CS 880: Complexity of Counting Problems	April 17, 2012

Lecture 25: More Structure for F

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This is a quick summary of the lecture.

- Recall that the rows of F are a column.
- We are going to apply Abelian group structure theory to say that F has a nice structure.

We have 3 facts:

- R_1: representation of $\mathbf{p}, \mathbf{t}, \mathbf{q}$.
- R_2: $C \in \mathbb{C}^{2m \times 2m}$ is a bipartization of $F \in \mathbb{C}^{2m \times 2m}$ where $M, N, C, \mathcal{D}$ satisfy the $\mathcal{U}$.
- R_3: Closed form of $F_{a,b} = \prod_{i,j} \omega_{q_{i,j}}^{a_{ij} b_{ij}}$.

We want to get the whole partition function as ω to a quadratic-form power.

We discussed the issues of a non-bipartite graph, but will not cover that topic.

We define for each $D^{[r]}$ the setsΛ_r and Δ_r. These sets are used to prove theorem 5.5 from the paper, saying that $EVAL(C, \mathcal{D})$ is sharp-P hard unless if the $\mathcal{L}$ properties are satisfied. We proved a very simple lemma, the equivalent of the Chinese Remainder Theorem of cosets.

Our goal then was to show that Δ_r is a coset. Consider the gadget on page 79. We replace all the edges with it, and once again we get a dramatic equation. However, that equation simplifies just as dramatically, and we can say that

$$A_{(0,u)(0,v)} = m^{4N} \left| \sum_{a \in \Delta_r} F_{u-v,a} \right|^2 . \tag{1}$$

In turn, you can understand the summation as an inner product:

$$A_{(0,u)(0,v)} = m^{4N} \left| \langle \chi, F_{u-v,\star} \rangle \right|^2 . \tag{2}$$

The value χ is 1 whenever $x \in \Delta_r$.

In considering the diagonals of the matrix, we realize that they are the largest value possible: $m^{4N} |\Delta_r|^2$. Therefore, every element in A is either 0 or the max value, by Bulatov-Grohe.

At this point we move in for the kill. We have a coset $\Phi = a + \langle \Delta_r - a \rangle$, and it is obvious that $\Delta_r \subset \Phi$. We prove, ultimately with a Fourier expansion, that Δ_r and Φ have the same cardinality, and therefore the same set.

Now we have two major notes:

1. $$|\langle \chi, F_{u,\star} \rangle| = n \implies \exists \alpha \in \mathbb{Z}_M \text{ s.t. } F_{u,x} = \omega_N^{\alpha} \forall x \in \Delta_r \tag{3}$$
2. otherwise,
$$\langle \chi, F_{u,\star} \rangle = 0 \implies \exists y, z \in \Delta_r \quad F_{u,y} \neq F_{u,z} . \tag{4}$$
This is a trivial observation.

Then we prove the following Lemma, 10.2: If $\exists \alpha F_{u,x} = \omega_N^{\alpha} \forall x \in \Delta_r$ then $F_{u,x} = \omega_M^{\alpha} \forall x \in \Phi$.

And concluded by proving Lemma 10.3: if $\exists u, z \in \Phi s.t. F_{u,y} \neq F_{u,z}$ then $\sum_{x \in \Phi} F_{u,x} = 0$.

After all this we conclude our major punchline, that $\Phi = \Delta_r$.