CS 880: Complexity of Counting Problems 04/10/2012

Lecture 21: Claims 8.2-8.4

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Last time we stoped at the claim 8.2, where we showed

$$\mathbf{D}_{2,a} = \lambda \mathbf{D}_{1,a} \quad a \in K \text{ and } \lambda \in \mathbb{C}, \\ \text{where } K = \{i \in [h] | \mathbf{D}_{1,i} \neq 0\}$$

We ultimately want to show that the equality is correct for every a. Let us recal the vanishing lemma A that we will use it for the first time in this lecture:
For a positive integer k and $1 \leq i \leq k$, let $\{x_{i,n}\}_{n\geq 1}$ be k infinite sequences of non-zero real numbers. In addition, let $\{x_{0,n}\}_{n\geq 1}$ be a sequence with $\{x_{0,n}\}_{n\geq 1} = 1$. The following is correct for all $0 \leq i < k$

$$\lim_{n\to\infty} \frac{x_{i+1,n}}{x_{i,n}} = 0.$$

Vanishing lemma A Let a_i and b_i be complex coefficients of $x_{i,n}$. Suppose

$$\exists 1 \leq l \leq k, \text{such that } a_i = b_i, \quad \forall 0 \leq i < l. \\ a_0 = b_0 = 1 \\ \text{Im}(a_l) = \text{Im}(b_l).$$

For *infinity many* n, $|\sum_{i=0}^{k} a_i x_{i,n}| = |\sum_{i=0}^{k} b_i x_{i,n}|$, then $a_l = b_l$.

We start this lecture by defining

$$K_2 = \{i \in [h] | \mathbf{D}_{2,i} \neq 0\}.$$

Note that K may not be a subset of K_2, in which for $a \in K$, $\mathbf{D}_{1,a} \neq 0$ but $\mathbf{D}_{2,a} = 0$ and by the claim 8.2 this means $\lambda = 0 \in \mathbb{C}$.
In addition, let

$$\mathcal{T}_g = \left\{ T_1 = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, T_2 = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, T_3 = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}, T_4 = \begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix} \right\}.$$

We want to show

$$\sum_{T\in\mathcal{T}_g} X_{v,T} = \sum_{T\in\mathcal{T}_g} X_{1,T} \quad \forall v \in A. \tag{1}$$

To do so we show that for every g'

$$\sum_{T\in\mathcal{T}_{g'}} X_{v,T} = \sum_{T\in\mathcal{T}_{g'}} X_{1,T} \qquad \text{where } 1 \leq g' < g \text{ and } v \in A.$$

And also if

$$Im\left(\sum_{T\in\mathcal{T}_g} X_{v,T}\right) = Im\left(\sum_{T\in\mathcal{T}_g} X_{1,T}\right),$$

then we can use the vanishing lemma part A to prove the equation (**??**).
Let us first consider all the $T \geq_\mu T_1$ which by definition have $\mu_b\mu_{b'} \geq \mu_1\mu_2$ and thus have one of

$$\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}1\\2\end{pmatrix}, \text{ or } \begin{pmatrix}2\\1\end{pmatrix}$$

as their first column.
Among these, the matrices that have at least one 1 in each row $\left(\text{recall } T = \begin{pmatrix} b & c \\ b' & c' \end{pmatrix}\right)$ have this neat property that $\mathbf{D}_1$ appears in both sum's of $X_{v,T}$

$$X_{v,T} = \left(\sum_{a\in[h]} \mathbf{D}_{b,a}\overline{\mathbf{D}_{c,a}}\mathcal{H}_{a,v}\right)\left(\sum_{a\in[h]} \mathbf{D}_{b',a}\overline{\mathbf{D}_{c',a}\mathcal{H}_{a,v}}\right).$$

Therefore we can consider two cases for the $X_{v,T}$, either $a \notin K$ and $X_{v,T} = 0$, or $a \in K$ and the other $\mathbf{D}$ term is a multiple of $\mathbf{D}_1$ so we are dealing with $\mathbf{D}_1\overline{\mathbf{D}_1}$.
On the other hand, we showed that for any $\mathcal{H}_{a,v} = \alpha$ is a root of unity whenever $a \in K$ and $v \in A$. Therefore $\mathcal{H}$ and its conjugate give us a product of a root of unity and its conjugate $\alpha\overline{\alpha}$ which is equal to 1. Since $\mathbf{D}_1$ is zero for $a \notin K$ we can have

$$X_{v,T} = \left(\sum_{a\in[h]} \mathbf{D}_{b,a}\overline{\mathbf{D}_{c,a}}\right)\left(\sum_{a\in[h]} \mathbf{D}_{b',a}\overline{\mathbf{D}_{c',a}}\right) = \|\mathbf{D}_1\|^4.$$

Therefore we can conclude that the equation **??** is correct for these matrices.
Next we need to show that the imaginary parts are the same for every $T \in \mathcal{T}_g$. In this case we need to consider T_3 and T_4 as well. From the above discussion $X_{v,T_1} = X_{1,T_1}$ and $X_{v,T_2} = X_{1,T_2}$ and we only need to consider T_3 and T_4 which occur as the following sum

$$\left(\sum_{a\in K} |D_{1,a}|^2\mathcal{H}_{a,v}\right)\left(\sum_{a=1}^{[h]} |D_{2,a}|^2\overline{\mathcal{H}_{a,v}}\right) + \left(\sum_{a\in K} |D_{2,a}|^2\mathcal{H}_{a,v}\right)\left(\sum_{a=1}^{[h]} |D_{1,a}|^2\overline{\mathcal{H}_{a,v}}\right). \quad (2)$$

From this sum it is clear that the conjugates cancel each other out and the sum ends up to be a real number. Hence the imaginary parts for every $T \in \mathcal{T}_g$ are the same and by using the vanishing lemma part A we conclude that

$$\sum_{T\in\mathcal{T}_g} X_{v,T} = \sum_{T\in\mathcal{T}_g} X_{1,T} \quad \forall v \in A.$$

Since the terms corresponding to T_1 and T_2 are equal ($X_{v,T_1} = X_{1,T_1}$ and $X_{v,T_2} = X_{1,T_2}$), therefore the sum becomes a sum on T_3 and T_4 terms that reaches the maximum possible amount

$$X_{v,T_3} + X_{v,T_4} = X_{1,T_3} + X_{1,T_4} = 2 \cdot \|\mathbf{D}_{1,*}\|^2 \|\mathbf{D}_{2,*}\|^2.$$

Let us consider the sum in (**??**) and consider the conditions that the addition reaches this maximum possible value. To reach this maximum, not only the coefficients of $\mathcal{H}$ must be constant, also they need to end up to give us a 2 (note that we consider the case when $\mathbf{D}_{1,*}\|^2\|\mathbf{D}_{2,*}\|^2 > 0$ because for the equal zero case the proof is trivial).
Let,

$$\mathcal{H}_{a,v} = \begin{cases} \beta_v & a \in K_2 \\ \alpha_v & a \in K \end{cases} \qquad \alpha_v, \beta_v \in \mathbb{C}, |\alpha_v| = |\beta_v| = 1,$$

therefore(recall that K may not be a subset of K_2),

$$\alpha_v \overline{\beta_v} + \overline{\alpha_v} \beta_v = 2,$$

which means $\alpha_v = \beta_v$.
At this point we proved the claim 8.3.
Claim 8.3. $\forall v \in A, \exists \alpha_v$ of norm 1, such that $\mathcal{H}_{a,v} = \alpha_v$ for all $a \in K_2 \cup K$.

We want to extend this to every a and show K_2 is in fact equal to K and we can have $\mathbf{D}_{2,*} = \mathbf{D}_{1,*}$ everywhere. To this goal, we first show that $|\mathbf{D}_{2,*}|^2 \perp \mathcal{H}_{a,v}$ for all $v \in B$. Obviously if $B = \emptyset$ the equation is true, so we can assume $B \neq \emptyset$.
Let,

$$T^* = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} \in \mathcal{T}_g.$$

In the property 3 we saw that

$$\sum_{T \in \mathcal{T}_i} X_{v,T} = 0 \quad i \in [d] \text{ and } v \in B,$$

therefore $\sum_{T \in \mathcal{T}_g} X_{v,T} = 0 \quad \forall v \in B$. We use this nice property and consider $T \in \mathcal{T}_g$ with $\mu_b \mu_{b'} = \mu_c \mu_{c'} = \mu_2^2$. To be in $\mathcal{T}_g$, a matrix can be constructed using the following columns

$$\begin{pmatrix} 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ s \end{pmatrix}, \text{ or } \begin{pmatrix} s \\ 1 \end{pmatrix},$$

where $s > 2$ such that $\mu_b \mu_{b'} = \mu_2 \mu_2$.
We divide the matrices in $\mathcal{T}_g$ into two cases as follows:
case 1: Let us consider matrices $T \in \mathcal{T}_g$ that have a row of the form (1 1), (1 2), or (2 1). Therefore

$$X_{v,T} = \left(\sum_{a=1}^{[h]} \mathbf{D}_{b,a} \overline{\mathbf{D}_{c,a}} \mathcal{H}_{a,v} \right) \left(\sum_{a=1}^{[h]} \mathbf{D}_{b',a} \overline{\mathbf{D}_{c',a} \mathcal{H}_{a,v}} \right) = 0,$$

we know that

1. $\mathbf{D}_1$ restricts a to $a \in K$.
2. on K, $\mathbf{D}_{1,*}$ and $\mathbf{D}_{2,*}$ are equal to a complex number (could be zero) times $\mathbf{D}_{1,*}$. therefore $\overline{\mathbf{D}_{1,*}} \cdot \mathbf{D}_{2,*}$ and $\mathbf{D}_{1,*} \cdot \overline{\mathbf{D}_{2,*}}$ are equal to $|\mathbf{D}_{1,*}|^2$ by a constant scale factor. Moreover,
3. $|\mathbf{D}_{1,*}|^2 \perp \mathcal{H}_{*,v}$ for all $v \in B$ (claim 8.1).

so the equality to zero is correct.
case 2: Next, let us consider the matrices $T \in \mathcal{T}_g$ without the above rows. Using the definition of the matrices in $\mathcal{T}_g$, these remaining matrices are

$$T^* = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}, T_1 = \begin{pmatrix} 1 & s \\ s & 1 \end{pmatrix}, T_2 = \begin{pmatrix} s & 1 \\ 1 & s \end{pmatrix}$$

Considering the X_v of these matrices shows that they have the conjugate-pair form with non-negative values

$$\left| X_{a=1}^{[h]} \mathbf{D}_{2,a} \overline{\mathbf{D}_{2,a}} \mathcal{H}_{a,v} \right|^2, \left| X_{a=1}^{[h]} \mathbf{D}_{1,a} \overline{\mathbf{D}_{s,a}} \mathcal{H}_{a,v} \right|^2, \left| X_{a=1}^{[h]} \mathbf{D}_{s,a} \overline{\mathbf{D}_{1,a}} \mathcal{H}_{a,v} \right|^2.$$

Since the sum of these non-negative values is equal to zero, then all are zeros.

$$X_{v,T^*} + X_{v,T_1} + X_{v,T_2} = 0.$$

In addition we can conclude that

$$|\mathbf{D}_{2,*}|^2 \perp \mathcal{H}_{*,v} \quad \forall v \in B$$

according to the same reasoning as before, this means

$$|\mathbf{D}_{2,*}|^2 \in span\{\mathcal{H}_{*,v} | v \in A\}$$

and therefore $|\mathbf{D}_{2,*}|^2$ is a constant on $K \cup K_2$. On the other hand, we defined K_2 such that $|\mathbf{D}_{2,*}|^2 \neq 0$ therefore this quantity is nonzero on K. Recall that on K, $\mathbf{D}_{2,*} = \lambda \mathbf{D}_{1,*}$ hence we can conclude that $\mathbf{D}_{2,a} = \lambda \mathbf{D}_{1,a}$ and also $K_2 \subset K$.
We have one more step to show $K_2 = K$. To this aim, we need to show that they have the same cardinality. Let χ_K be the characteristic vector for K where

$$\chi_K = \begin{cases} 1 & \text{on } K \\ 0 & O/W \end{cases}$$
$$\Rightarrow \chi_K = \sum_{v \in A} x_v \mathcal{H}_{*,v} \quad x_v \in \mathbb{C}$$

and also by using the claim 8.3 we can have:

$$\begin{aligned} x_v \|\mathcal{H}_{*,v}\|^2 &= < \chi_K, \mathcal{H}_{*,v} > \\ &= \sum_{a \in K} \overline{\mathcal{H}_{a,v}} \\ &= |K| \overline{\alpha_v} \quad \forall v \in A. \\ \Rightarrow |x_v| h &= |K| \forall v \in A. \end{aligned}$$

Thus

$$|K| = \|\chi_K\|^2 = \sum |x_v| \|\mathcal{H}_{*,v}\|^2 = |A|. \left(\frac{|K|}{h}\right)^2 = \frac{|A||K|^2}{h}$$

$$\Rightarrow |K| = \frac{h}{|A|}.$$

The exact same process for K_2 gives us the equality of the cardinalities:

$$|K| = |K_2| = \frac{h}{|A|}.$$

By this we extended the claim 8.2 and showed the correctness of claim 8.4 as we define it here:
Claim 8.4 There exists some complex number λ, such that $\mathbf{D}_{2,*} = \lambda \mathbf{D}_{1,*}$

The next step will be to extend this to have $\mathbf{D}_{l,*} = \lambda \mathbf{D}_{1,*}$. We do not go through this in the class but the explanations in the paper are clear to follow and understand.