Scheduling with Precedence Constraints

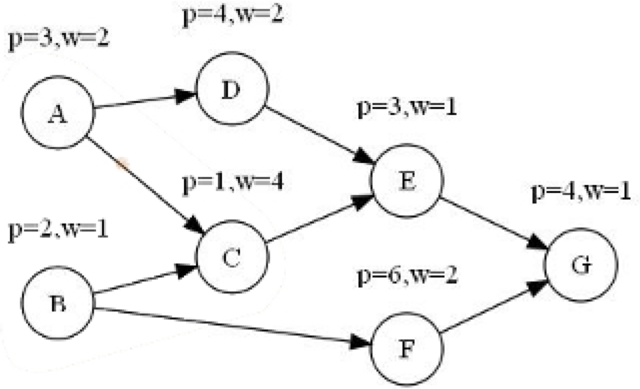
James Jolly and Pratima Kolan

November 25, 2009

Precedence Graphs in Task Scheduling

- typically DAGs
- vertices are tasks
  - processing time p_i
  - weight w_i
- edges are data depedencies
  - $i \longrightarrow j$, i precedes j

Precedence Graphs in Task Scheduling



Precedence Graph Scheduling Problems

Objectives:

- minimize makespan
- minimize weighted completion time
- maximize throughput

Consider:

- release times
- resource constraints

Focus: Minimize Weighted Completion Time

- G is a DAG
- n tasks
- task i has weight w_i
- task i finishes at c_i
- minimize $\sum_{i=1}^{n} w_i c_i$
  under precedence constraints
  (NP-Hard)

Our Goal

- show two approximations
- construct single-machine schedule
- convert single-machine schedule to multi-machine schedule

Single-Machine Scheduling With Precedence

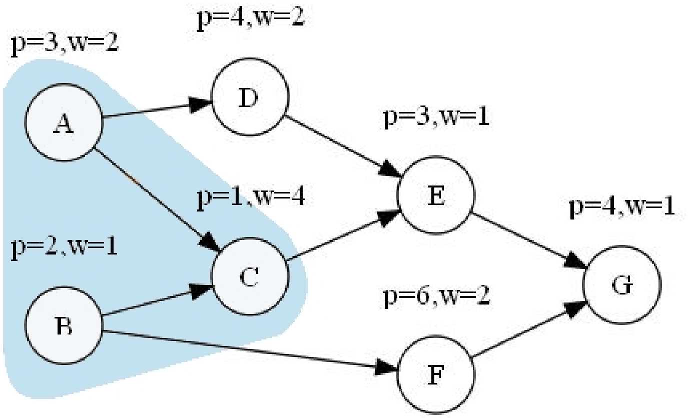
- breaks tasks into groups
  (P-time)
- ranks groups
- schedules each group in order of increasing rank
  (P-time, $\alpha = 2$)

Sub-DAG Rank

- given collection of tasks $T = \{t_1, t_2, ..., t_k\}$
- importance of scheduling T first
- $R(T) = \dfrac{\displaystyle\sum_{i=1}^{k} p_i}{\displaystyle\sum_{i=1}^{k} w_i}$

Precedence Closed Sub-DAG

- every task inside only depends on other tasks inside



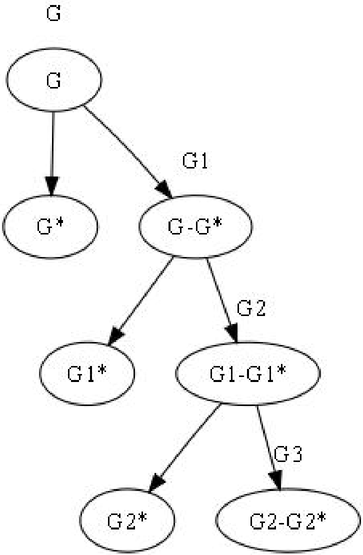
Minimal Rank Precedence Closed Sub-DAG, G*

- Properties:
  - feasible schedule for G^* is 2OPT
  - there exists an optimal schedule S of G where the optimal schedule for G^* comes as a segment starting at time 0

2 Approximation schedule for G^*

- if G^* has rank α, then any subgraph of G^* has rank higher than α
  $\forall j \in G^*, \frac{\sum_{1 \le i \le j} p_i}{\sum_{1 \le i \le j} w_i} \ge \alpha$
  $$\begin{aligned} \mathrm{OPT} &= \sum_j w_j C_j \\ &\ge \textstyle\sum_j w_j \sum_{i \le j} \alpha w_i = \alpha \sum_j (w_j)^2 + \sum_{i \le j} w_i w_j \\ &= \alpha (W(G^*))^2 - \frac{(W(G^*))^2}{2} = \frac{\alpha (W(G^*))^2}{2} = \frac{P(G^*) W(G^*)}{2} \end{aligned}$$
- any schedule with no idle time has weighted completion time of at most $P(G)W(G)$

Overview of the Algorithm



Approximation Factor

- Total weighted completion time of G is:
  $\gamma(G^*) + p(G^*)w(G - G^*) + \gamma(G - G^*)$
- How do we find G^*?

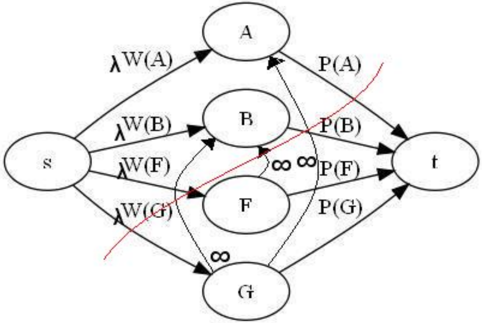
G^* Construction

construct a graph G_λ
solve it by finding min-cut of the graph
use min-cut to find sub-DAG of rank at most λ

- vertex set $V = T \cup \{(s, t)\}$
- add an edge from source s to every job with cost on it equal to λw_i
- add an edge from every job to the sink t with cost equal to processing time of the job
- for every precedence constraint between two vertices t_1, t_2 in G, then we add an edge from t_2 to t_1 having infinite cost

G^* Construction

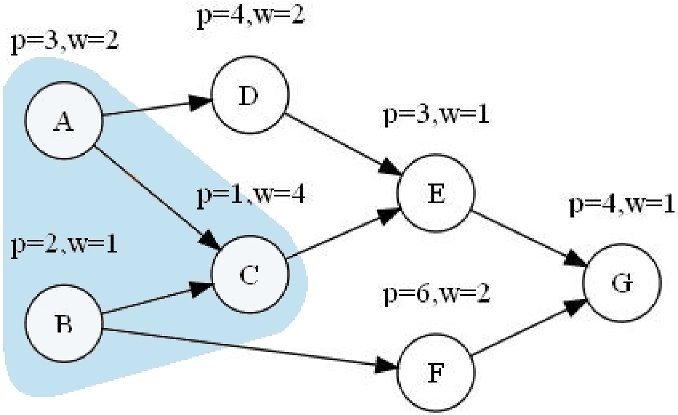
if there exists any cut (A, B) in G_λ whose value is bounded $\lambda w(G)$ then subgraph $A - \{s\}$ is precedence closed and that the rank of $A - \{s\}$ is at most λ



λ Values

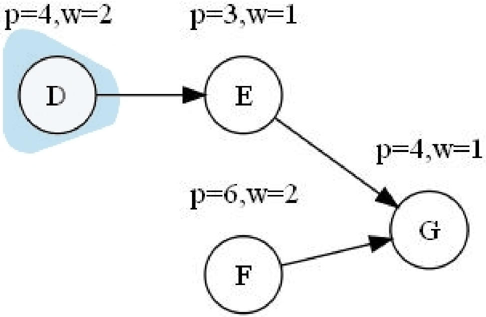
- how do we increase λ?
- λ_{min} = minimum rank of any vertex
- λ_{max} = rank of the graph
- perform binary search

Execution Step 1



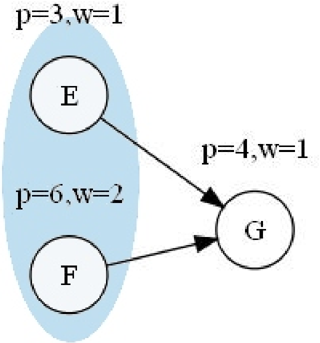
Here $\{A, B, C\}$ form a minimal rank precedence closed subgraph.

Execution Step 2



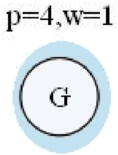
Here $\{D\}$ forms a minimal rank precedence closed subgraph.

Execution Step 3



$\{F, E\}$ is a minimal rank precedence closed subgraph.

Execution Step 4



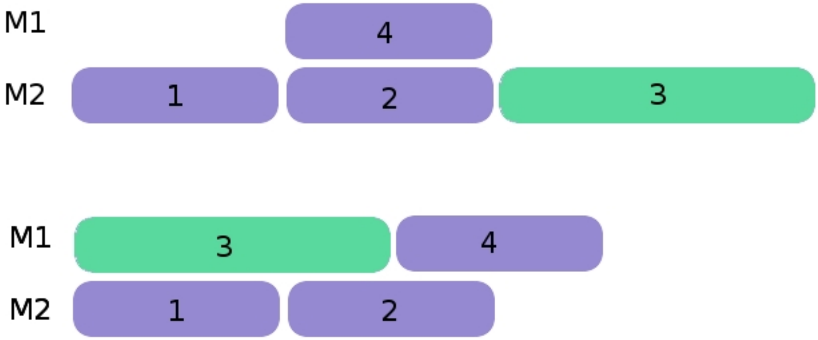
$\{G\}$ forms minimal rank precedence closed subgraph.

Multi-Machine Scheduling With Precedence

- requires feasible single-machine schedule as input
- uses identical machines $M = \langle m_1, m_2, \ldots, m_k \rangle$
- weighs parallelism increases against input schedule ordering

Delay List Intuition

- schedule lowest rank ready tasks next
- sometimes beneficial to schedule tasks out of order
  - take advantage of an idle processor
  - may bump back an important process



Delay List Algorithm

$t = 0$

if a machine m in M is idle, then:

    if the first task i in V is ready:

        schedule i on m

        mark all idle time up to start time of task i

    otherwise:

        scan through V, pick the first task i that is ready

        if $\beta * P_i \leq$ sum of all unmarked idle time

            schedule task i on m

    $t = t + 1$

Choosing β

- low β, avoid processor downtime, more out-of-order scheduling
- high β, accept more downtime, more faithful scheduling
- input schedule quality and weight variance important factors

Conclusions

- can produce approximate single-machine schedules
- single-machine schedules beget multi-machine schedules
- weight and processing time distributions help tuning both
- immense pratical significance