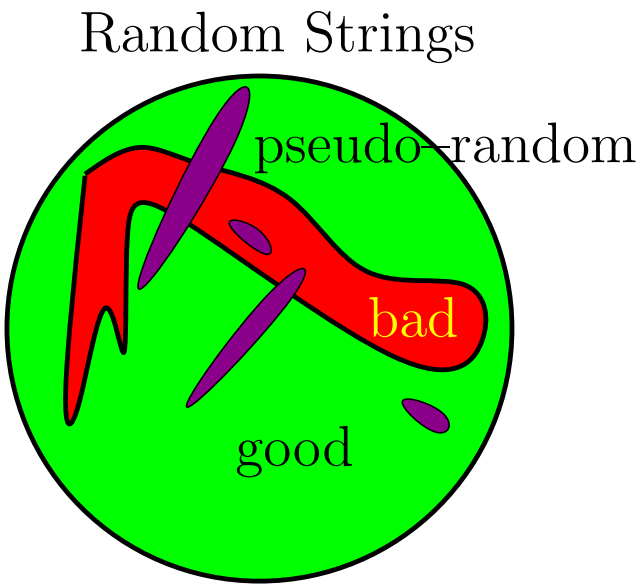

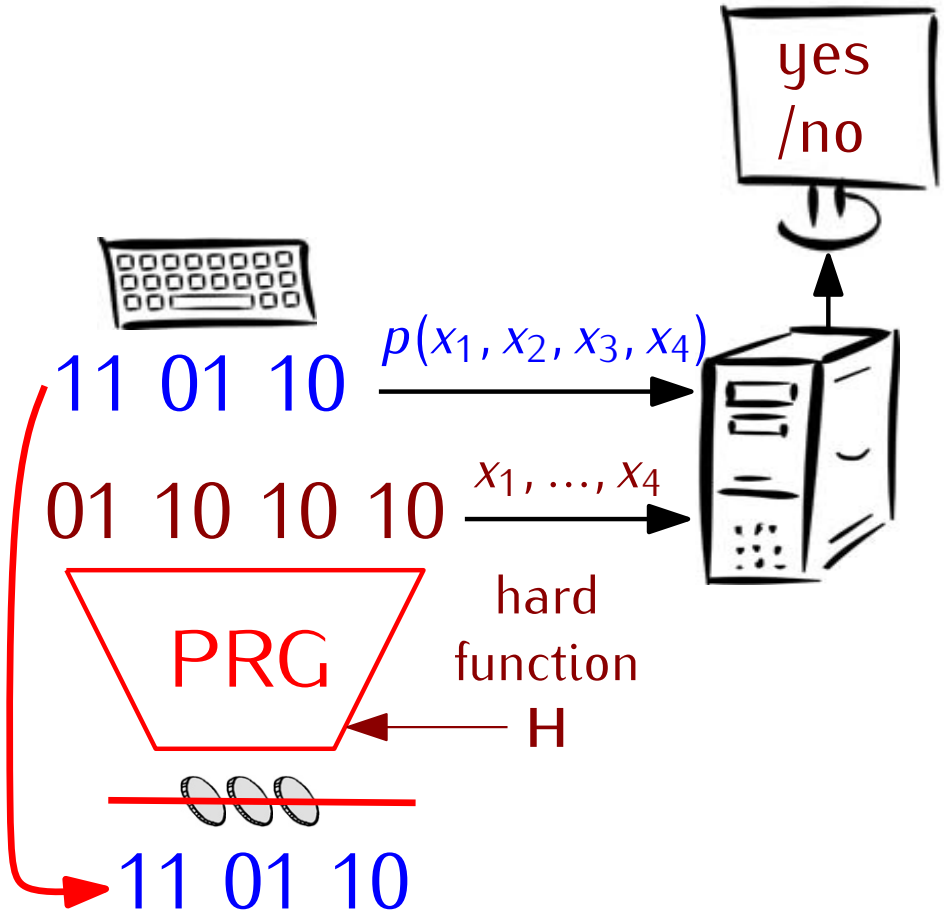

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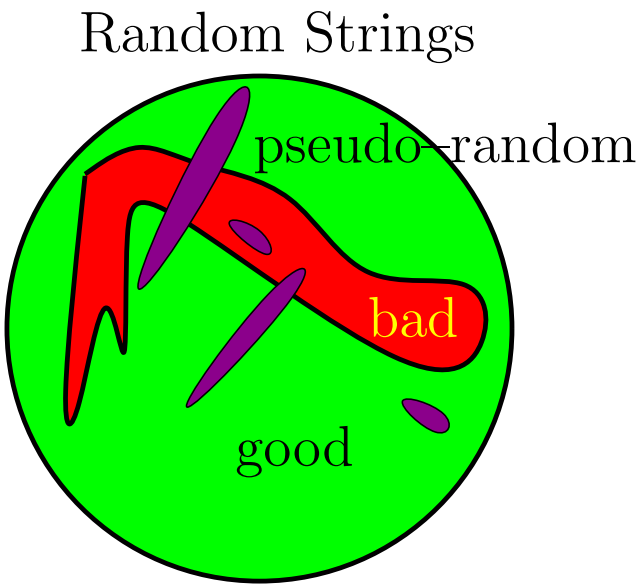

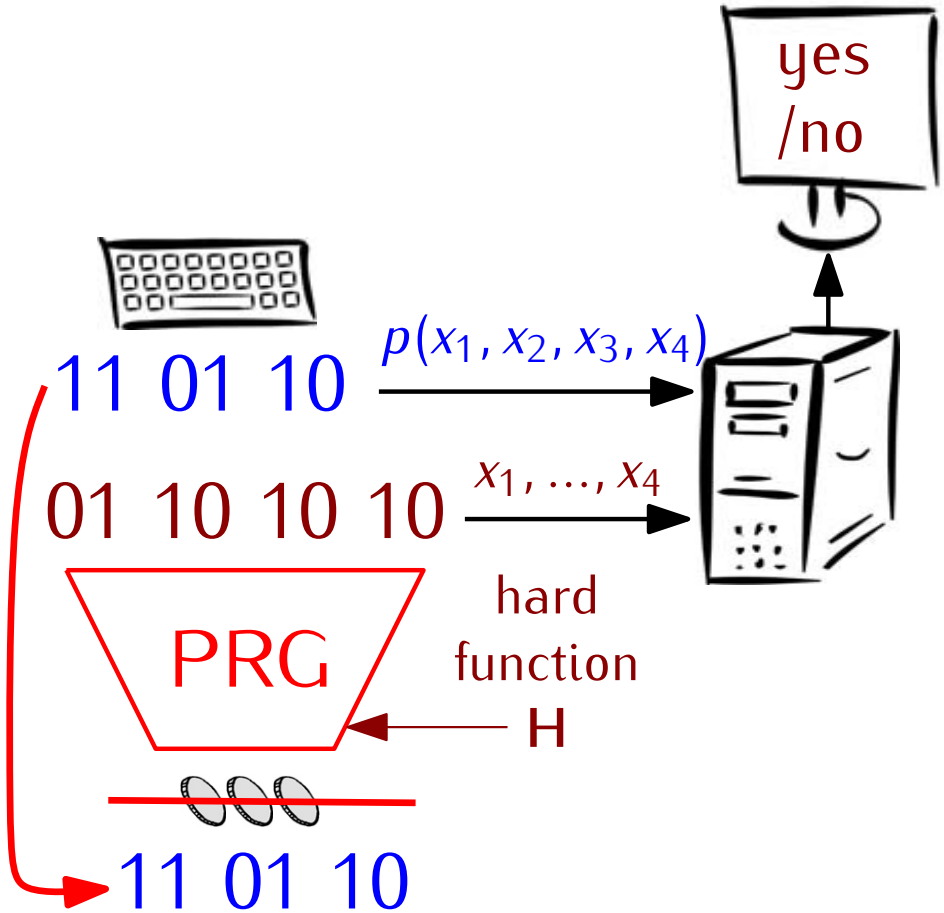

Seed length n, poly stretch

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Randomized algorithm M(x, r) computing language L

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Derandomization – Yes! [NW, ...]

Standard Use of PRG's vs. Typ-Correct

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Standard Derandomization	**Typ-Correct Derandomization**
[Nisan & Wigderson, ...]	

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- Similar conditional results for AM, BPL, ...

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Other Settings

Multi-party communication protocols

Circuit Lower Bounds and Derandomization

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[Kabanets-Impagliazzo]

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- Simpler proof for everywhere-correct setting

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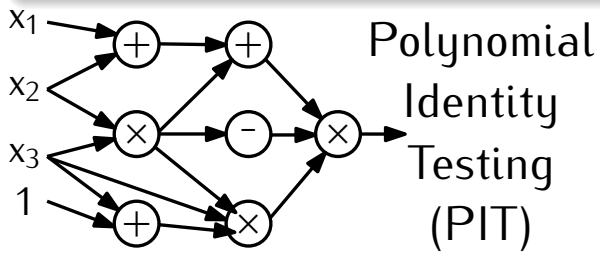

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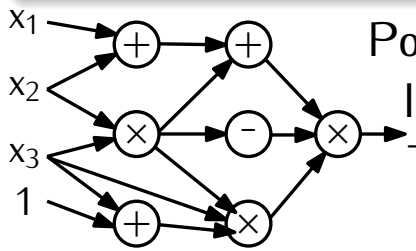

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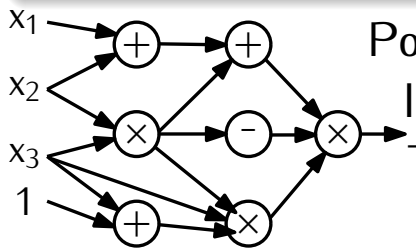

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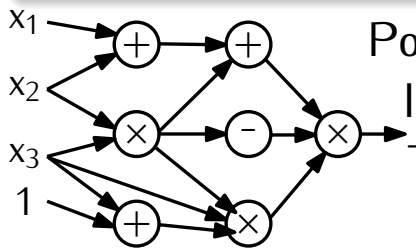

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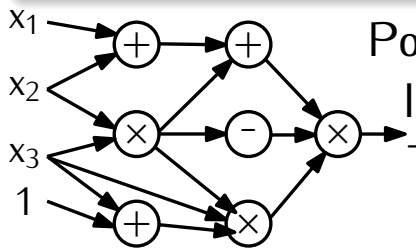

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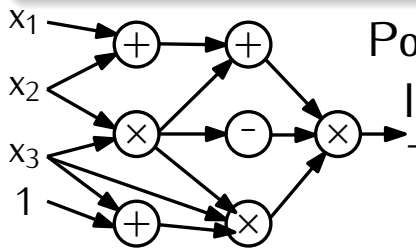

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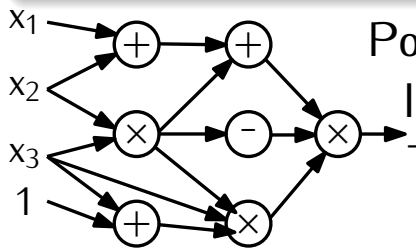

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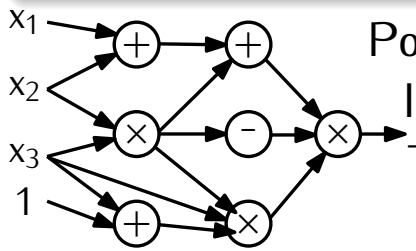

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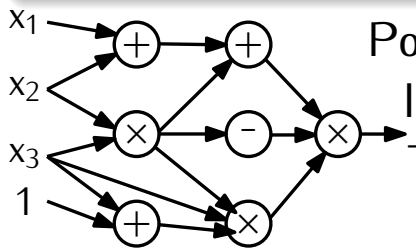

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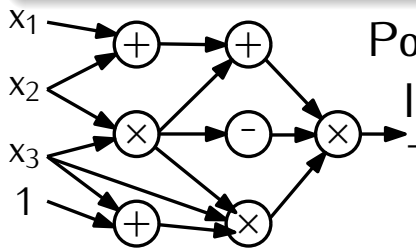

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- What if assumed PIT alg is only typically-correct?

Typically-Correct Derand Implies Circuit Lower Bounds

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NSUBEXP algorithm M computes PIT with $\leq 2^{n^\epsilon}$ errors
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    - Hierarchy theorems for randomized algorithms

Outline

How Powerful is Randomness?

- "Typically-correct" Derandomization
- Hierarchy Theorems for Randomized Algorithms
- Derandomizing Monotone Computations

Hierarchy Theorems

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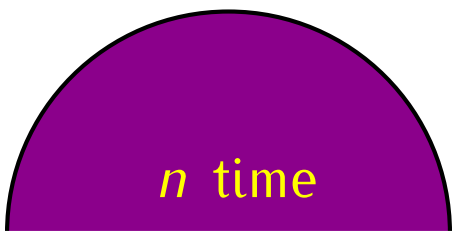

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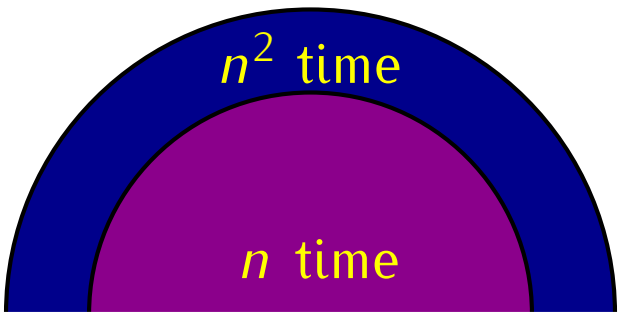

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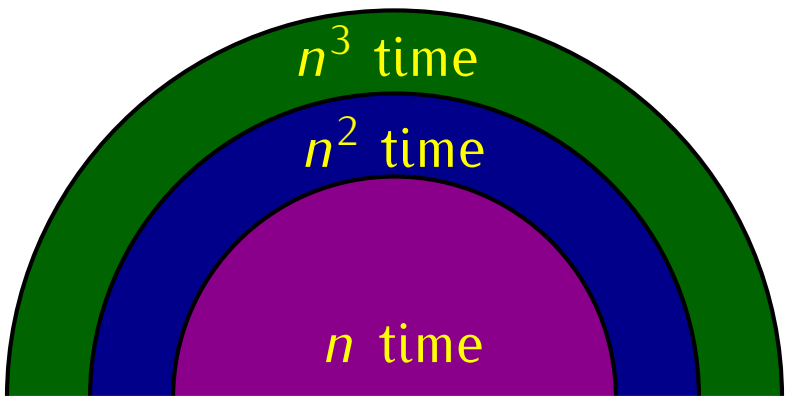

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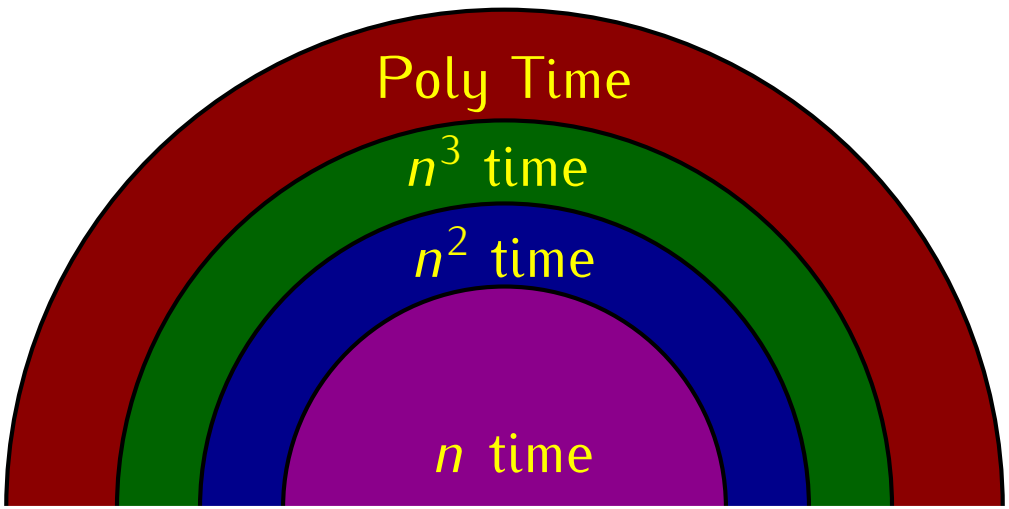

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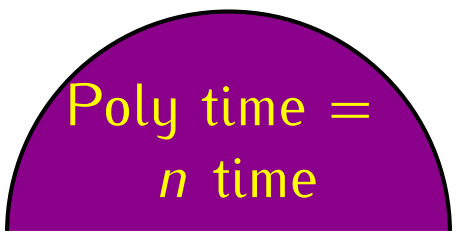

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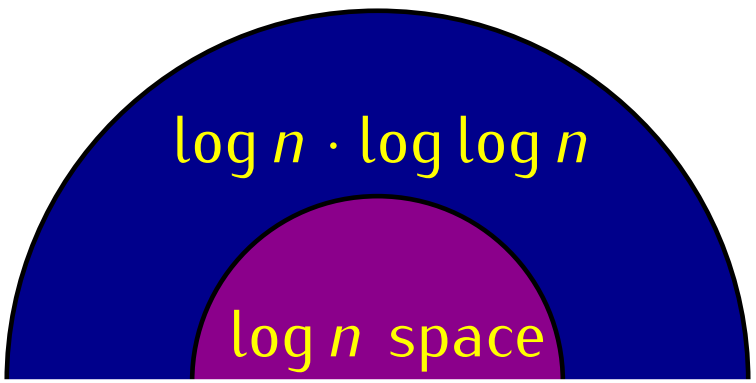

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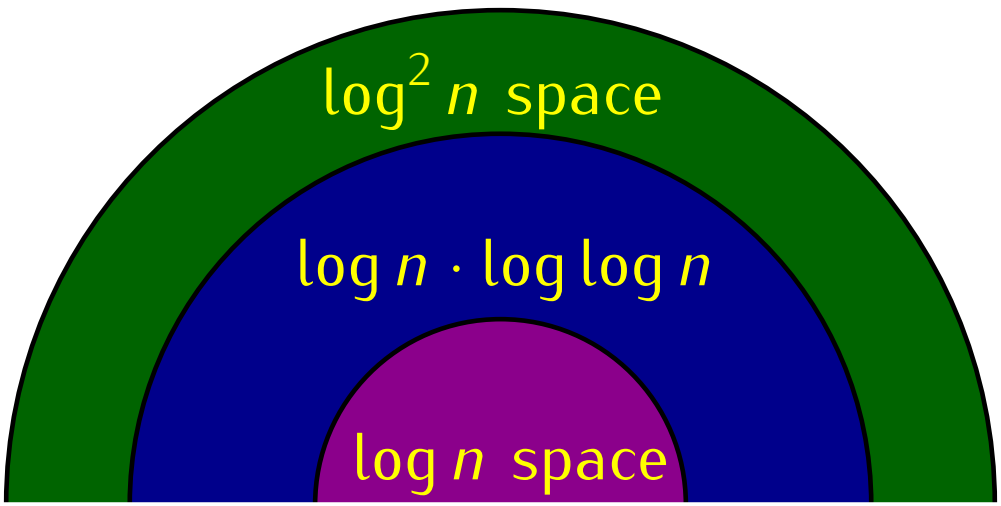

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- Hierarchies without need for derandomization?

Hierarchy Theorems for Deterministic Algorithms

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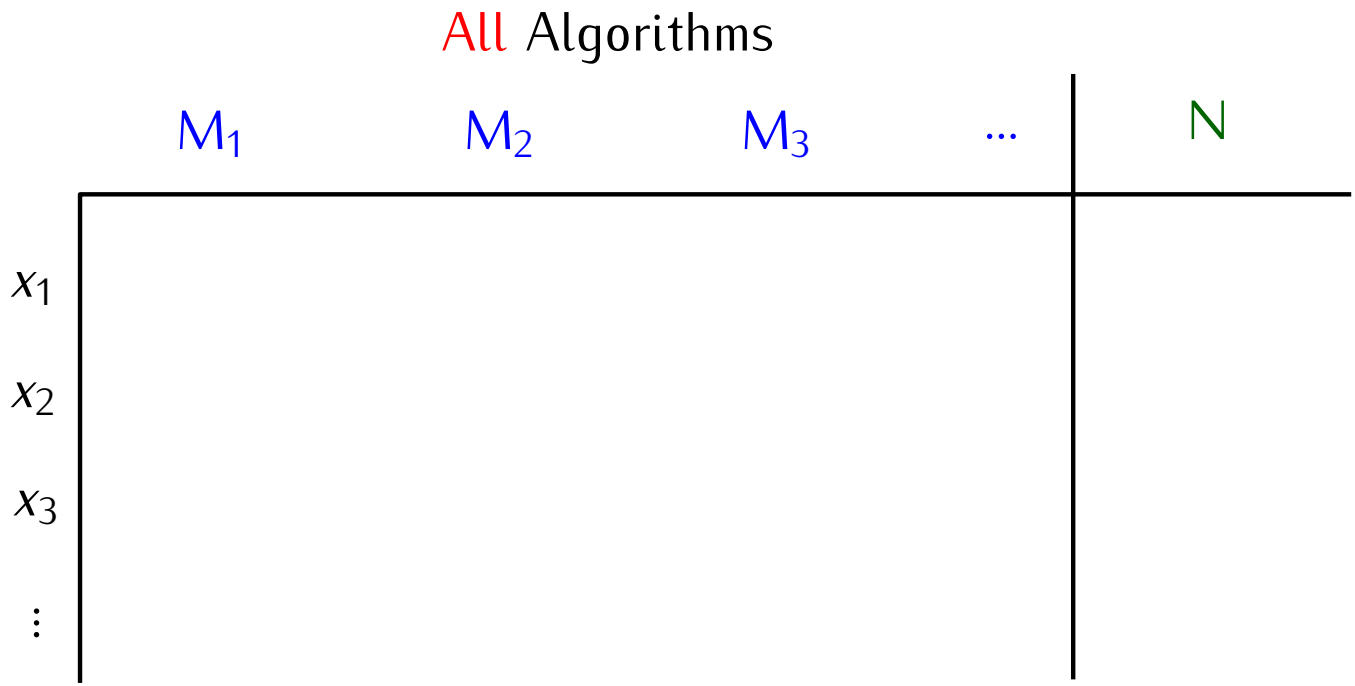
All Algorithms

M_1 M_2 M_3 ...

N

Hierarchy Theorems for Deterministic Algorithms



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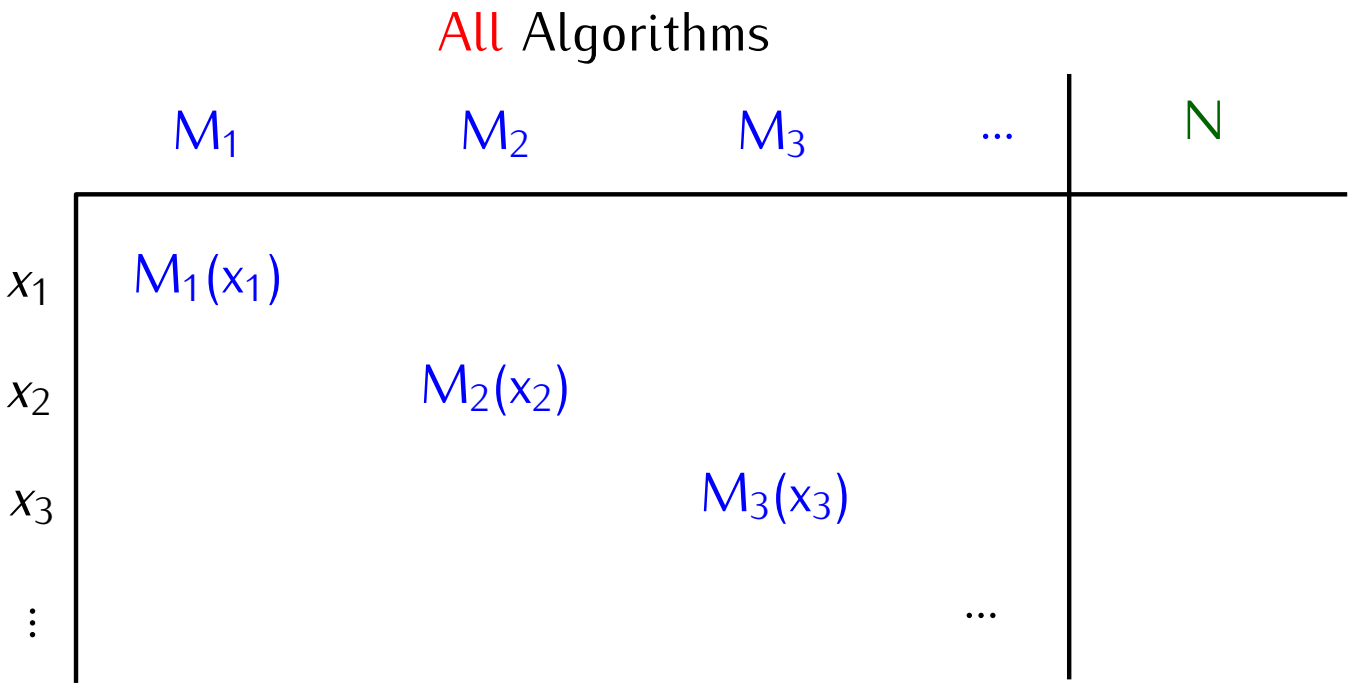

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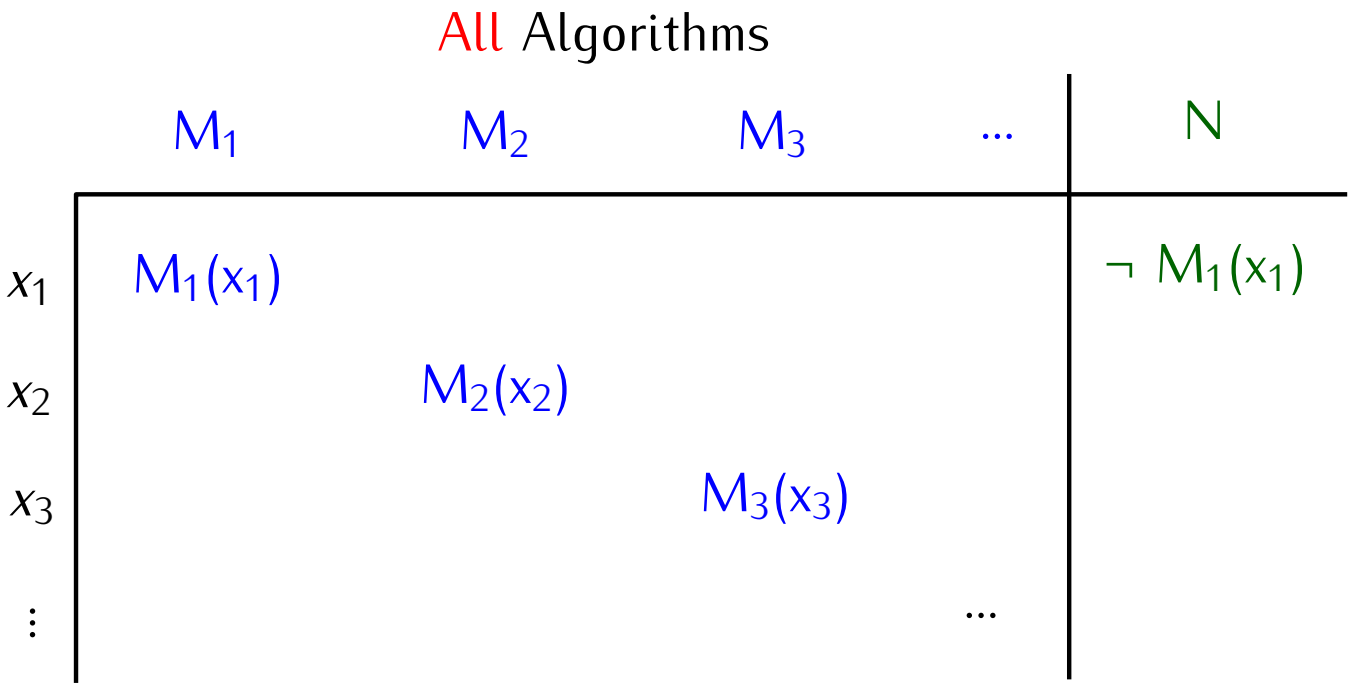

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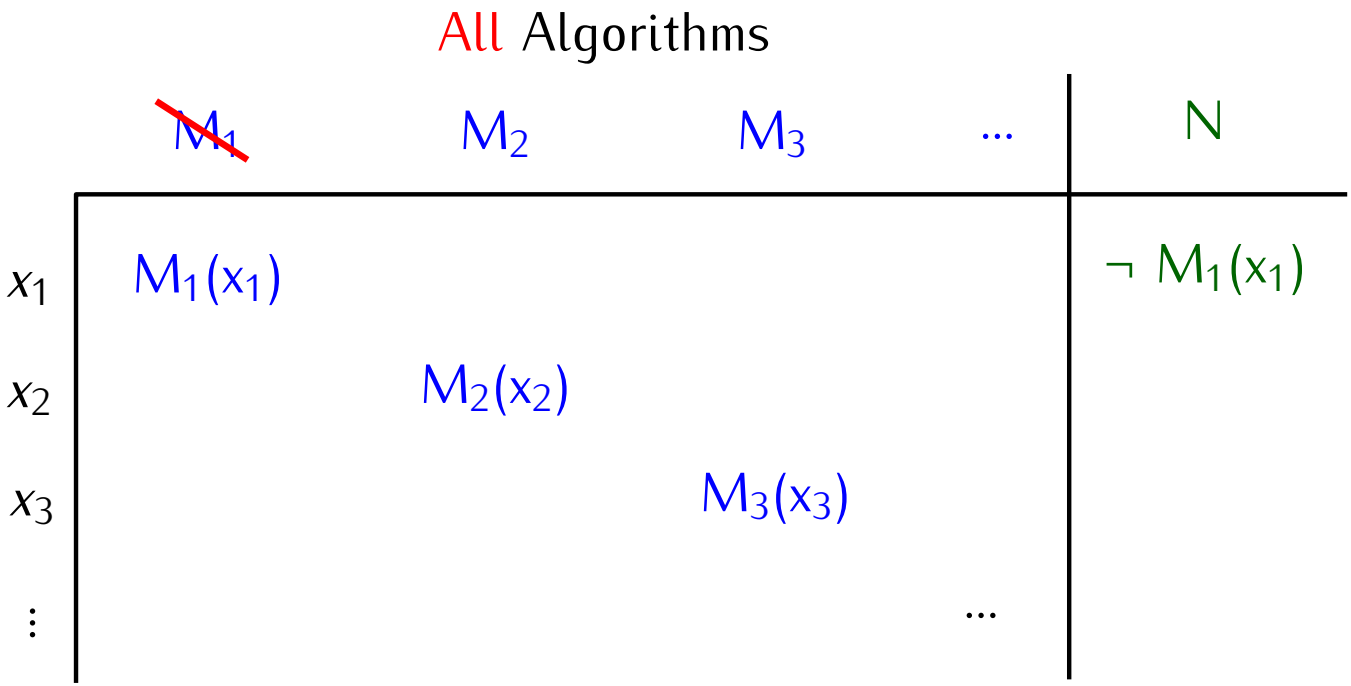

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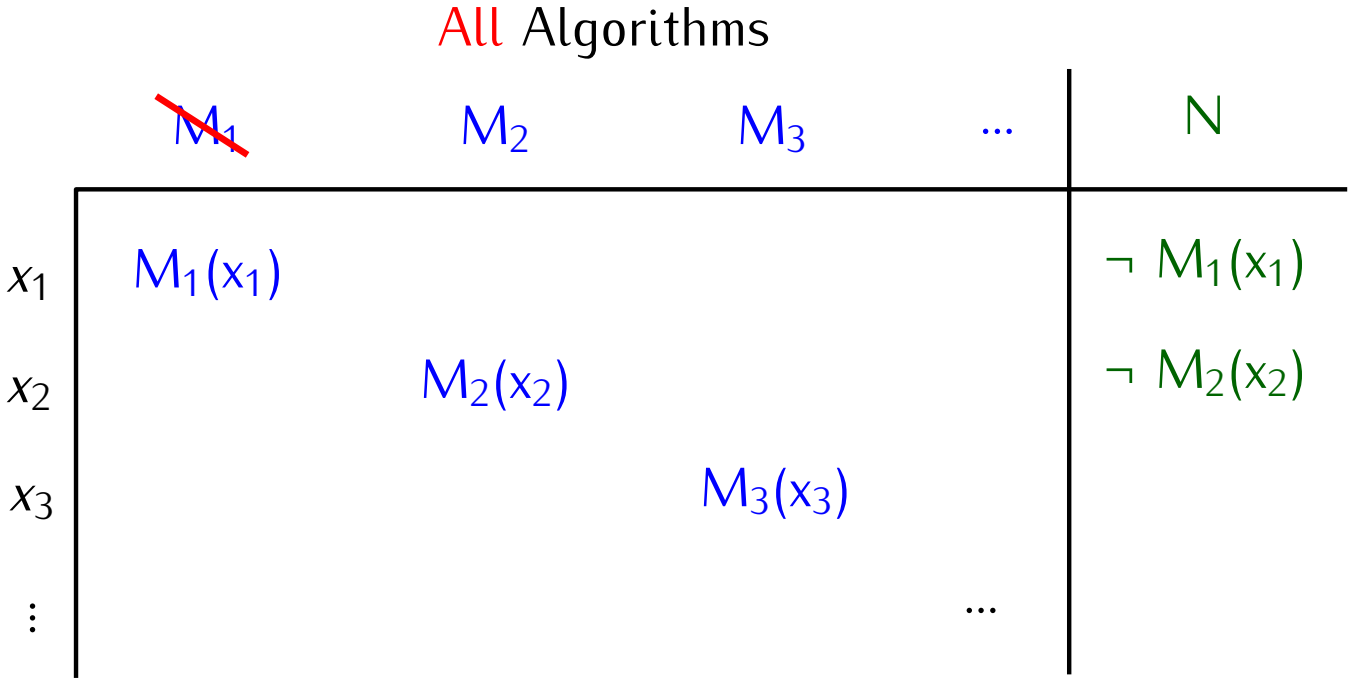

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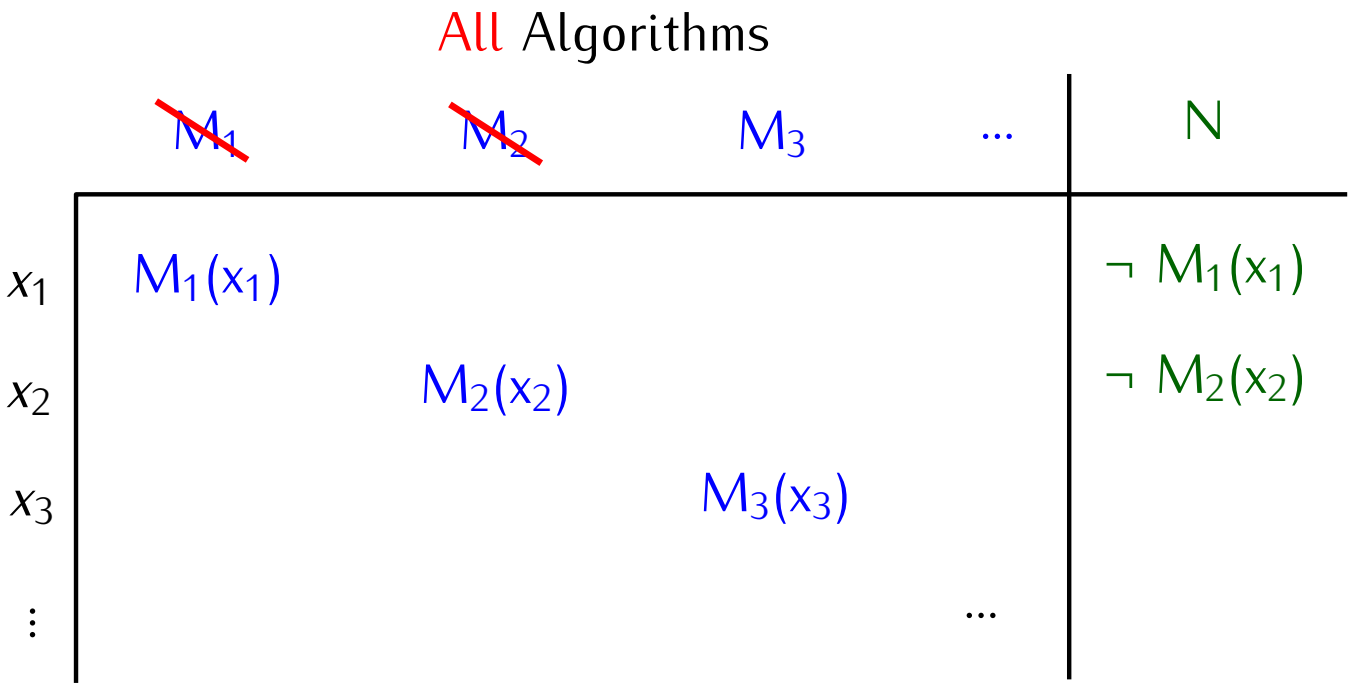

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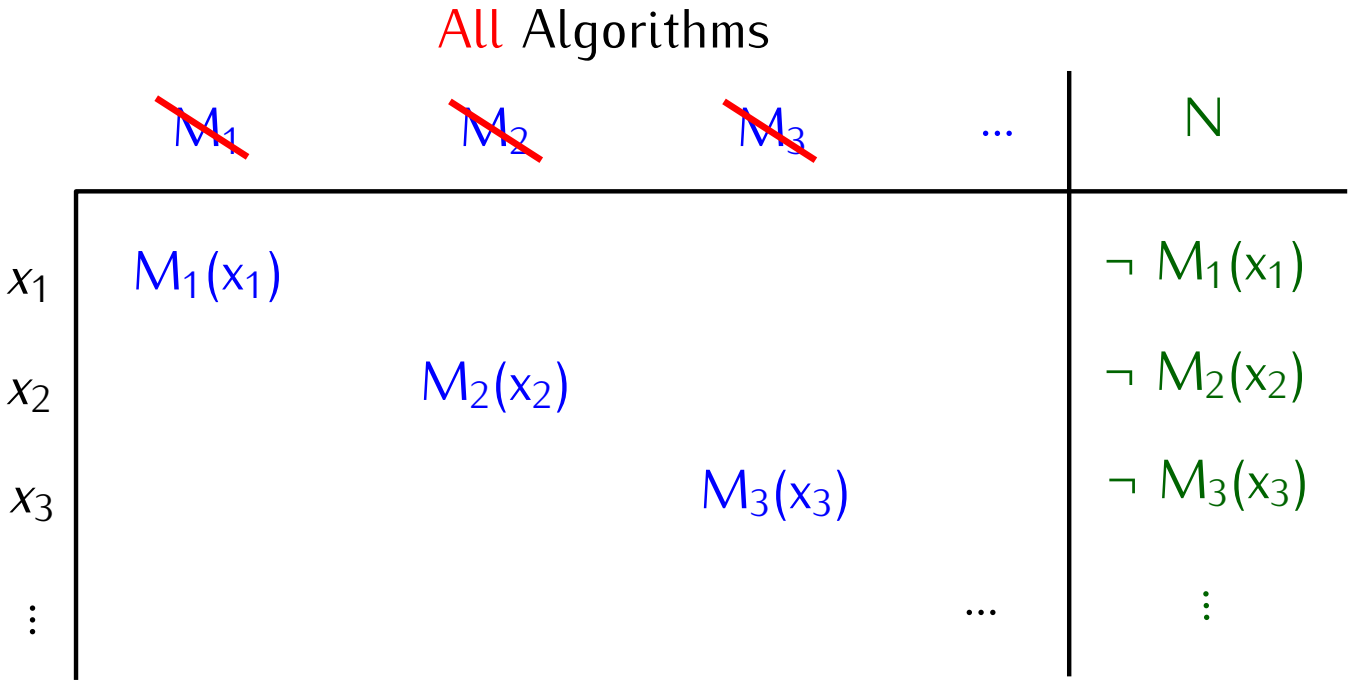

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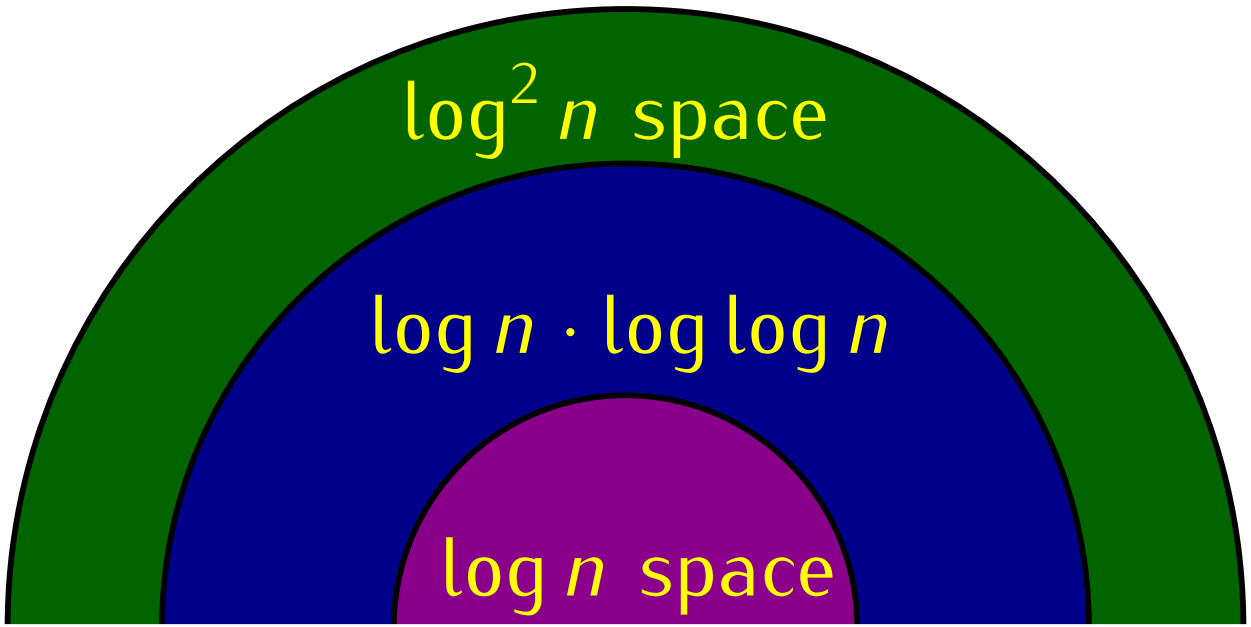

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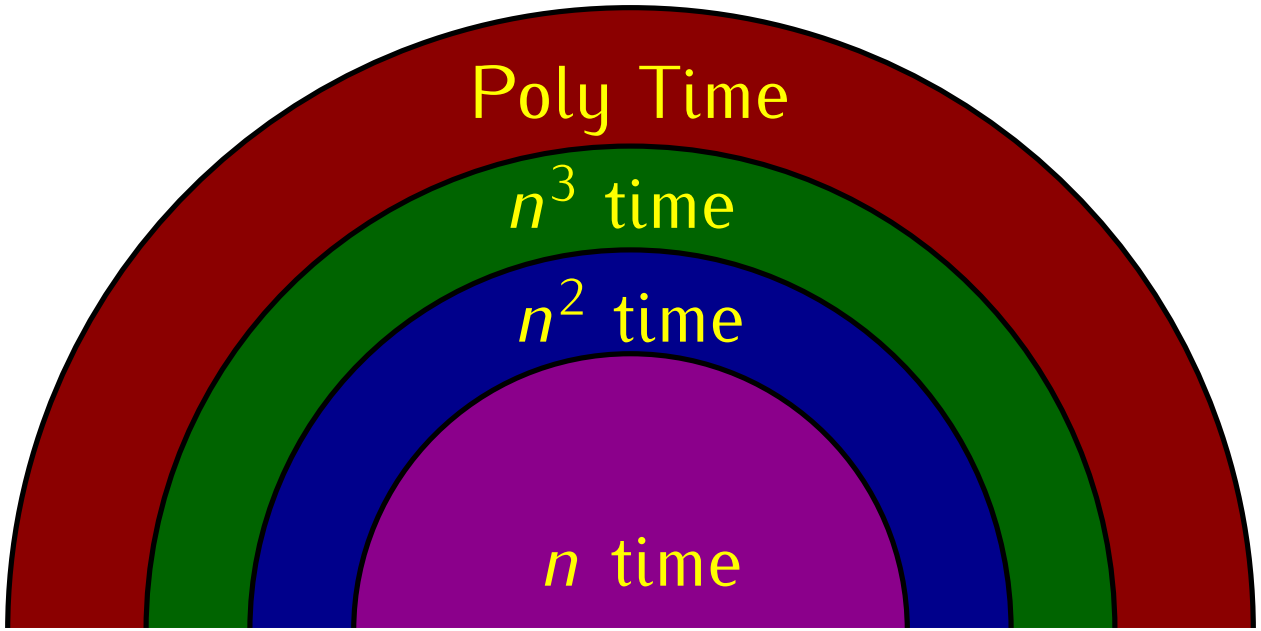

Hierarchy Theorems for **Bounded-Error** Rand Algs?

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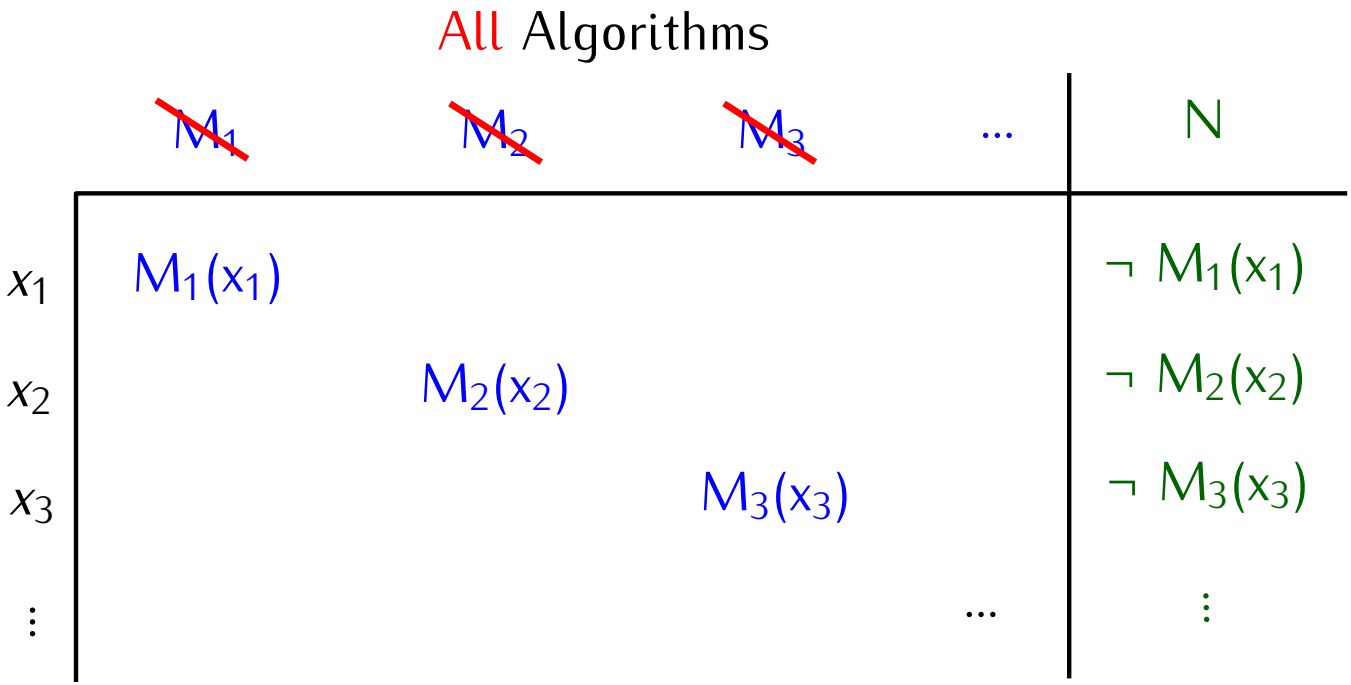

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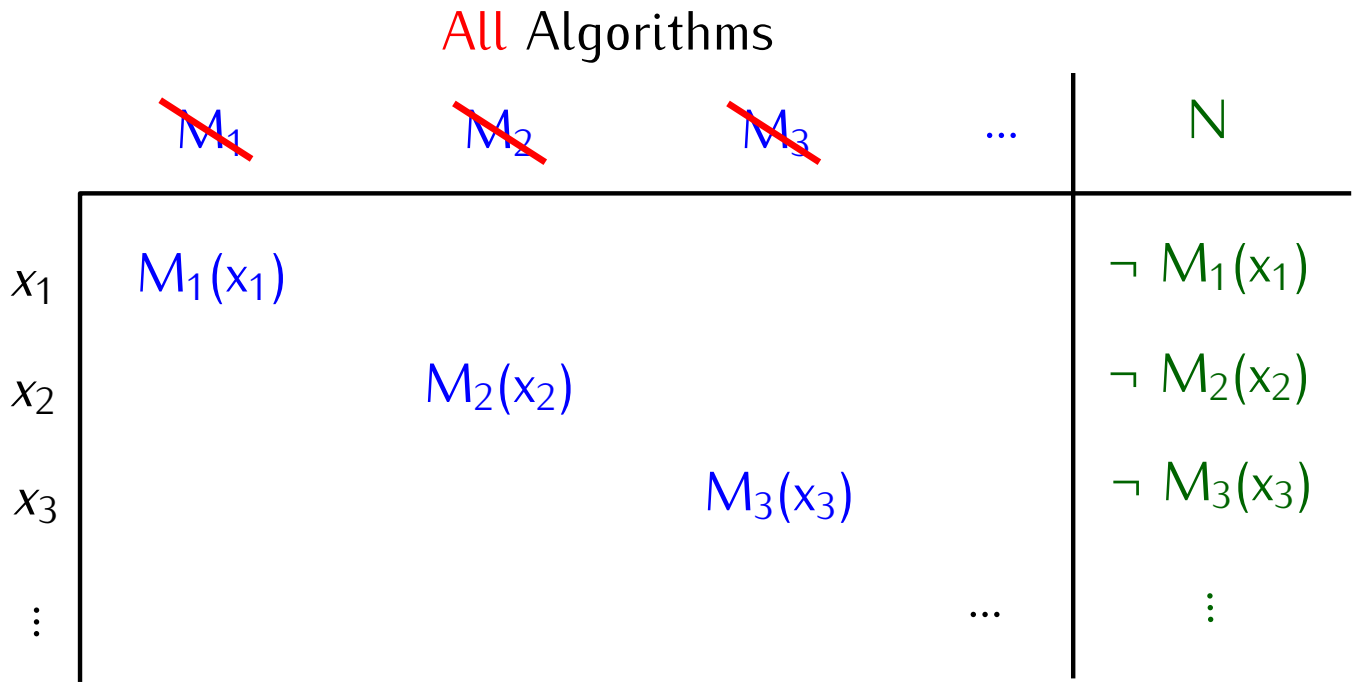

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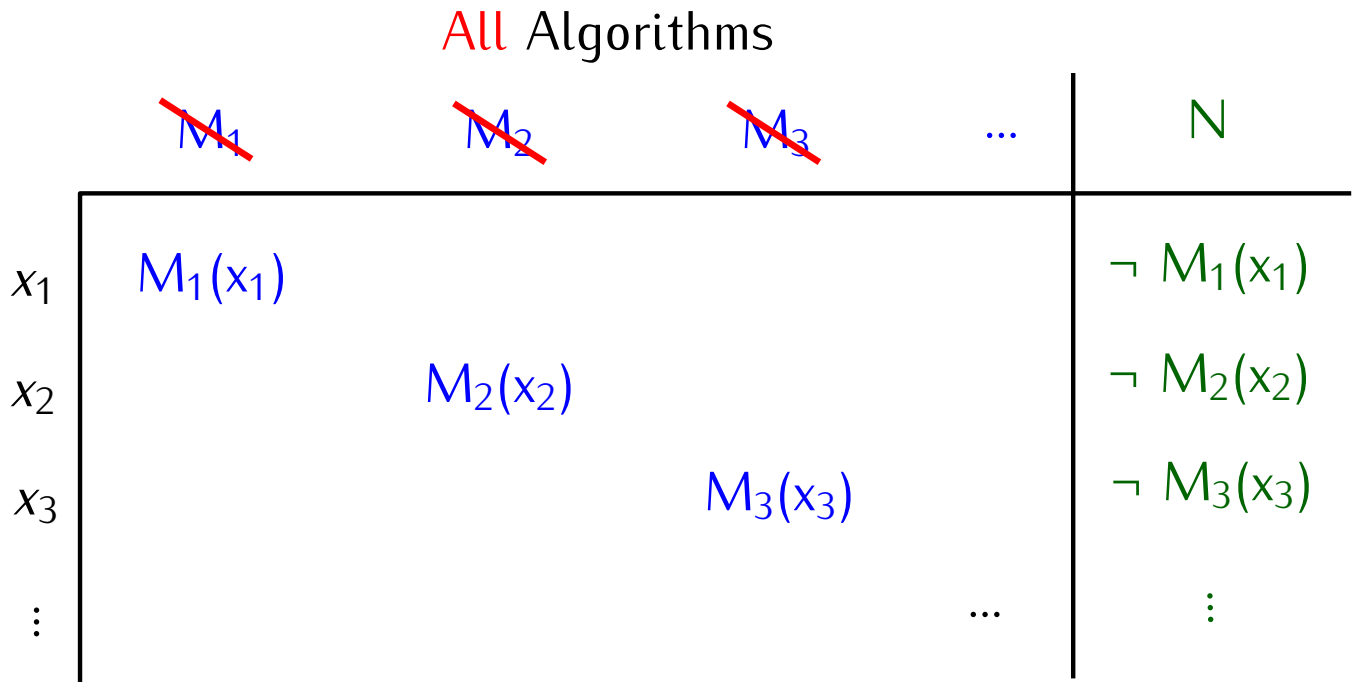

What if $\Pr[M_1(x_1) = \text{“yes”}] \approx .5$?$\Rightarrow$ N not *bounded error*

Best-known Hierarchies Theorems for Randomized Algs

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- **Best possible: SPACE(s') $\not\subseteq$ SPACE(s) for any s'$=\omega$(s)**

Tight Hierarchies with One Bit of Advice, Log Space

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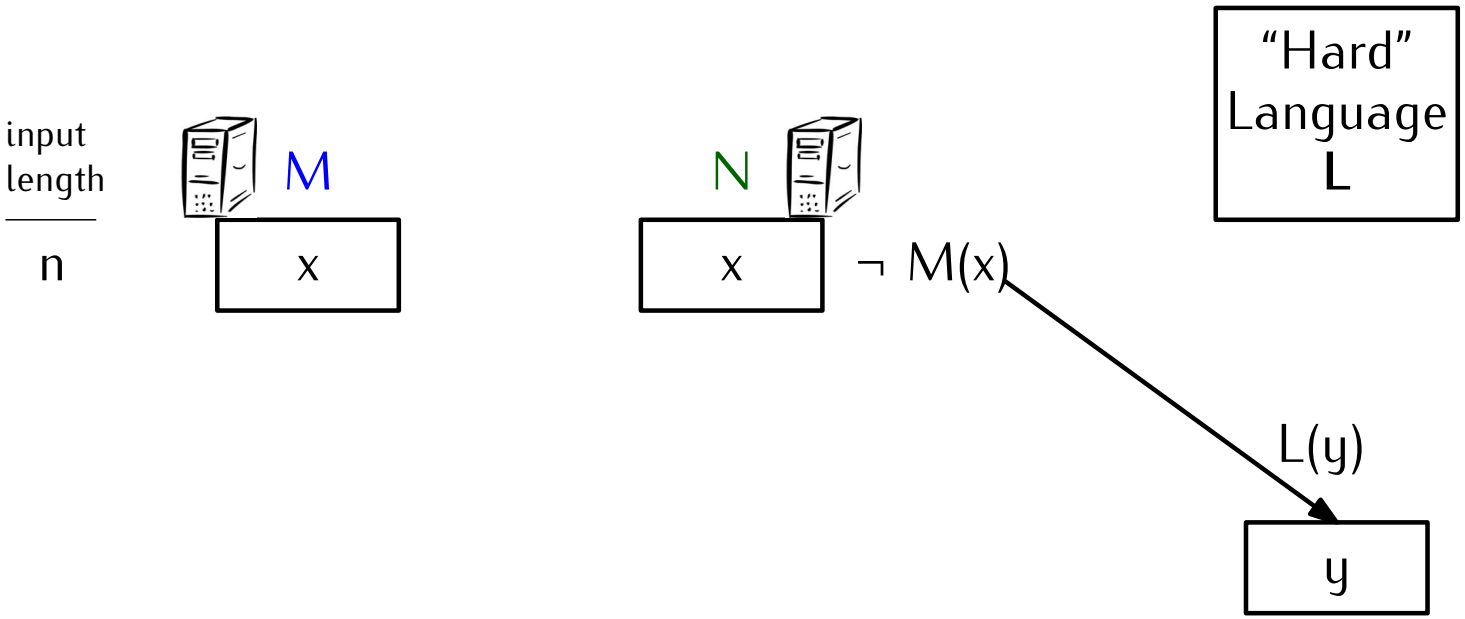

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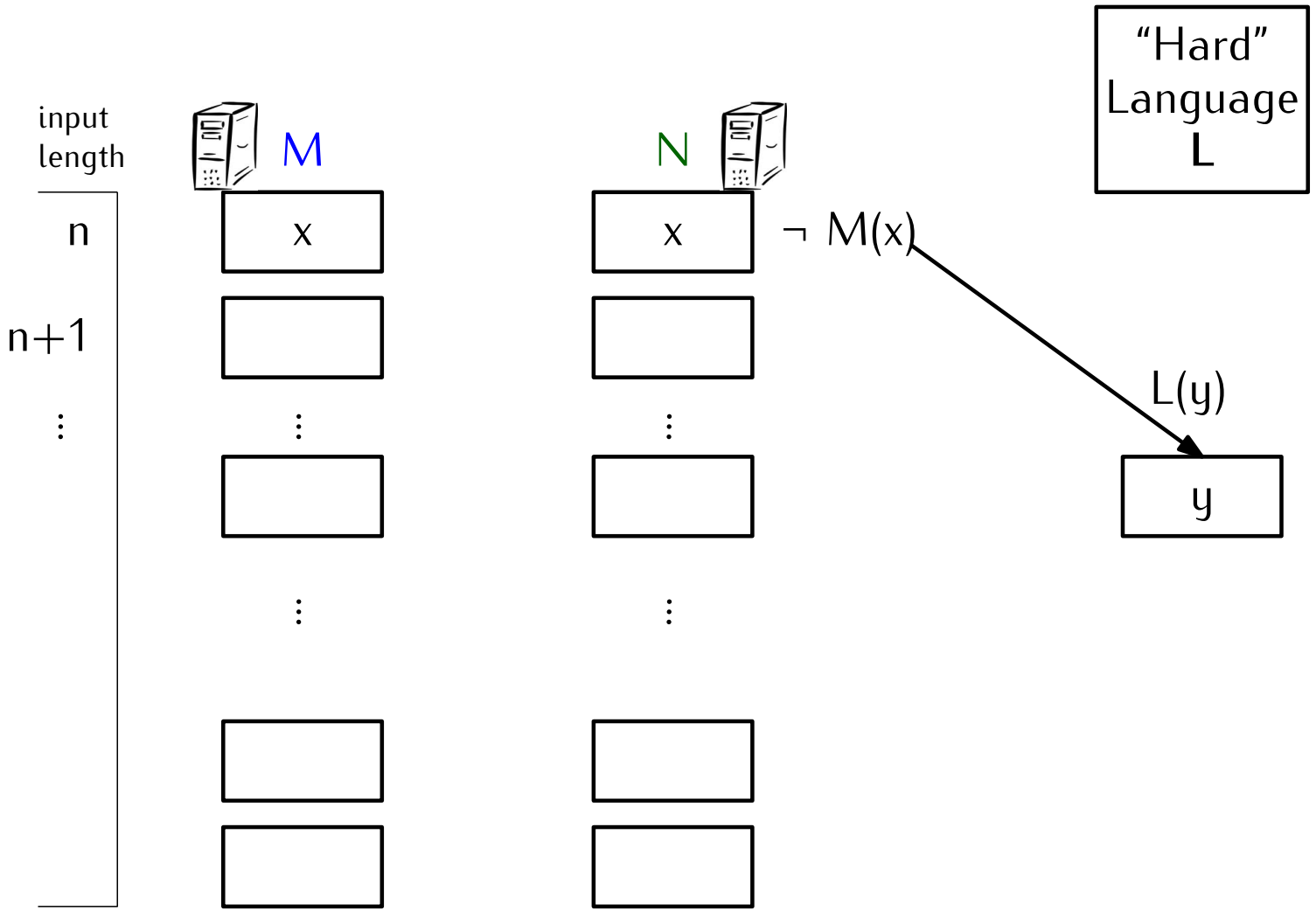

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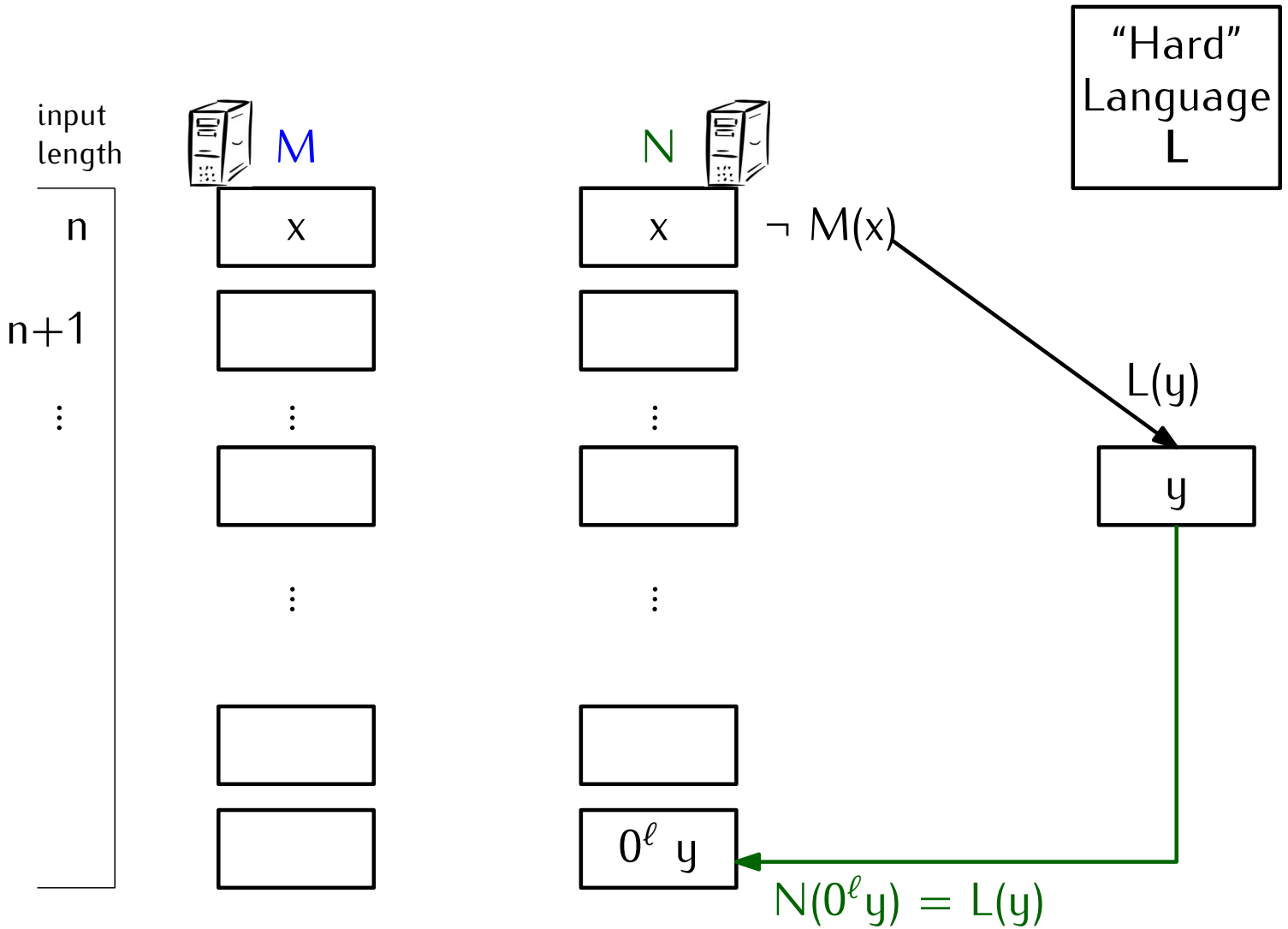

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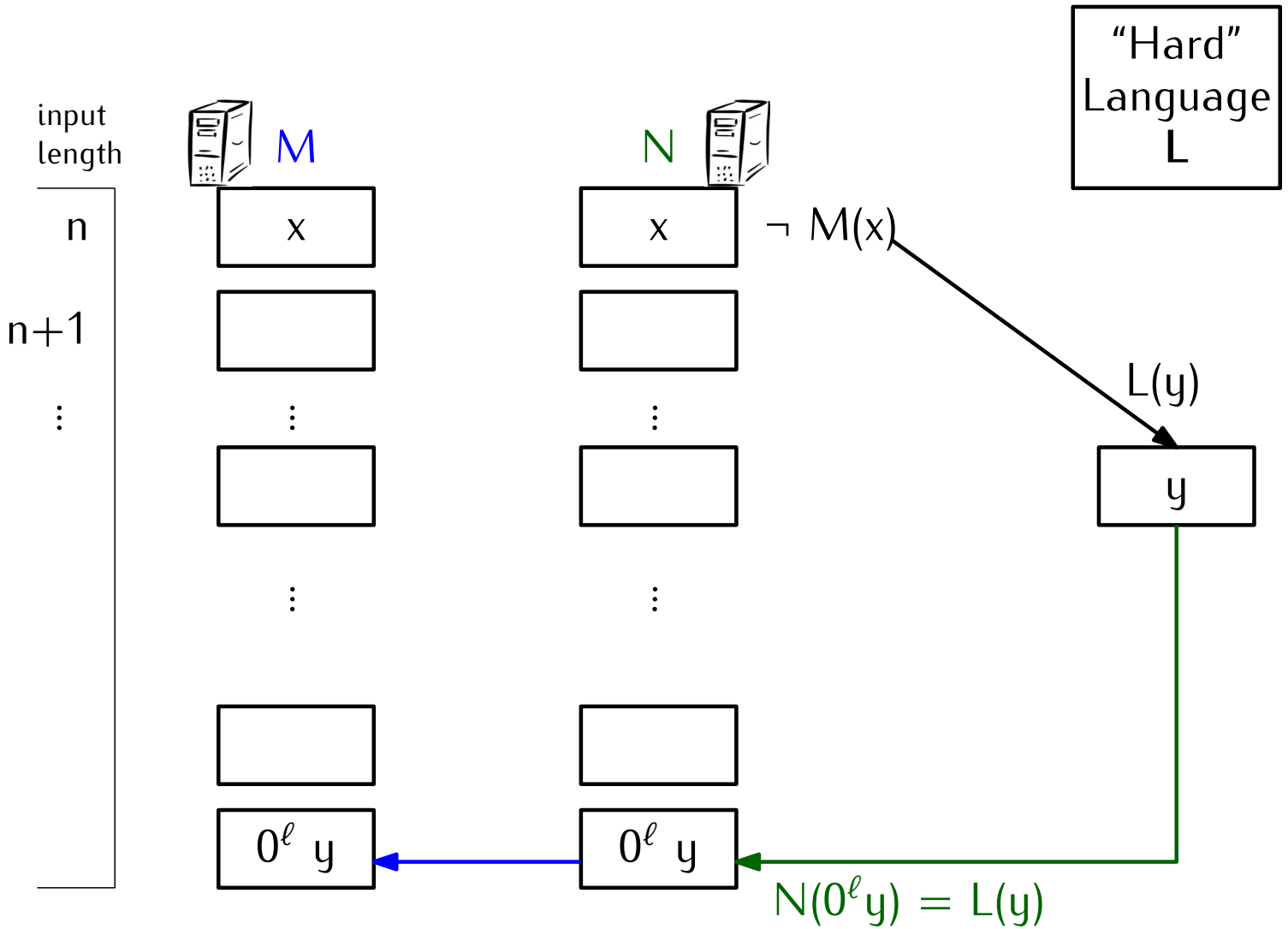

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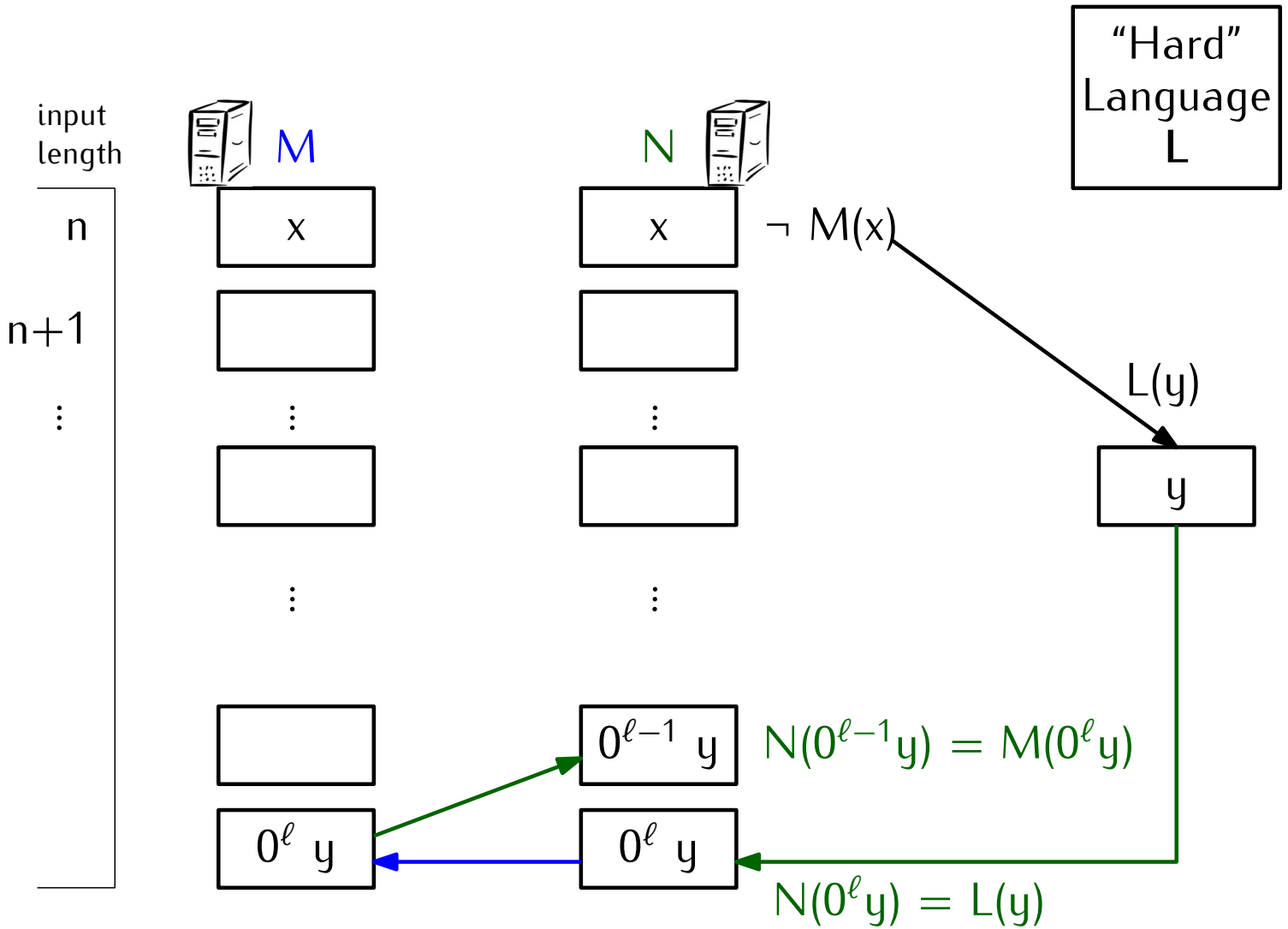

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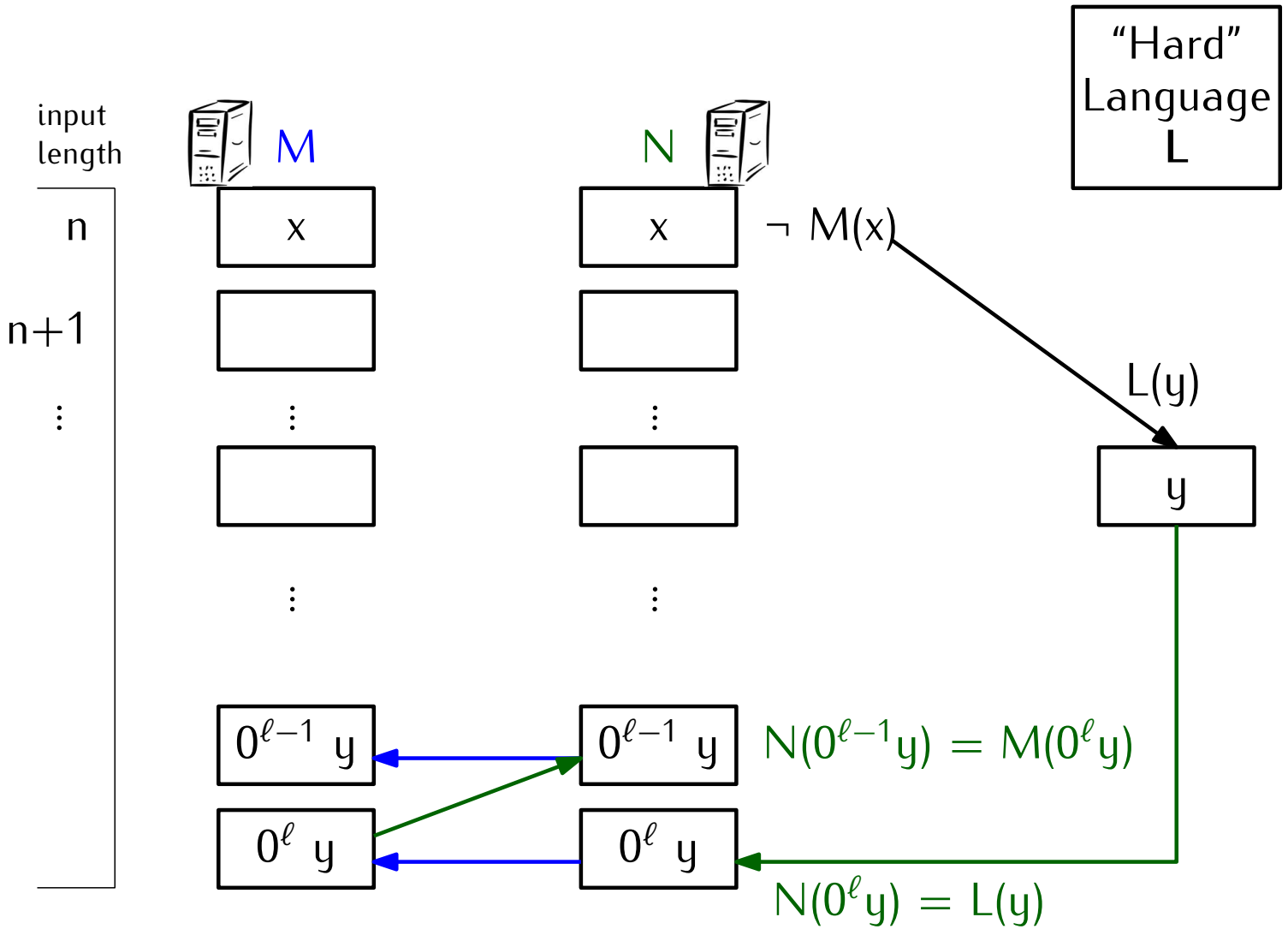

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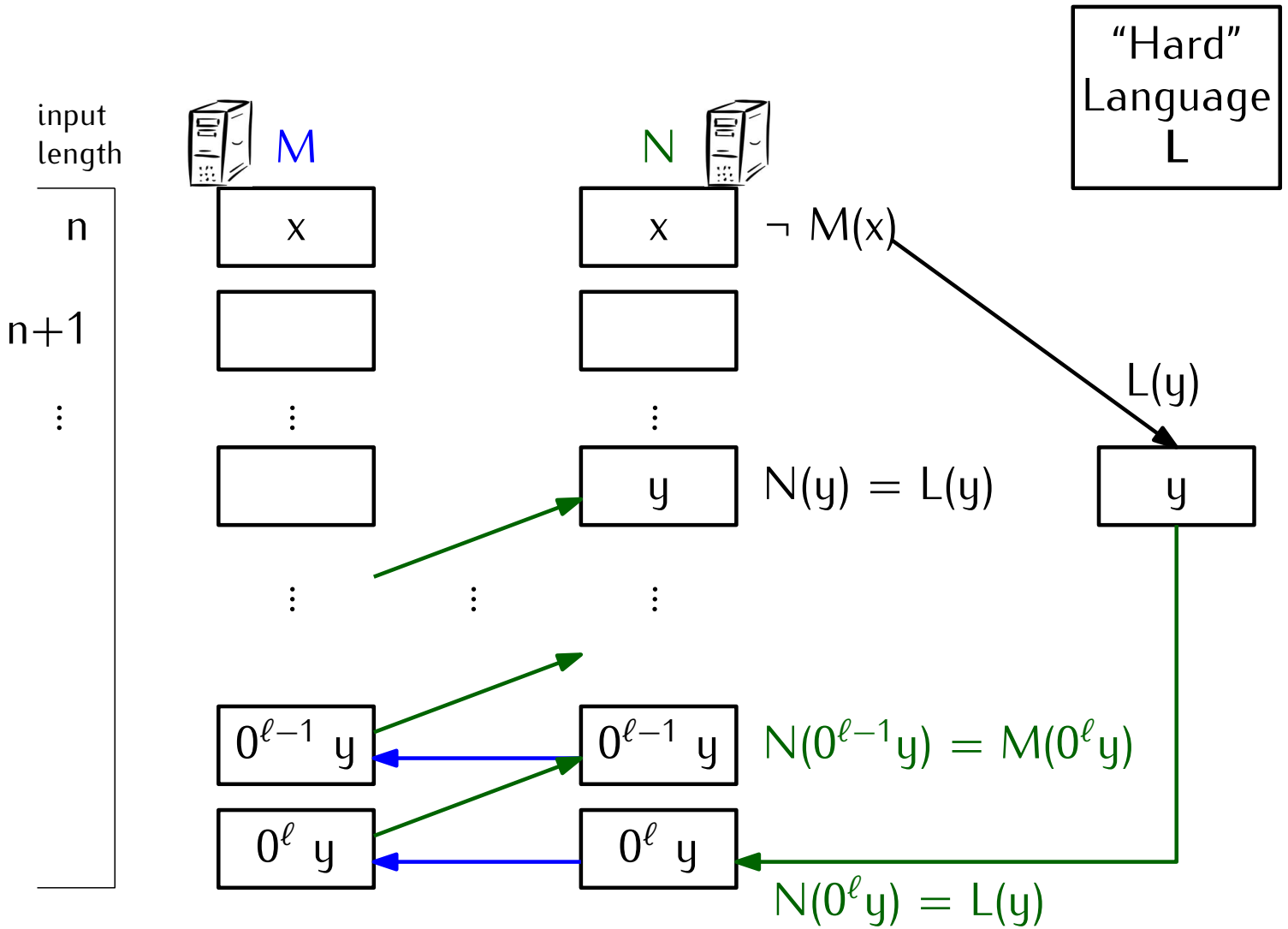

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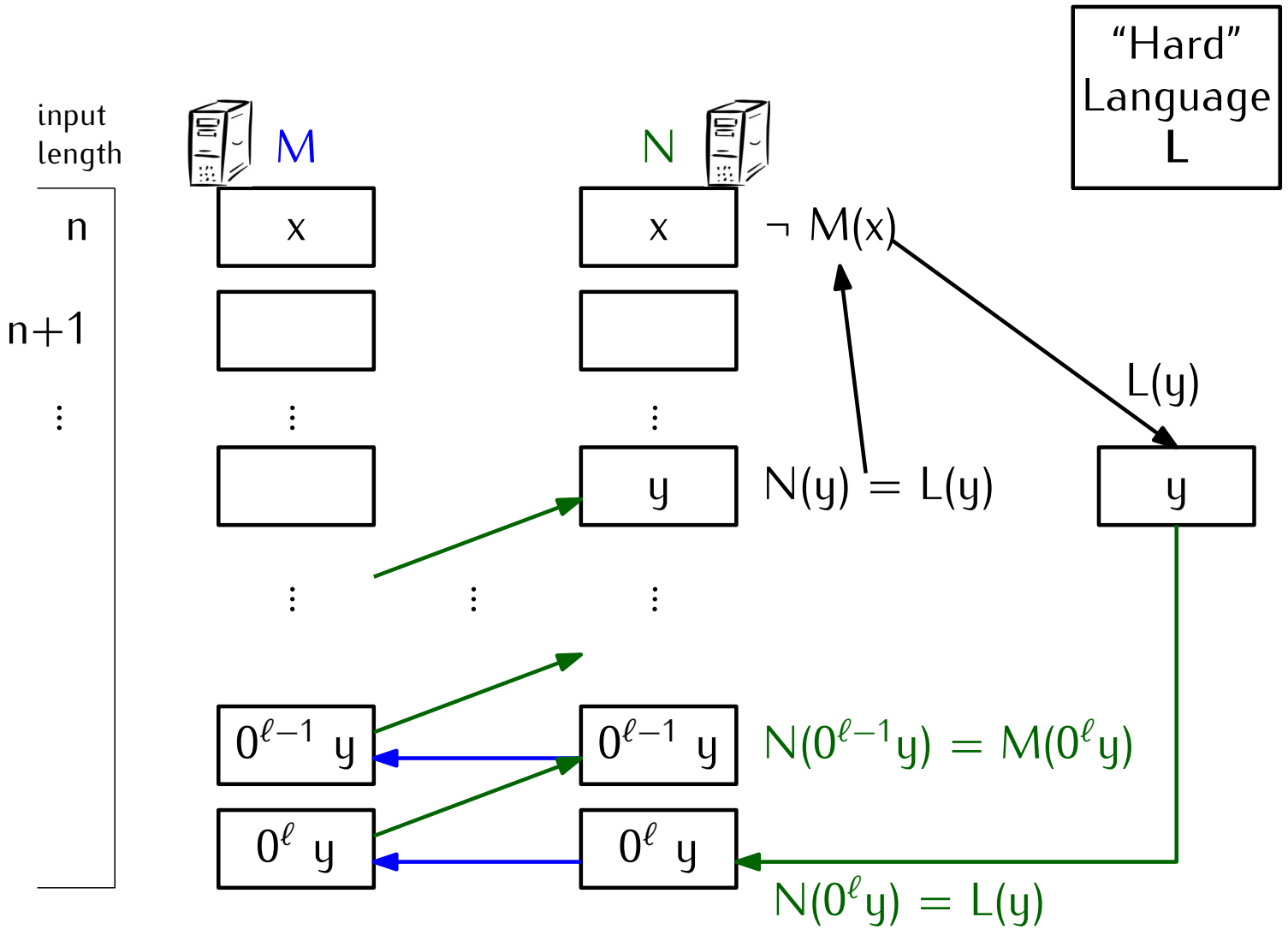

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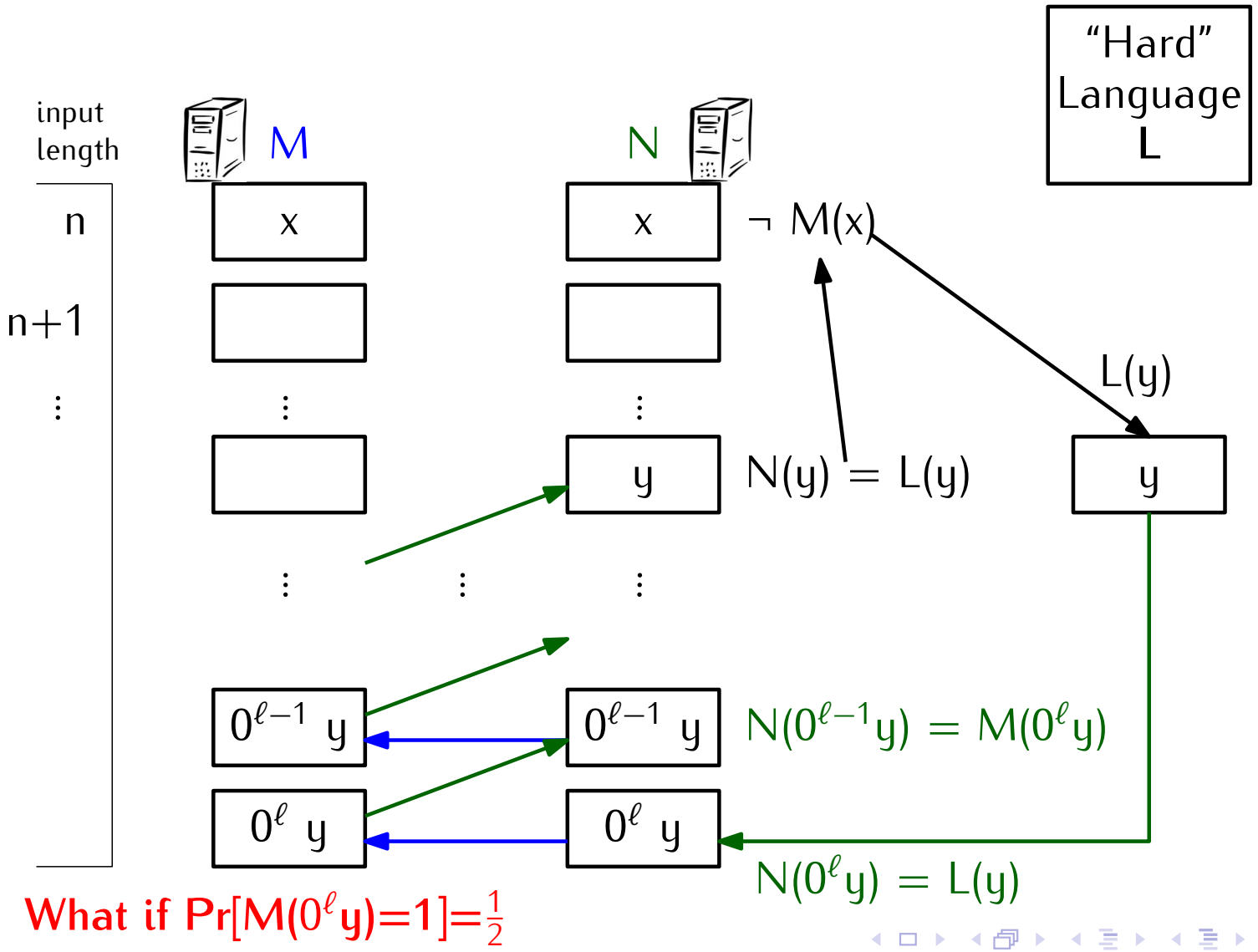

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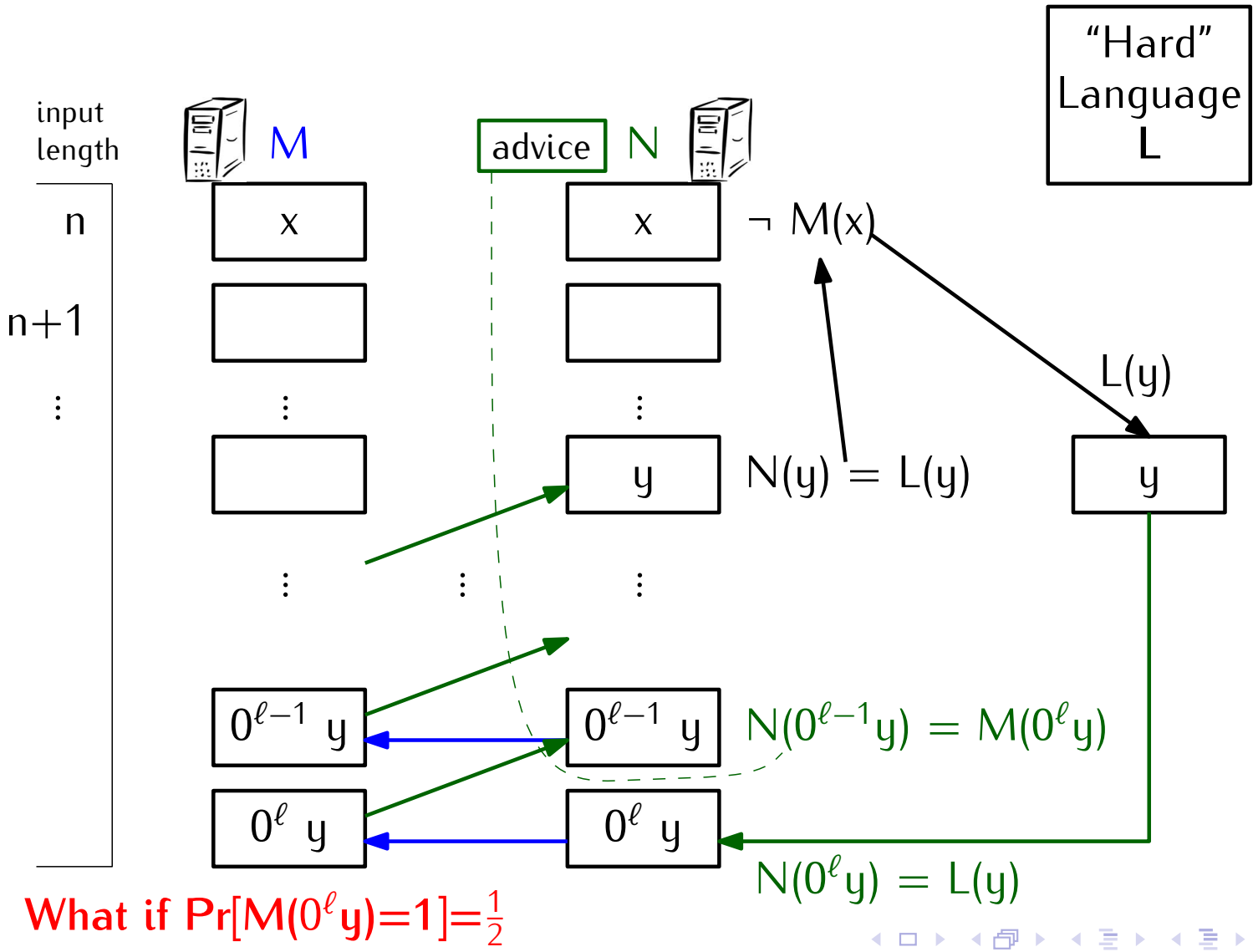

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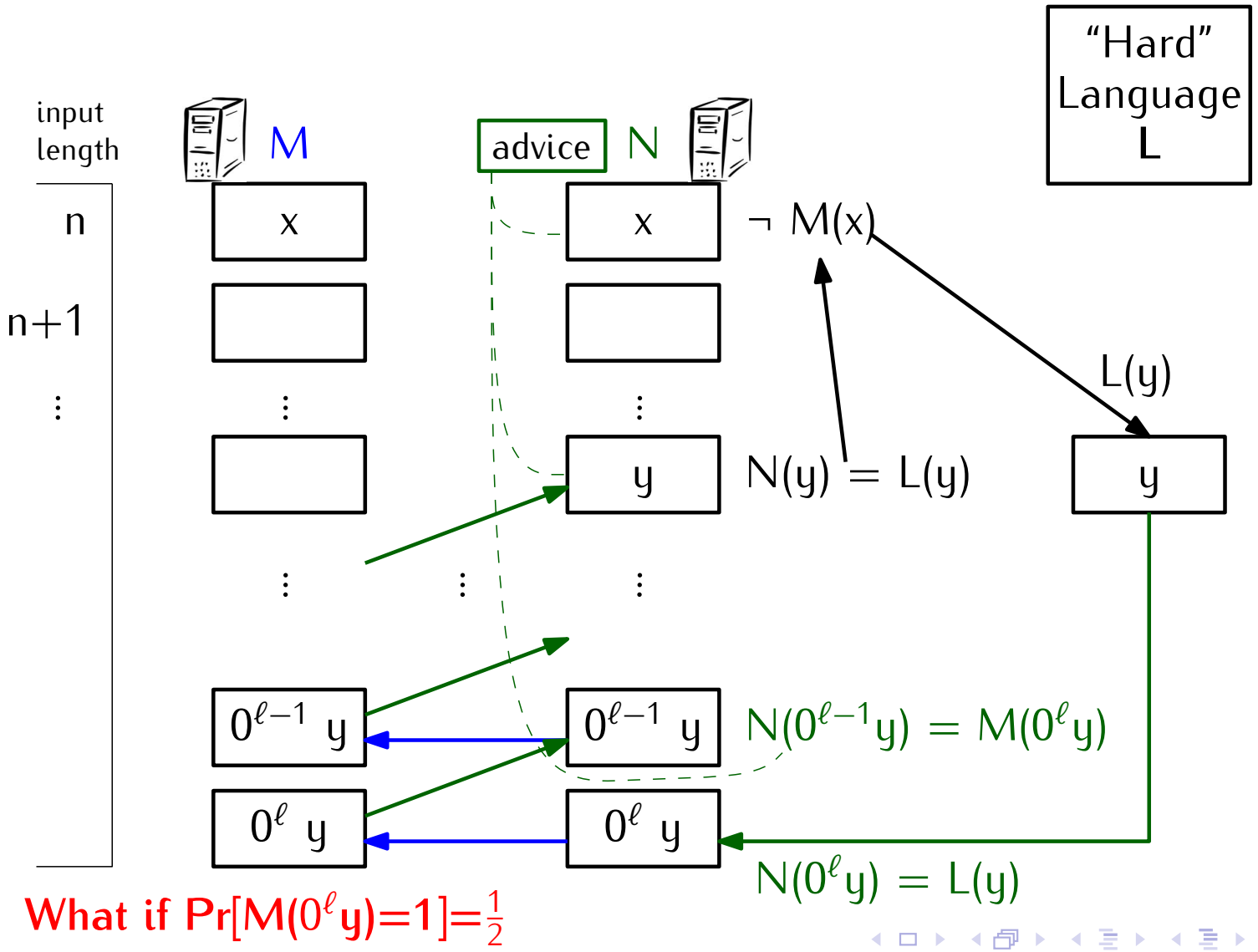

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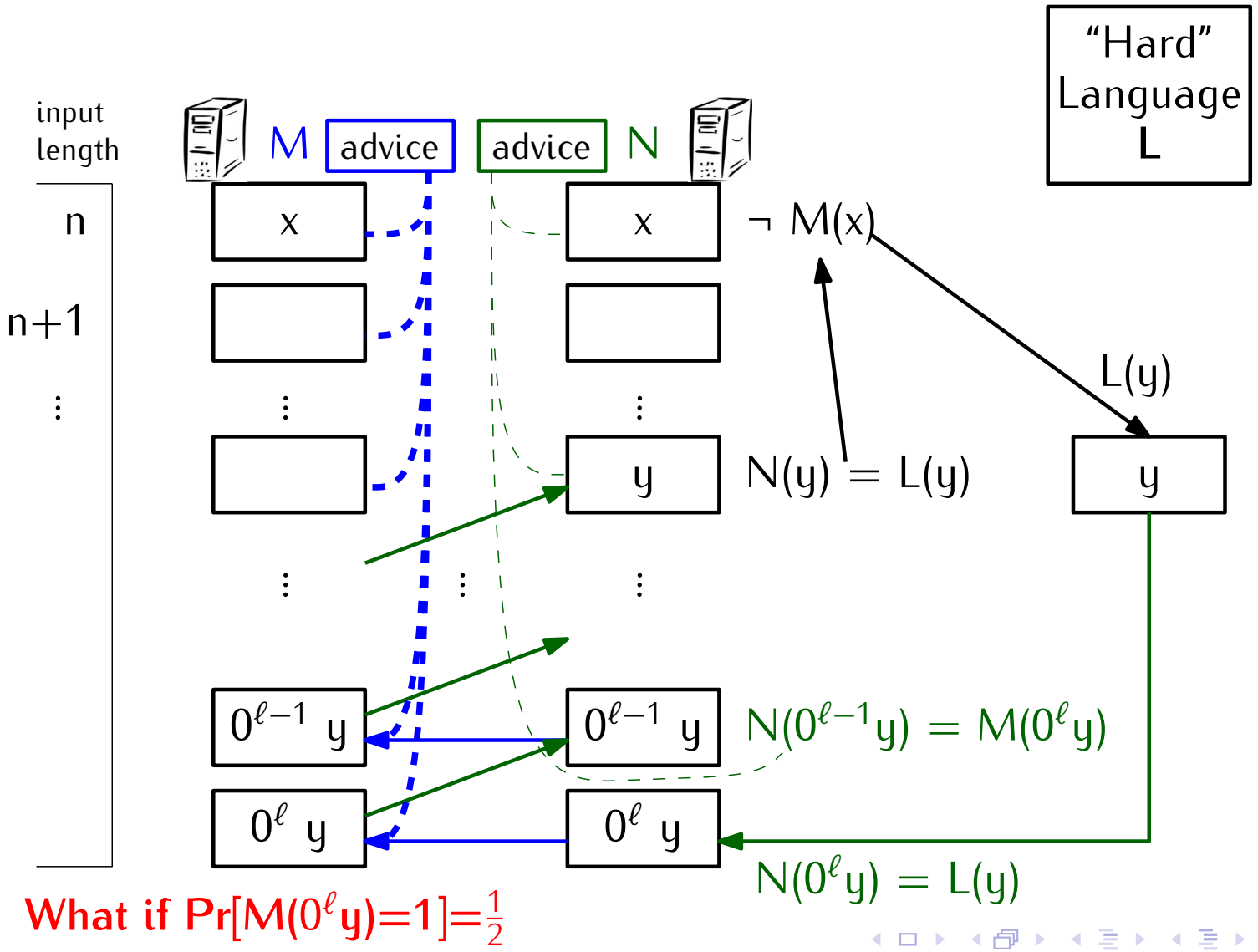

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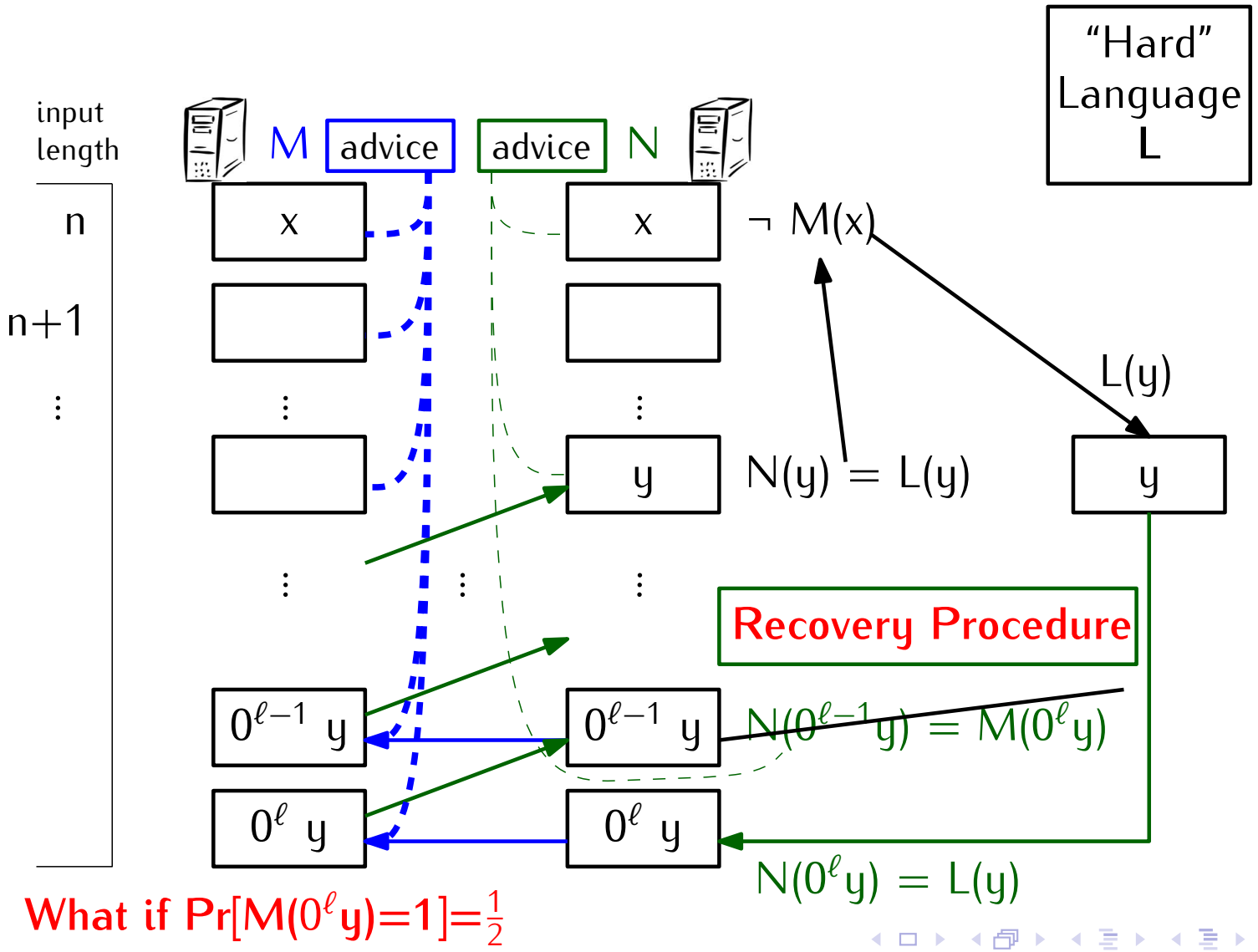

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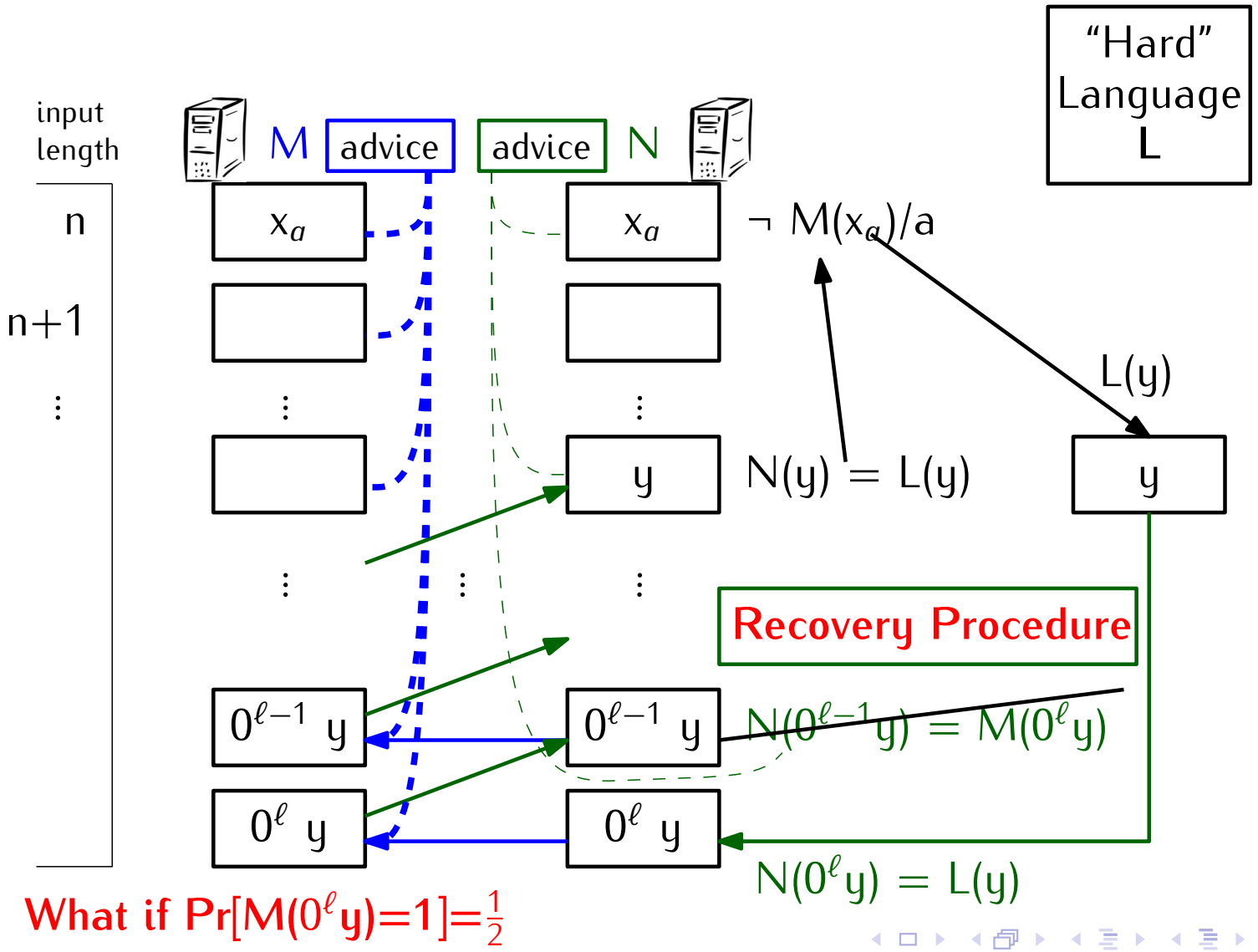

Recovery Procedure for L

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- **Input:** y, list of randomized machines

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- **Input:** y, list of randomized machines
- **Output:** L(y), using small space, with bounded error
- **Pre-condition:** at least one machine in list computes L on instances of length $|y|$, using small space, with bounded error

Hard Language L – $\langle D,x,t,j \rangle$

Hard Language L – ⟨D,x,t,j⟩

Computation Tableau of machine D on input x

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internal state work tape contents

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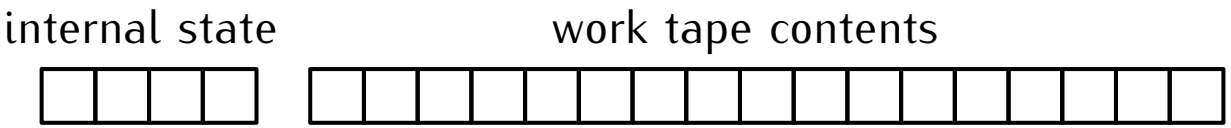

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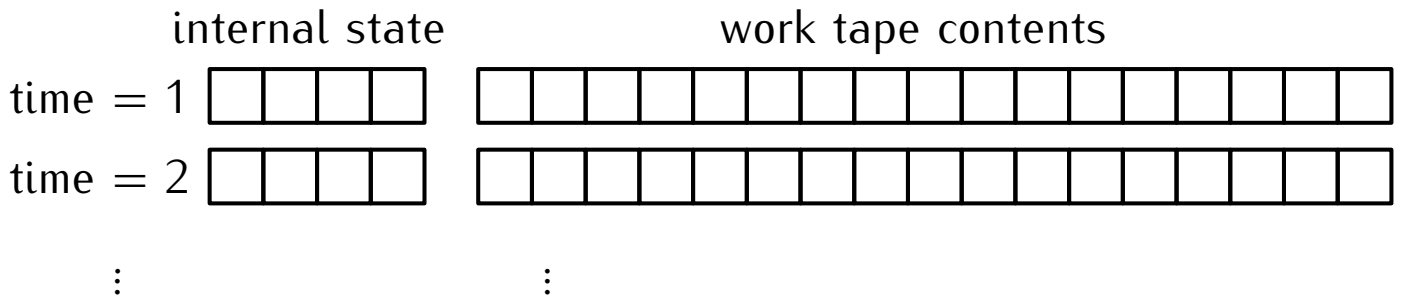

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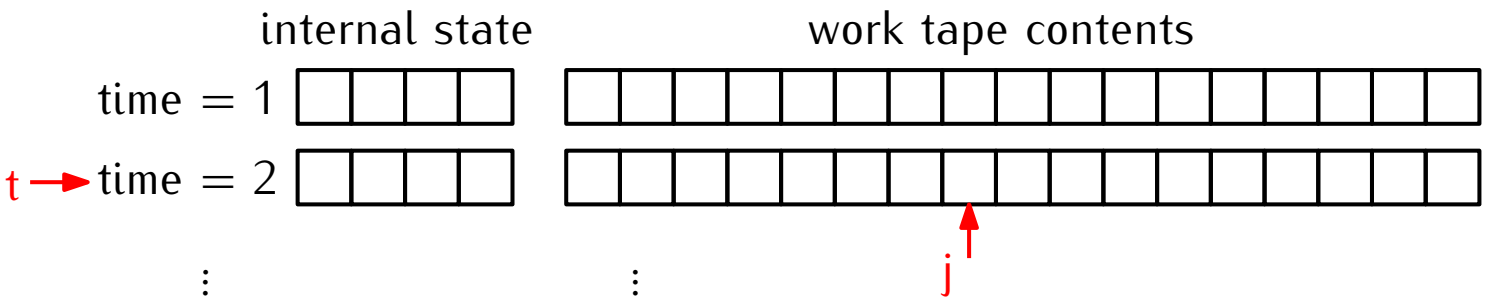

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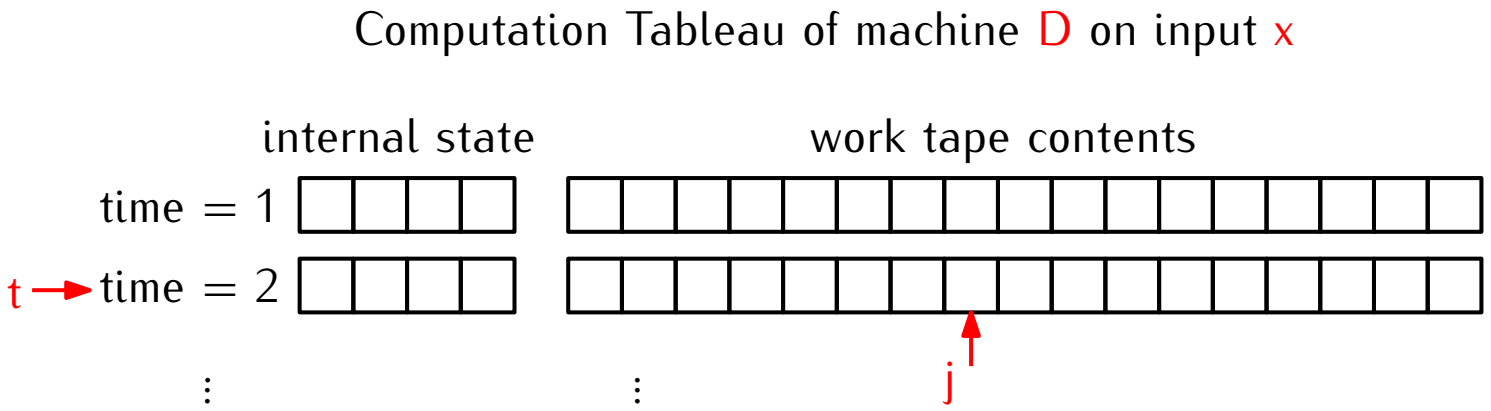

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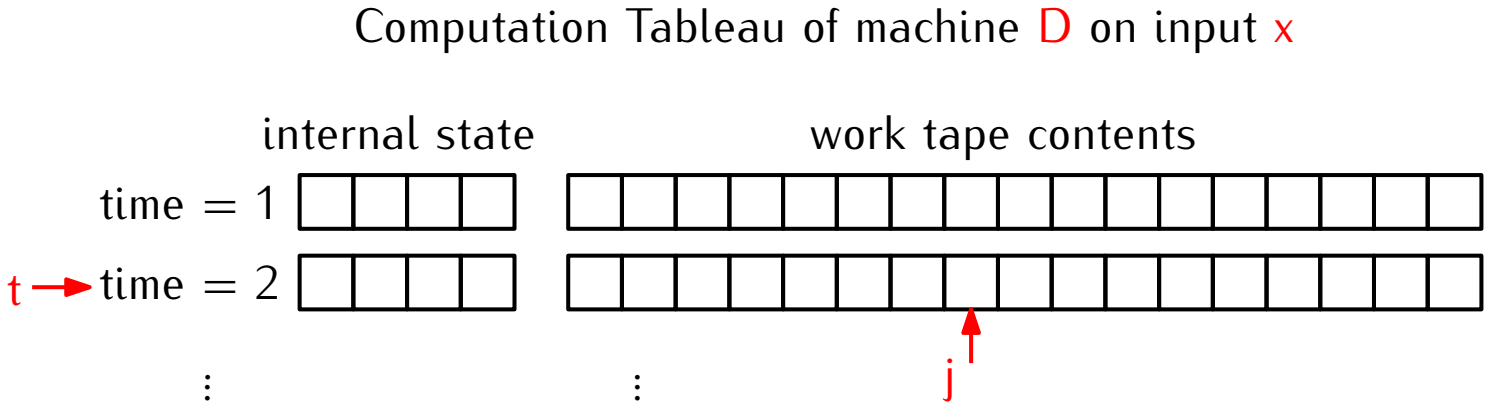

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- Space-efficient Recovery Procedure for L

Recovery Procedure

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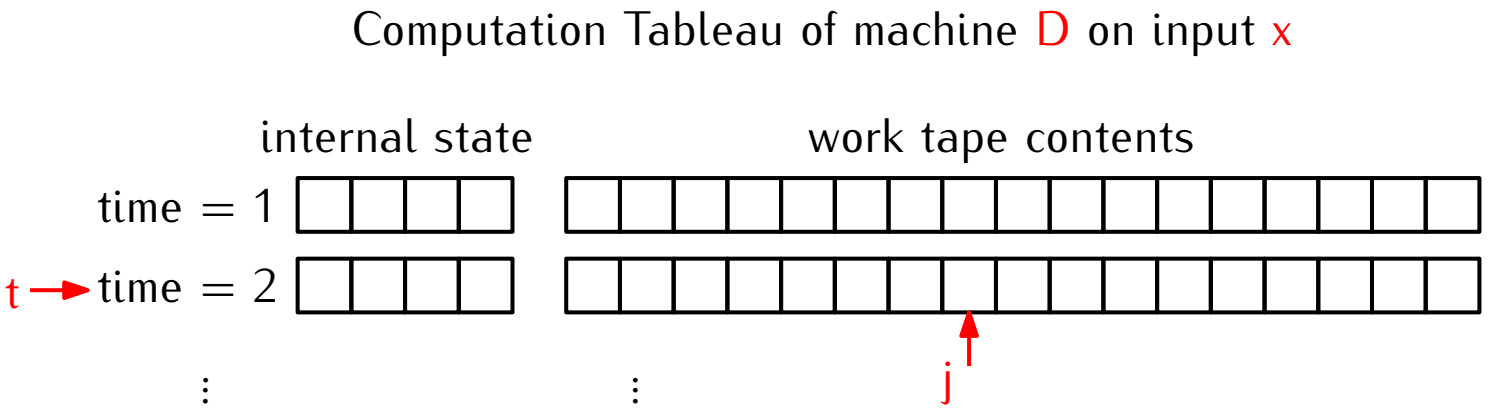

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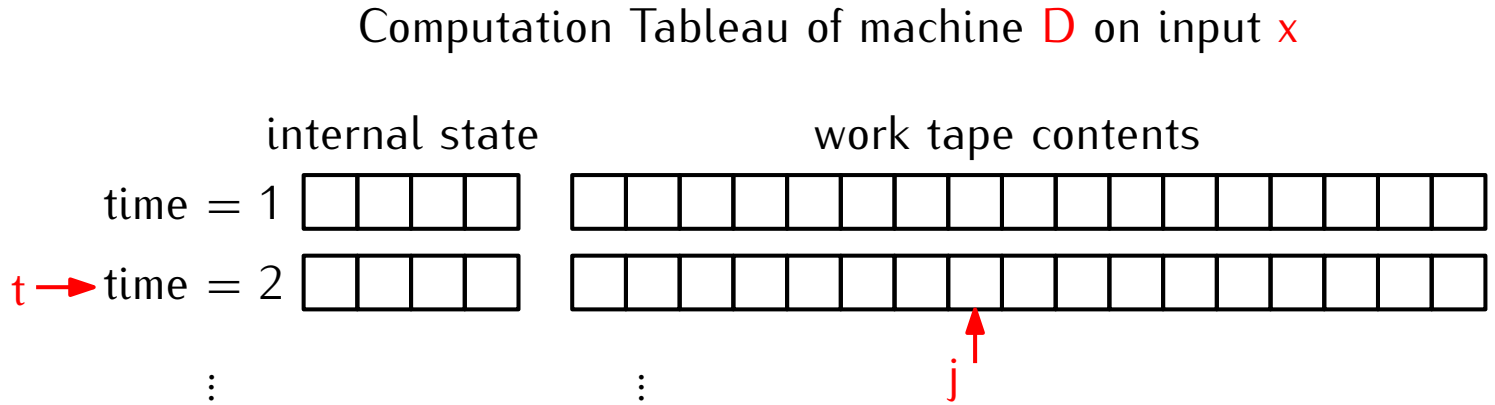

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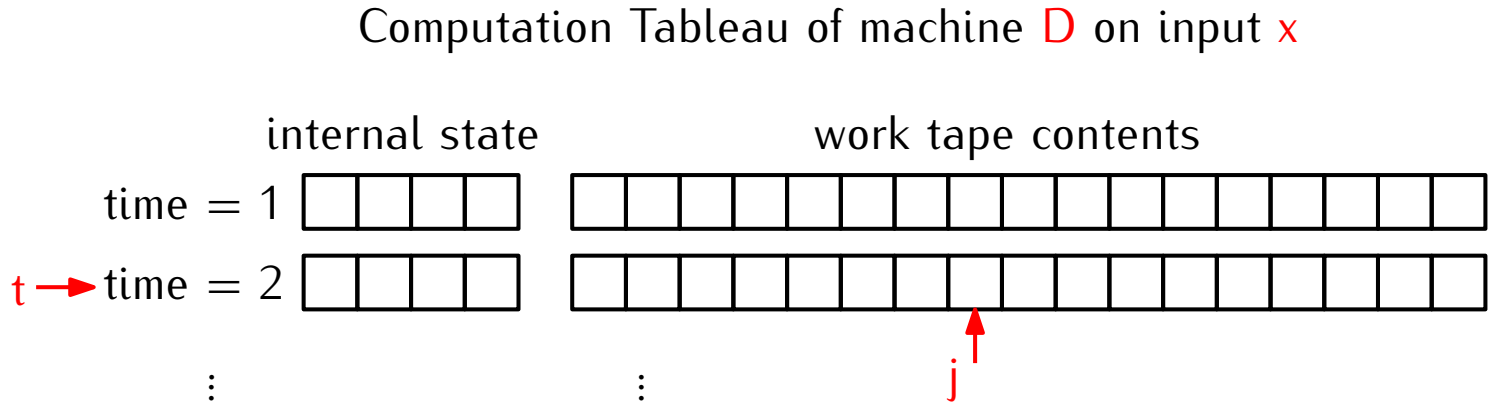

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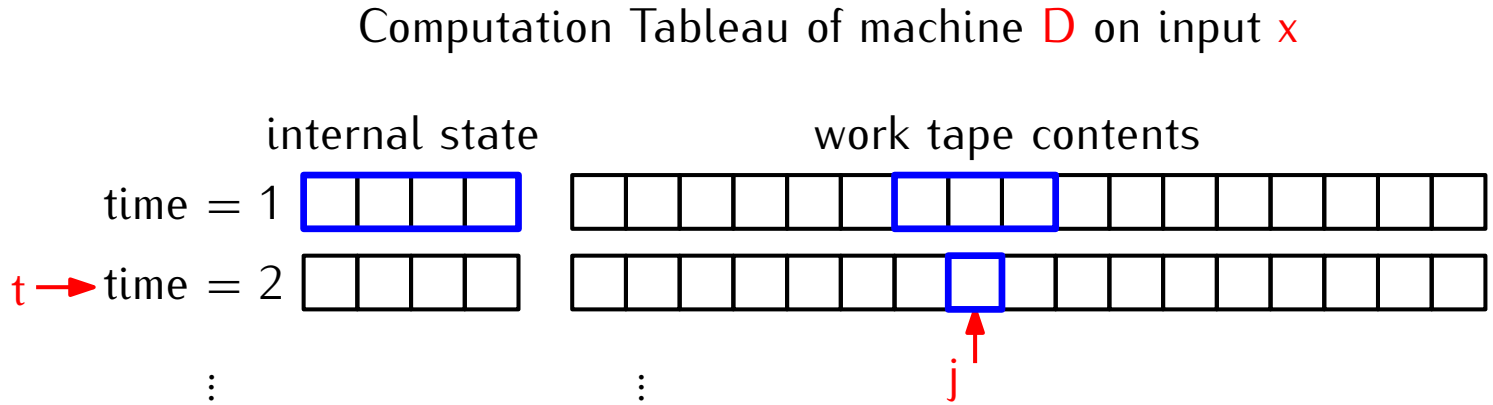

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  - Local consistency test

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SPACE(s')/1 $\not\subseteq$ SPACE(log n)/log n, for any s'=ω(log n)

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  - Zero-error recovery procedure for L based on inductive counting [I88, S88]

Outline

How Powerful is Randomness?

- "Typically-correct" Derandomization
- Hierarchy Theorems for Randomized Algorithms
- Derandomizing Monotone Computations

Derandomizing Monotone Computations

Monotone Graph Properties

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Does G have a 3-clique/triangle?

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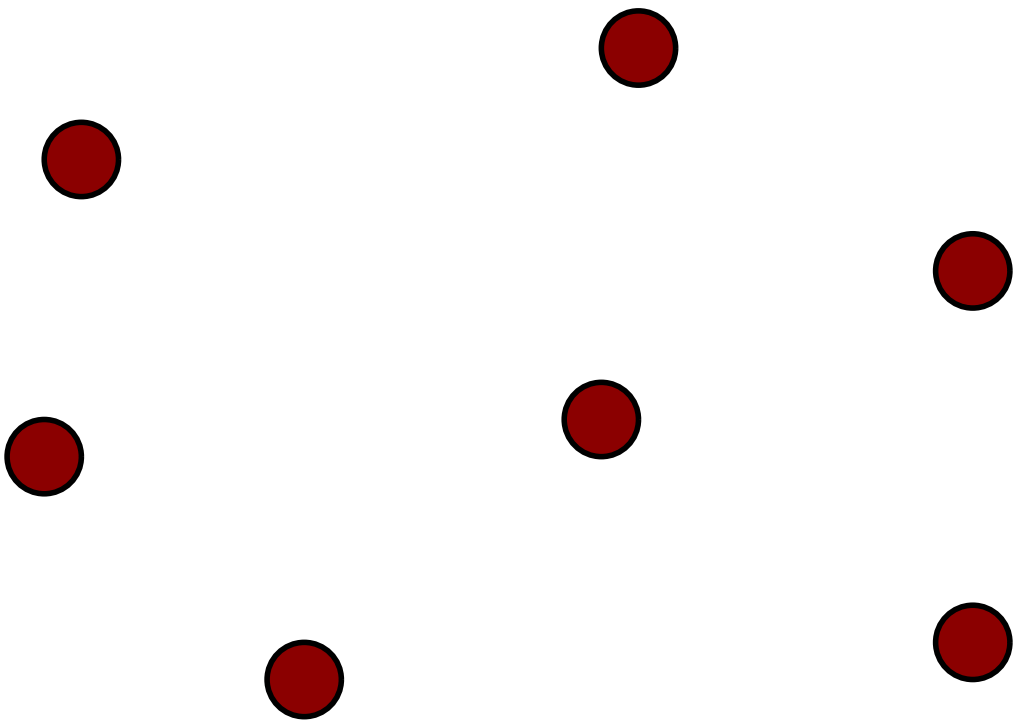

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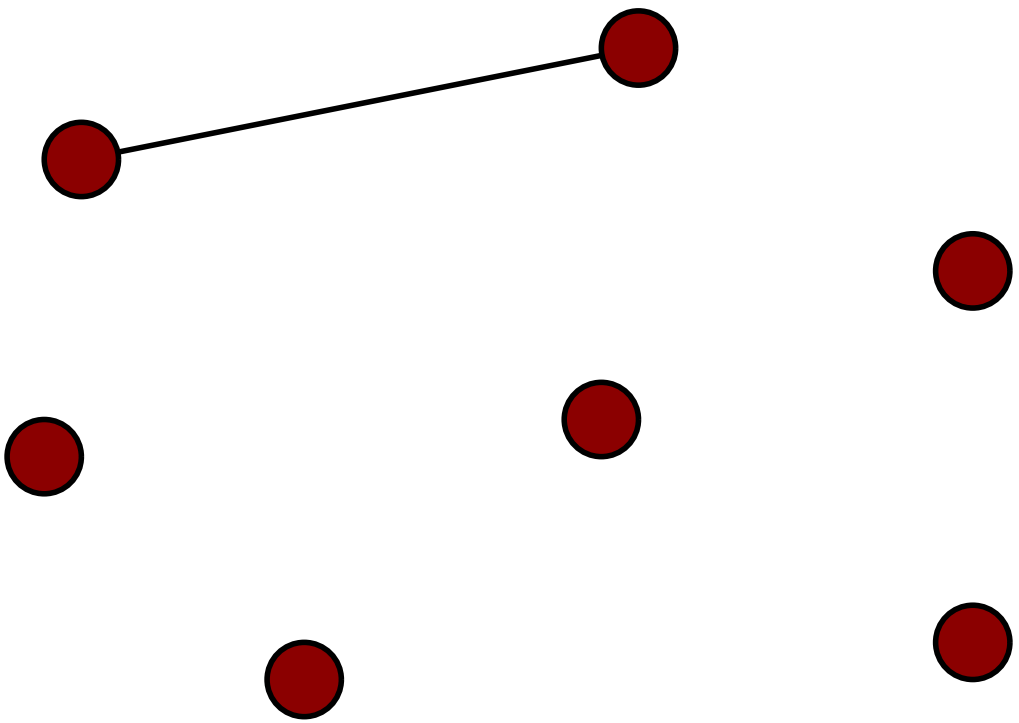

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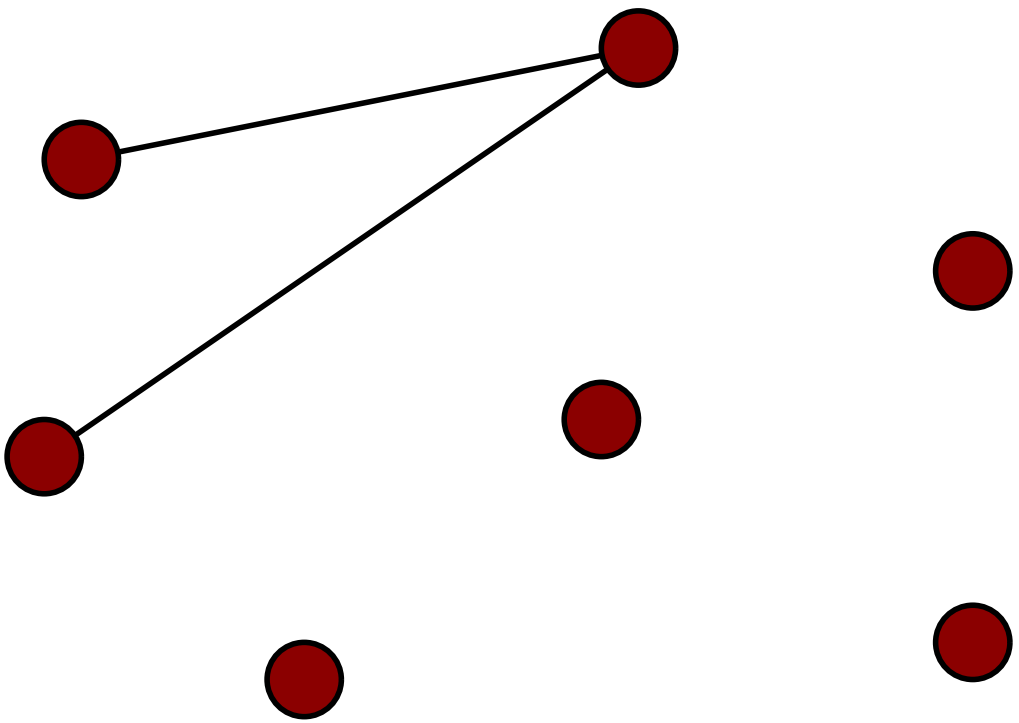

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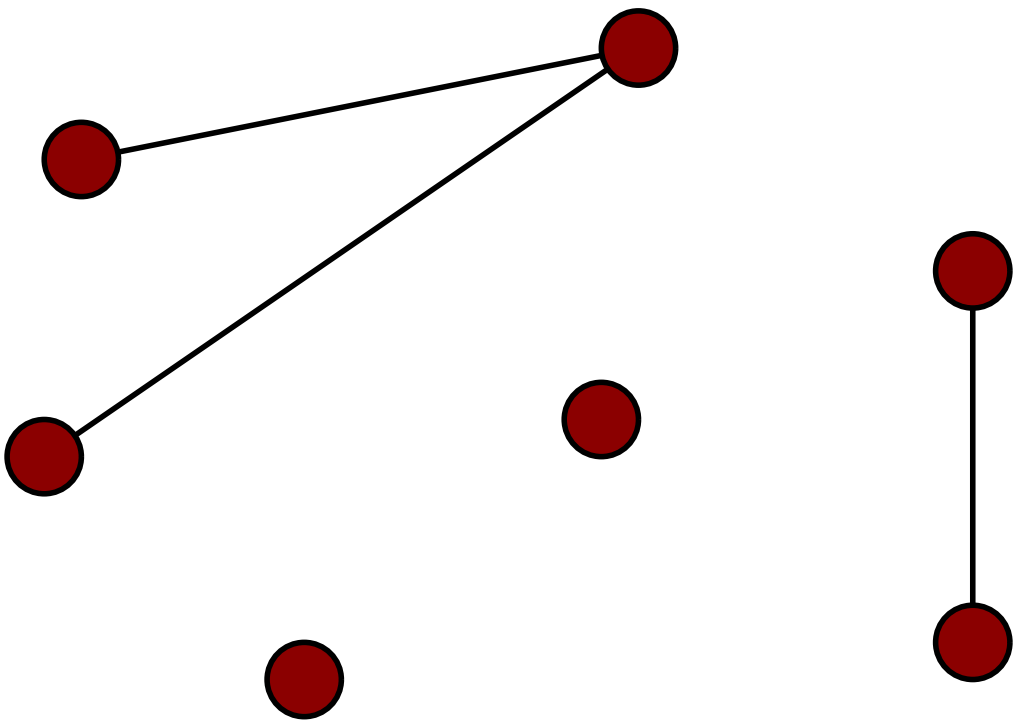

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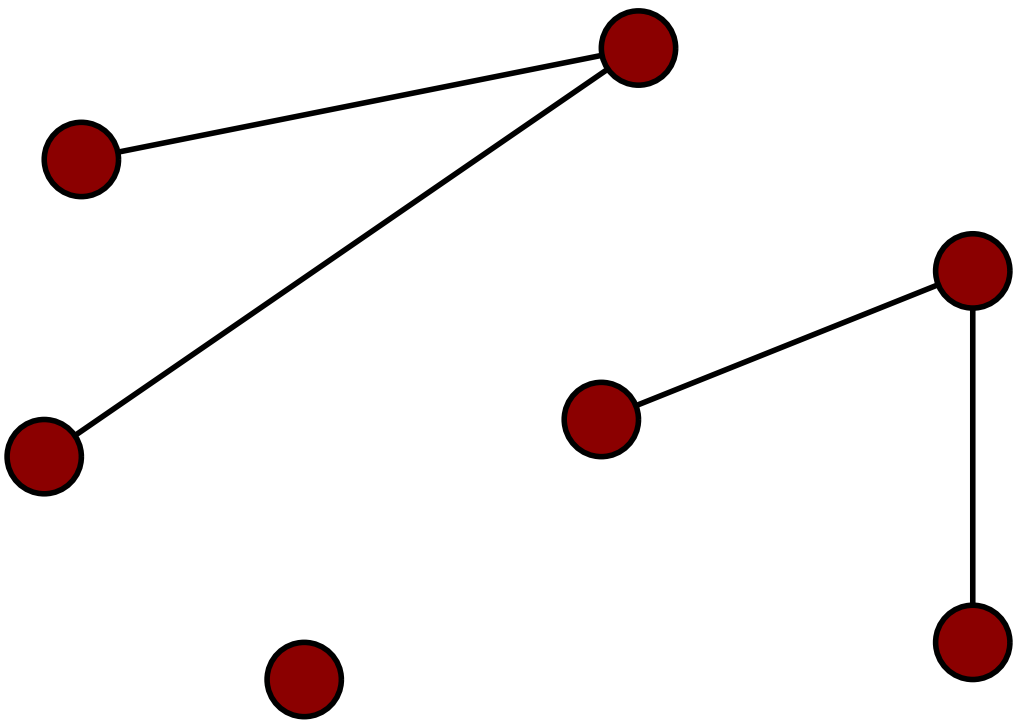

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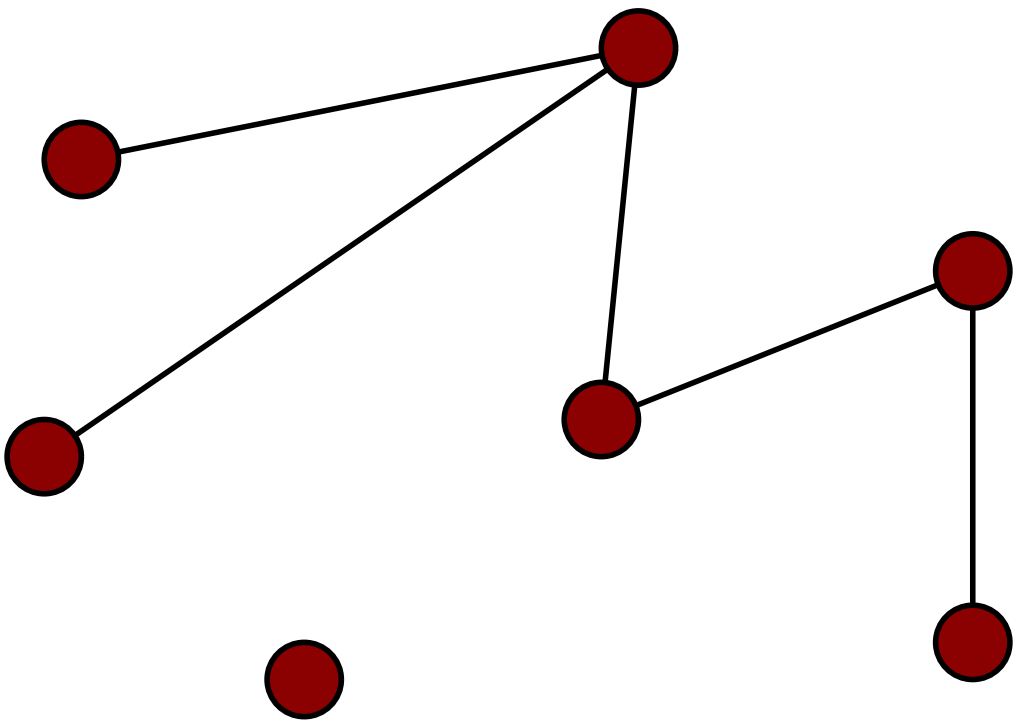

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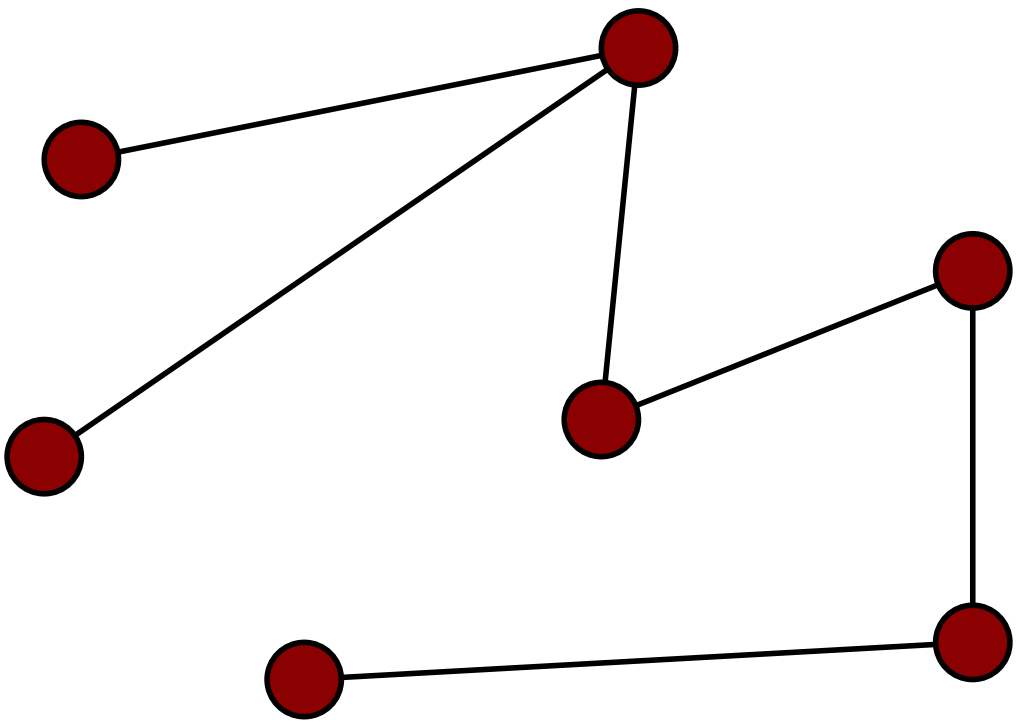

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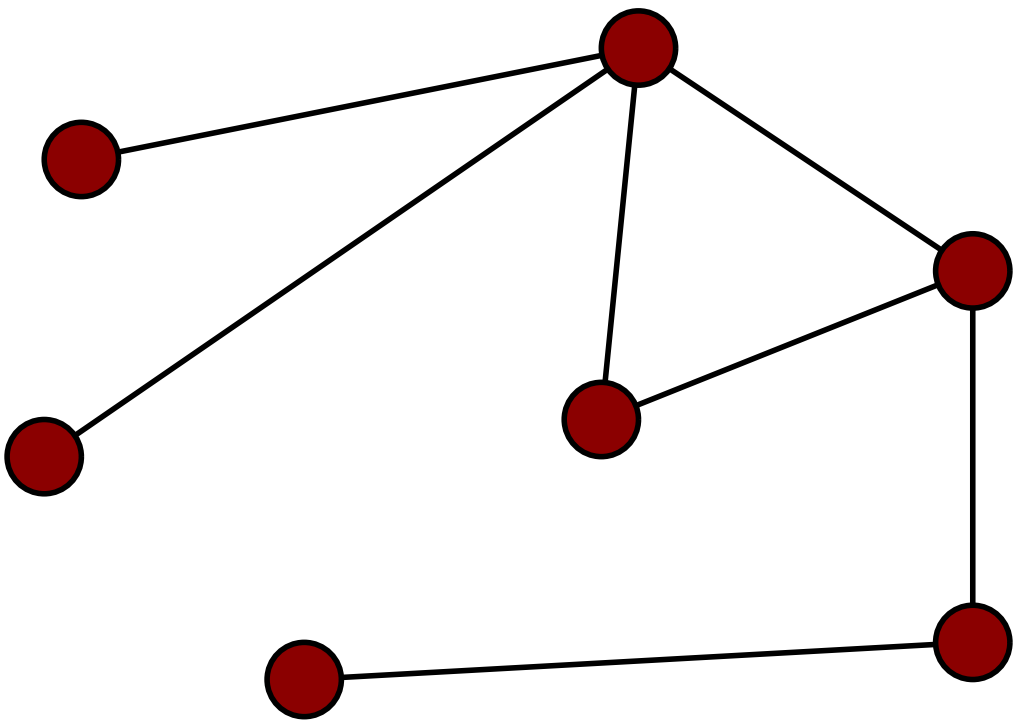

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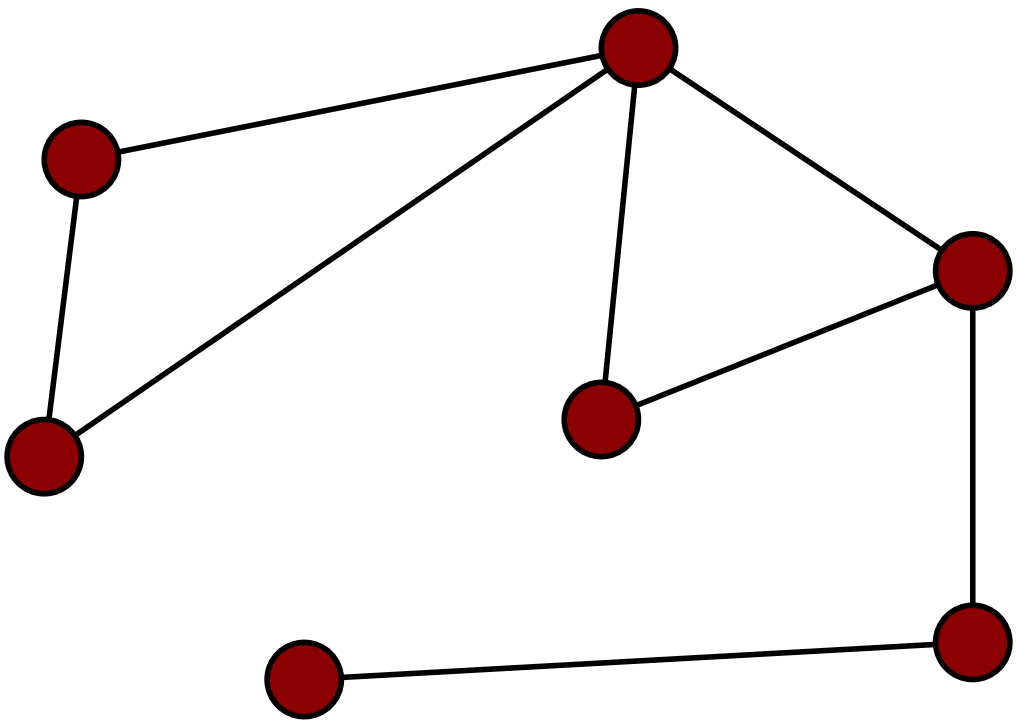

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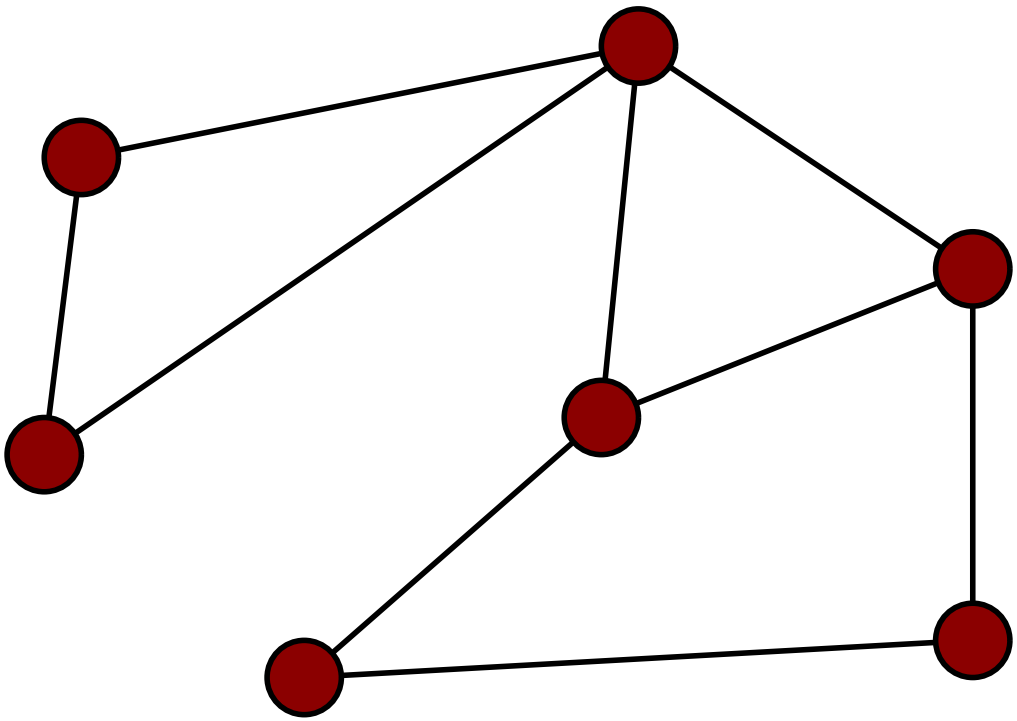

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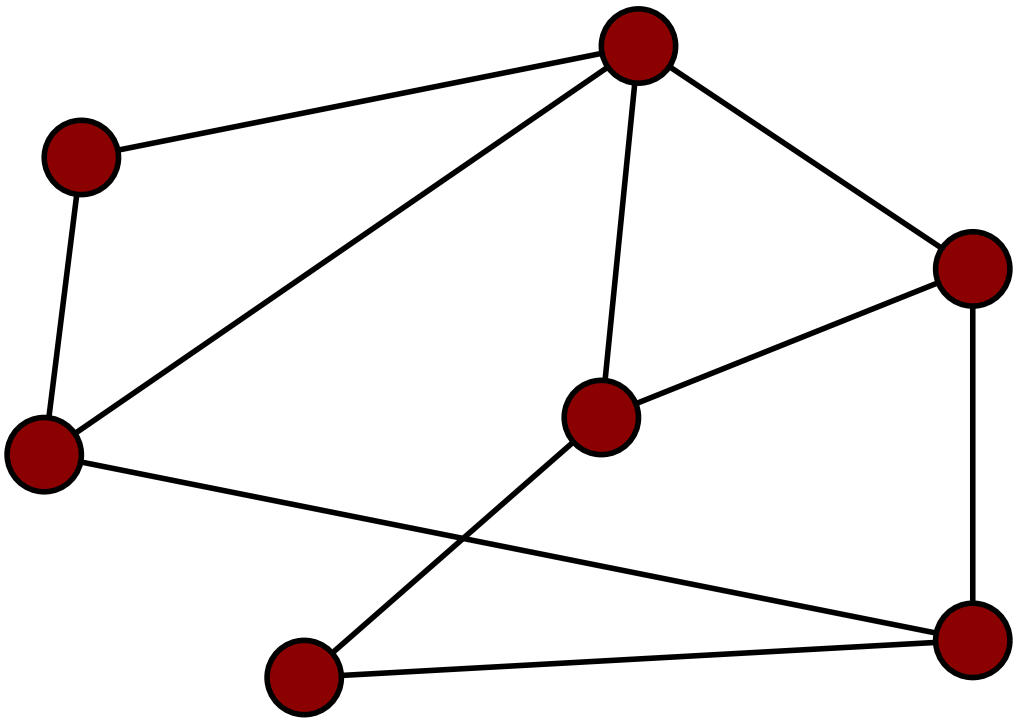

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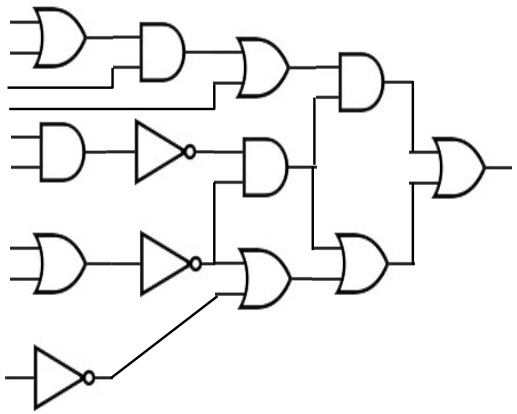

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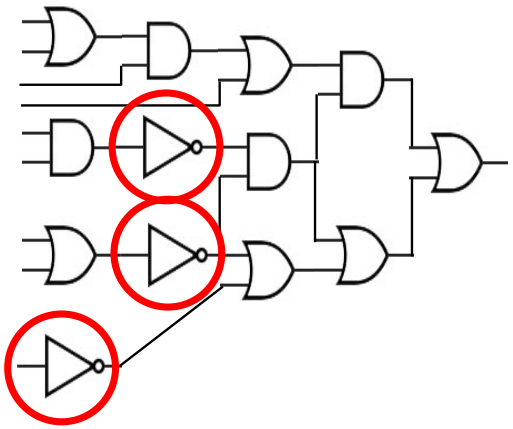

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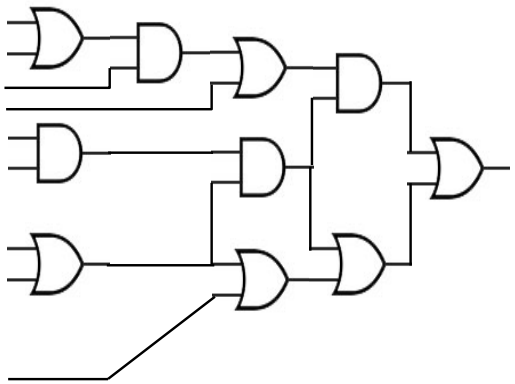

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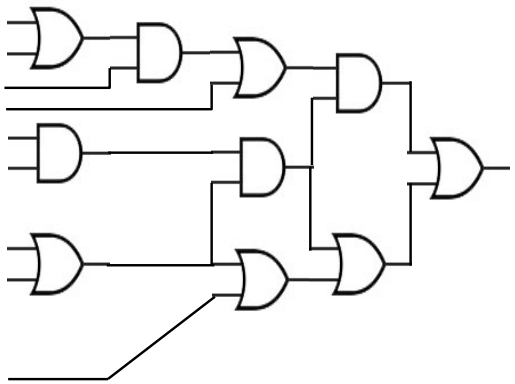

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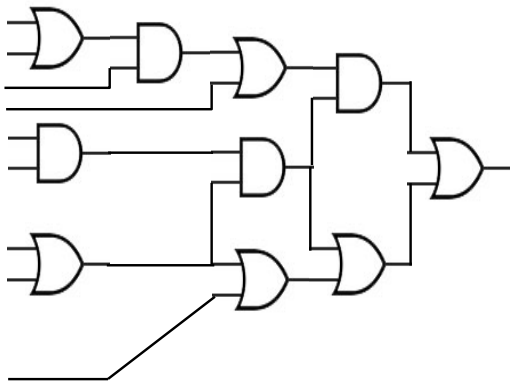

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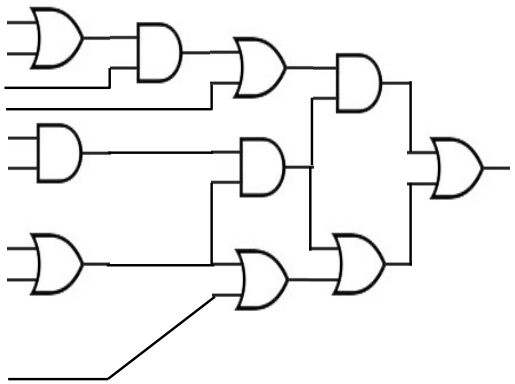

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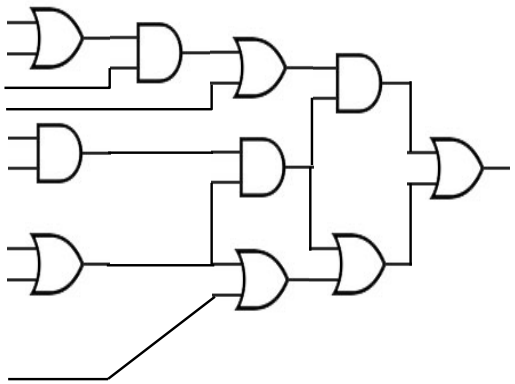

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Results

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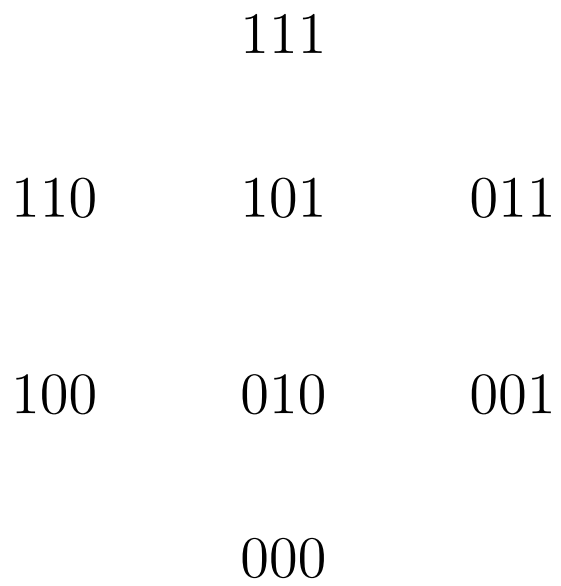

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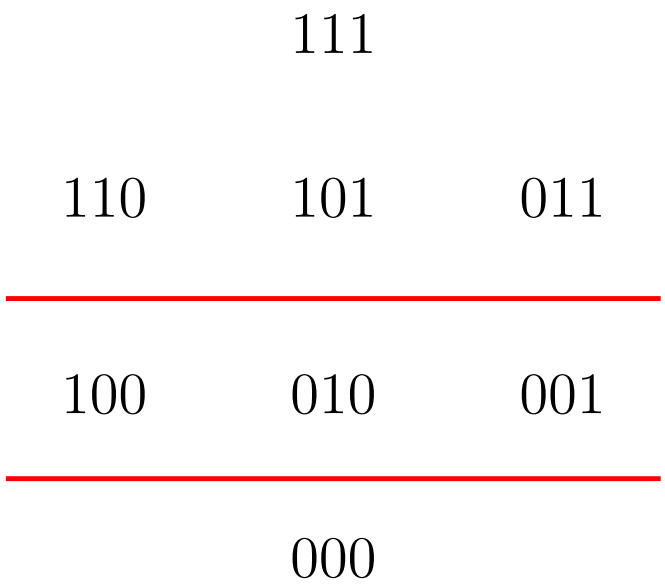
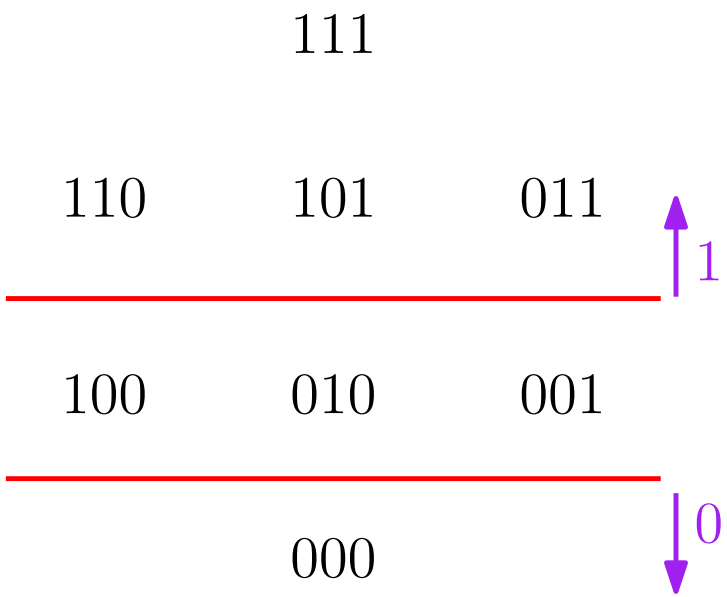


Monotone Slice Functions



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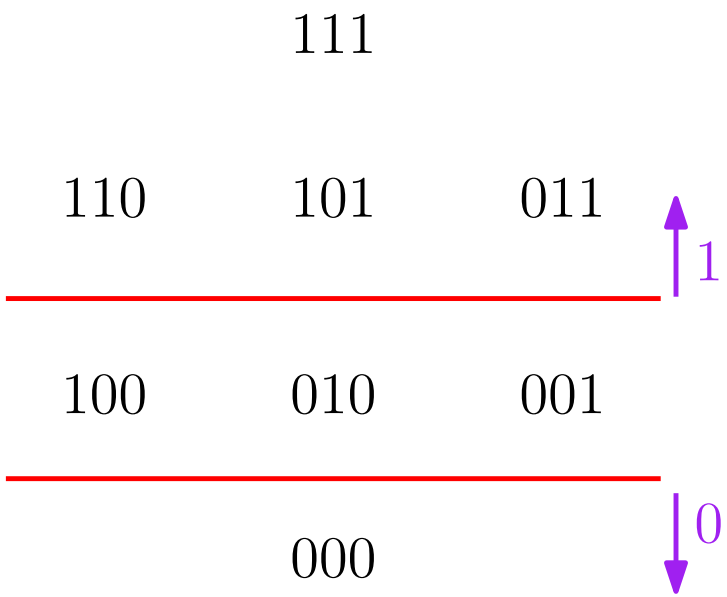

- i-th bit $= 0$

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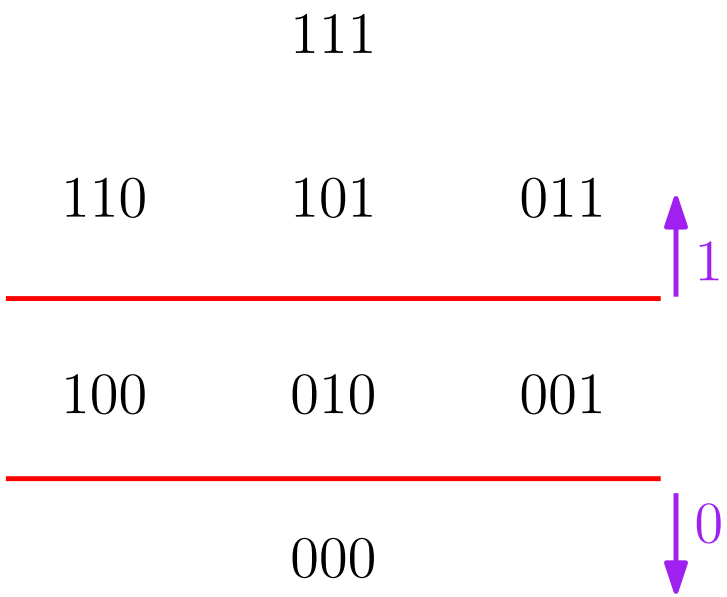

- i-th bit $= 0$ iff at least one other bit is 1

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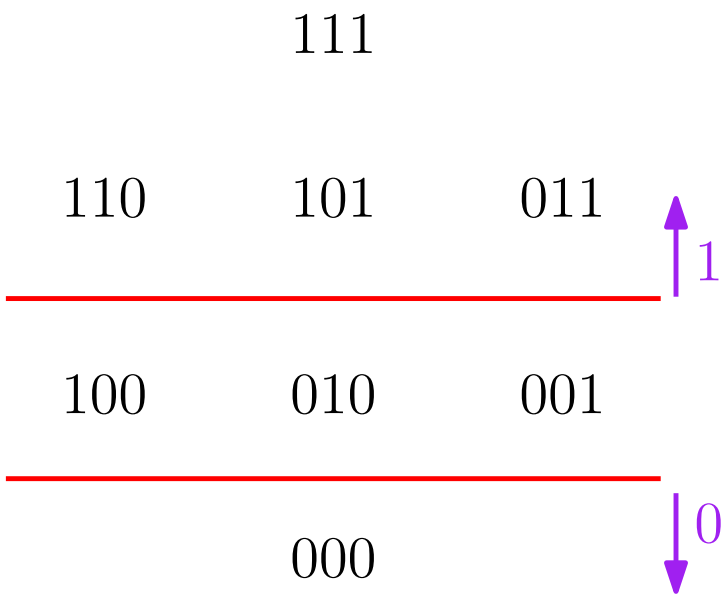

- i-th bit $= 0$ iff at least one other bit is 1
- **Slice function $\Rightarrow$ monotone and general circuit size nearly the same**

General versus Monotone Distinguishers

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Distribution $\mathcal{D}$, general circuit C such that
$|\Pr_{X \leftarrow \mathcal{D}}[C(X) = 1] - \Pr_{X \leftarrow U_n}[C(X) = 1]| \geq \epsilon$

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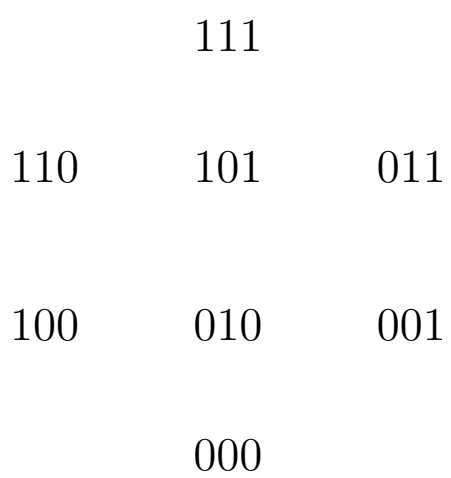

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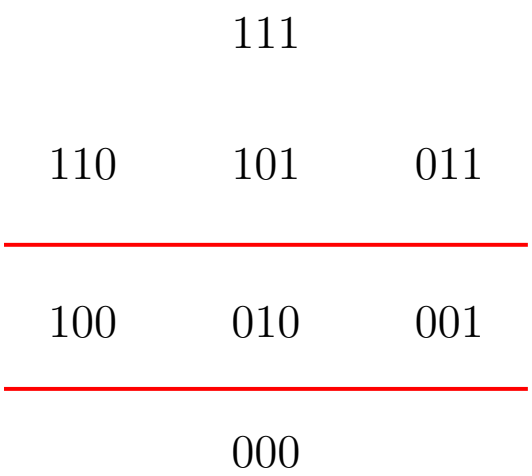

- C $(\frac{\epsilon}{n+1})$-distinguishes for x with |x|=k

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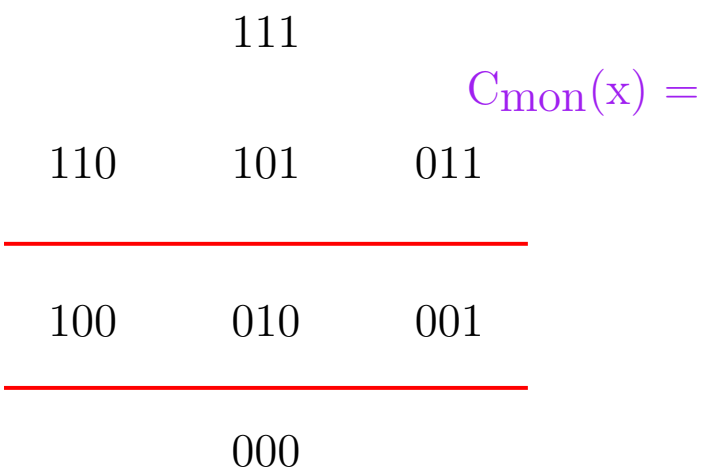

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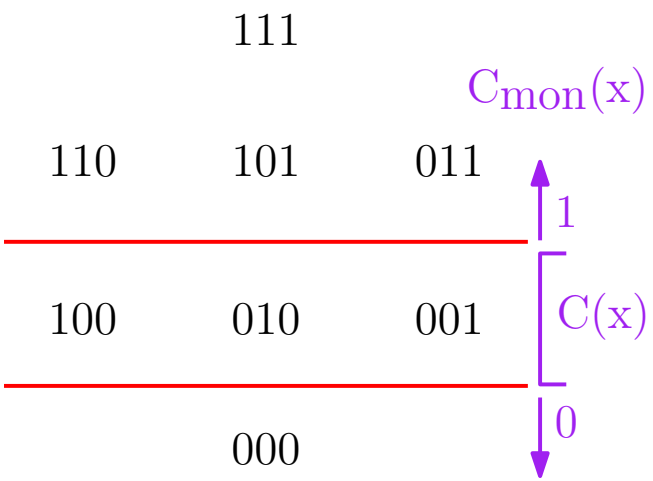

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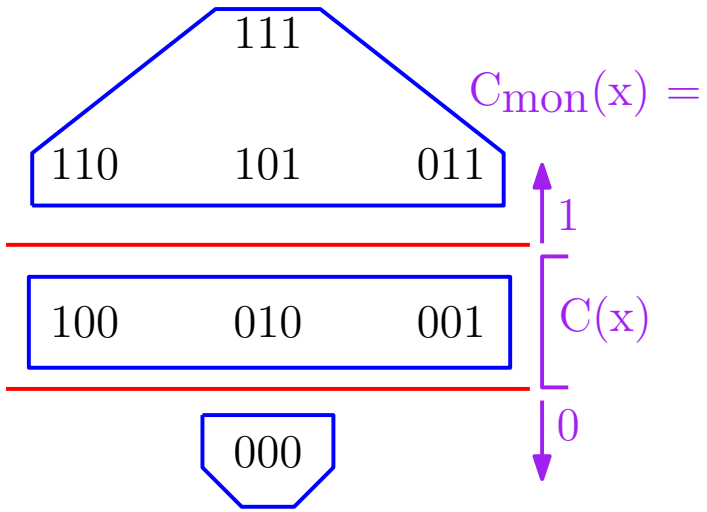

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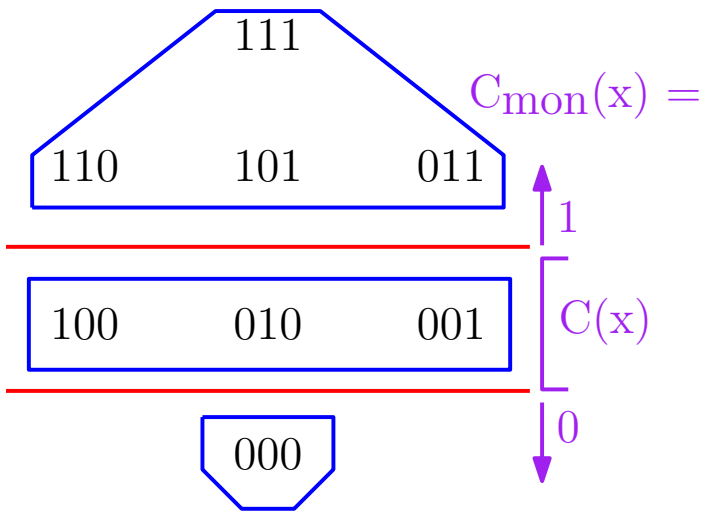

- C $(\frac{\epsilon}{n+1})$-distinguishes for x with |x|=k
- If threshold is not $\epsilon' = (\frac{\epsilon}{2(n+1)})$-distinguisher, then C_{mon} is

Recap

How Powerful is Randomness?

- "Typically-correct" Derandomization
- Hierarchy Theorems for Randomized Algorithms
- Derandomizing Monotone Computations

The End, Thank You!

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Slides will be available at:

http://www.kinnejeff.com/ (or E-mail me)