Corrigendum: The complexity of counting graph homomorphisms

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Abstract

We close a gap in the proof of Theorem 4.1 in our paper "The complexity of counting graph homomorphisms" [Random Structures and Algorithms **17** (2000), 260–289].

Our paper [2] analysed the complexity of counting graph homomorphisms from a given graph G into a fixed graph H. This problem was called $\#H$. A crucial step in our argument, Theorem 4.1, was to prove that the counting problem $\#H$ is $\#P$-complete if H has a connected component H_ℓ such that the counting problem $\#H_\ell$ associated with H_ℓ is $\#P$-complete. The first step of our proof claimed the existence of a positive integer r such that every entry of ${A_\ell}^r$ was positive, where A_ℓ is the adjacency matrix of H_ℓ. However, as Leslie Goldberg has recently pointed out to us, no such r exists when H_ℓ is bipartite. This claim was not used elsewhere in the paper, but its invalidity leaves a gap in the proof of Theorem 4.1. We give here an amended proof which shows how to deal with bipartite connected components of H.

Before presenting the proof we note that Bulatov and Grohe [1] have extended our result to the problem of computing a weighted sum of homomorphisms to a weighted graph H. This problem is equivalent to the problem of computing the partition function of a spin system from statistical physics, and to the problem of counting the solutions to a constraint satisfaction problem whose constraint language consists of two equivalence relations.

In the rest of the paper, the equation numbers correspond to those in [2]. The following interpolation result is well-known (a proof was given in [2]) and we state it again here for ease of reference.

Lemma 3.2 *Let $w_1, \ldots, w_r$ be known distinct nonzero constants. Suppose that we know values $f_1, \ldots, f_r$ such that*

$$f_s = \sum_{i=1}^{r} c_i \, {w_i}^s$$

for $1 \le s \le r$. The coefficients $c_1, \dots, c_r$ can be evaluated in polynomial time.

Let $[A^r]_{ij}$ denote the (i,j)th entry of the matrix A^r. The following is well-known but for completeness we provide a proof.

Lemma 4.2 *Let H be a connected graph with adjacency matrix A. If H contains an odd cycle then there exists a positive even integer r such that every entry of A^r is positive. Otherwise H is bipartite and there exists a positive even integer r such that $[A^r]_{ij}$ is positive whenever vertices i, j belong to the same side of the bipartition.*

Proof. Suppose that the vertex set of H is $C = \{1, \dots, k\}$. The value of $[A^r]_{ij}$ counts the number of walks from i to j in H of length exactly r. First suppose that H contains an odd cycle of length ℓ. Fix a vertex v in this odd cycle and let m_i be the length of the shortest path in H from vertex i to v, for $1 \le i \le k$. For $1 \le i \le j \le k$, define

$$m_{ij} = \begin{cases} m_i + m_j & \text{if} \quad m_i + m_j \quad \text{is even,} \\ m_i + m_j + \ell & \text{if} \quad m_i + m_j \quad \text{is odd.} \end{cases}$$

Then there is a walk from i to j of length m_{ij} in H. Note that m_{ij} is even and that there is a walk in H from i to j of length $m_{ij} + 2s$ for all integers $s \ge 0$. So setting $r = \max\{m_{ij}\}$ ensures that every entry of A^r is positive. (Of course this may also be true for some smaller value of r, which need not be even.)

If H has no odd cycle then H is bipartite and clearly there does not exist an integer $r > 0$ such that every entry of A^r is positive. Suppose that the vertex bipartition of H is $C^{(1)} \cup C^{(2)}$. Define m_{ij} to be the length of the shortest path from i to j for all pairs of vertices $i, j \in C^{(t)}$ for $t = 1, 2$. Then m_{ij} is even and there exists a walk from i to j in H of length $m_{ij} + 2s$, for all integers $s \ge 0$. Setting $r = \max\{m_{ij}\}$ ensures that $[A^r]_{ij} > 0$ whenever $i, j \in C^{(t)}$ for $t = 1, 2$, as required. □

Theorem 4.1 *Suppose that H is a graph with connected components $H_1, \dots, H_T$. If $\#H_\ell$ is #P-complete for some ℓ such that $1 \le \ell \le T$, then $\#H$ is #P-complete.*

Proof. Let A, A_ℓ be the adjacency matrix of H, H_ℓ respectively, for $1 \le \ell \le T$. We first assume that no component of H is bipartite, and subsequently we show how the proof must be modified to deal with bipartite components. Fix a positive integer r such that ${A_\ell}^r$ has only positive entries, for $1 \le \ell \le T$. The existence of r is guaranteed by applying Lemma 4.2 to each component of H and taking the maximum value of r. Note that r can be found in constant time. We show how to perform polynomial-time reductions from EVAL(A_ℓ) to EVAL(A), for $1 \le \ell \le T$. This is sufficient, since at least one of the problems EVAL(A_ℓ) is #P-hard, by assumption, and EVAL(A) is clearly in #P.

Let G be a given graph. We wish to calculate the values of $Z_{A_\ell}(G)$ in polynomial time, for $1 \le \ell \le T$. For $1 \le s \le T$, form the graph G_s with edge bipartition $E' \cup E''$ from G as follows. Let $v \in V$ be an arbitrary vertex of G. Take s copies of G, placing all these edges in E'. Let $\{v_i, v_j\} \in E''$ for $1 \le i < j \le s$, where v_i is the copy of v in the ith copy of G. These graphs can be formed from G in polynomial time. Now for $1 \le s \le T$ and $1 \le p \le s^{2k^2}$, form the graph $(S_r T_p)'' G_s$ by taking the p-thickening of each edge in E'', and then forming the r-stretch of each of these $ps(s-1)/2$ edges. That is, between v_i and v_j we have p copies of the path P_r, for $1 \le i < j \le s$. These graphs can be formed from G_s in polynomial time. Let $Z_A(G, c)$ be defined by

$$Z_A(G, c) = |\{X \in \Omega_H(G) \mid X(v) = c\}|.$$

Then

$$\begin{aligned} & Z_A((S_r T_p)'' G_s) \\ = & \sum_{\ell=1}^{T} \sum_{X:\{v_1,\ldots,v_s\}\mapsto C_\ell} \prod_{1\le i\le s} Z_{A_\ell}(G, X(v_i)) \prod_{1\le i<j\le s} \left({A^r}_{X(v_i)X(v_j)}\right)^p \qquad (12) \\ = & \sum_{w\in\mathcal{W}^{(s)}(A)} c_w\, w^p, \qquad (13) \end{aligned}$$

where $\mathcal{W}^{(s)}(A)$ is defined by

$$\mathcal{W}^{(s)}(A) = \left\{ \prod_{1\le i<j\le s} {A^r}_{X(v_i)X(v_j)} \mid X : \{v_1, \ldots, v_s\} \mapsto C_\ell \text{ for some } \ell, 1 \le \ell \le T \right\} \setminus \{0\}.$$

The set $\mathcal{W}^{(s)}(A)$ can be formed explicitly in polynomial time. Arguing as in (7), the set $\mathcal{W}^{(s)}(A)$ has at most s^{2k^2} distinct elements, all of which are positive. Suppose that we knew the values of $Z_A((S_r T_p)'' G_s)$ for $1 \le p \le |\mathcal{W}^{(s)}(A)|$. Then, by Lemma 3.2, the values c_w for $w \in \mathcal{W}^{(s)}(A)$ can be found in polynomial time. Adding them, we obtain $f_s = f_s(G) = \sum_{w\in\mathcal{W}^{(s)}(A)} c_w$. This value is also obtained by putting $p = 0$ in (13). Equating this to the value obtained by putting $p = 0$ in (12), we see that

$$\begin{aligned} f_s &= \sum_{\ell=1}^{T} \sum_{X:\{v_1,\ldots,v_s\}\mapsto C_\ell} \prod_{1\le i\le s} Z_{A_\ell}(G, X(v_i)) \\ &= \sum_{\ell=1}^{T} Z_{A_\ell}(G)^s. \end{aligned}$$

For ease of notation, let $x_\ell = Z_{A_\ell}(G)$ for $1 \le \ell \le T$. We know the values of $f_s = \sum_{\ell=1}^{T} {x_\ell}^s$ for $1 \le s \le T$. Let ψ_s be the sth elementary symmetric polynomial in variables $x_1, \ldots, x_T$, defined by

$$\psi_s = \sum_{1\le i_1<\cdots<i_s\le T} x_{i_1} \cdots x_{i_s}$$

for $1 \le s \le T$. Now

$$f_s - \psi_1 f_{s-1} + \cdots + (-1)^{s-1} f_1 \psi_{s-1} + (-1)^s s \psi_s = 0$$

for $1 \le s \le T$ (this is Newton's Theorem, see for example [3, p. 12]). Using these equations, we can evaluate ψ_s for $1 \le s \le T$ in polynomial time. But $x_1, \ldots, x_T$ are the roots of the polynomial

$$g(z) = z^T - \psi_1 z^{T-1} + \cdots + (-1)^{T-1}\psi_{T-1} z + (-1)^T \psi_T.$$

Since this is a polynomial with integral coefficients, the roots can be found in polynomial time using the algorithm of Lenstra, Lenstra and Lovász [4]. Thus we obtain the set of values $\{Z_{A_\ell}(G) \mid 1 \le \ell \le T\}$.

Let $N = |\{Z_{A_\ell}(G) \mid 1 \le \ell \le T\}|$. If $N = 1$ then all the values of $Z_{A_\ell}(G)$ are equal. Thus we know the value of $Z_{A_\ell}(G)$ for $1 \le \ell \le T$, as required. Otherwise, search for a connected graph Γ, with the minimal number of vertices, such that $|\{Z_{A_\ell}(\Gamma) \mid 1 \le \ell \le T\}| = N$. We know that Γ exists, since it is a minimal element of a nonempty set with a lower bound (the empty graph), using partial order on graphs defined by the number of vertices and inclusion. Moreover, Γ depends only on H. Therefore we can find Γ by exhaustive search, in constant time. (This constant may very well be huge, but we are not seeking a practical algorithm.) We also know the values $Z_{A_\ell}(\Gamma)$ for $1 \le \ell \le T$. Let $\sim$ be the equivalence relation on $\{1, \ldots, T\}$ such that $Z_{A_\ell}(\Gamma) = Z_{A_s}(\Gamma)$ if and only if $\ell \sim s$. Let π be the partition of $\{1, \ldots, T\}$ consisting of the equivalence classes of $\sim$. Write $\pi = I_1 \cup \cdots \cup I_N$ and let $\mu_j = |I_j|$ for $1 \le j \le N$. Finally, let $\mu = \max\{\mu_j \mid 1 \le j \le N\}$. Assume without loss of generality that $j \in I_j$ for $1 \le j \le N$. That is, the first N values of $Z_{A_\ell}(\Gamma)$ form a transversal of the N equivalence classes.

We perform a second reduction, which is an adaptation of the one just described. For $1 \le s \le \mu$ and $1 \le t \le N$, form the graph $G_{(s,t)}$ with edge bipartition $E' \cup E''$ as follows. Let w be an arbitrary vertex in Γ, and recall the distinguished vertex v in G. Take s copies of G and t copies of Γ, placing all these edges in E'. Let $V^* = \{w_1, \ldots, w_s\} \cup \{v_1, \ldots, v_t\}$, where w_i is the copy of w in the ith copy of Γ and v_j is the copy of v in the jth copy of G. Finally, let E'' be the set of all possible edges between the vertices in V^*. Form the graph $(S_r T_p)'' G_{(s,t)}$ for $1 \le p \le (s+t)^{2k^2}$, $1 \le s \le \mu$ and $1 \le t \le N$, by replacing each edge in E'' by p copies of the path P_r. Arguing as in the first reduction, the values of $Z_A((S_r T_p)'' G_{(s,t)})$ for $1 \le p \le (s+t)^{2k^2}$ can be used to produce the values

$$f_{(s,t)}(G) = \sum_{\ell=1}^{T} Z_{A_\ell}(G)^s \, Z_{A_\ell}(\Gamma)^t \tag{14}$$

for $1 \le s \le \mu$, $1 \le t \le N$, in polynomial time.

We can rewrite (14) as

$$f_{(s,t)}(G) = \sum_{j=1}^{N} \left(\sum_{\ell \in I_j} Z_{A_\ell}(G)^s \right) Z_{A_j}(\Gamma)^t.$$

For each fixed value of s, we know the value $f_{(s,t)}(G)$ for $1 \le t \le N$. First suppose that $Z_{A_\ell}(\Gamma) \ne 0$ for $1 \le \ell \le T$. Using Lemma 3.2, we obtain the coefficients ${c^{(s)}}_j =$

$\sum_{\ell\in I_j} Z_{A_\ell}(G)^s$, for $1 \le j \le N$, in polynomial time. We can do this for $1 \le s \le \mu$. Now suppose without loss of generality that $Z_{A_1}(\Gamma) = 0$. Then Lemma 3.2 only guarantees that we can find ${c^{(s)}}_j$ for $2 \le j \le N$, in polynomial time. However, we know the set $\{Z_{A_\ell}(G) \mid 1 \le \ell \le T\}$. Therefore we can form the value $c^{(s)} = \sum_{\ell=1}^{T} Z_{A_\ell}(G)^s$ in polynomial time, for $1 \le s \le \mu$. Then

$$ {c^{(s)}}_1 = \sum_{\ell\in I_1} Z_{A_\ell}(G)^s = c^{(s)} - \sum_{j=2}^{N} {c^{(s)}}_j. $$

Thus in both cases we can find the values of

$$ {c^{(s)}}_j = \sum_{\ell\in I_j} Z_{A_\ell}(G)^s $$

for $1 \le j \le N$ and $1 \le s \le \mu$, in polynomial time. Arguing as above, using Newton's Theorem, we can find the set of values $\{Z_{A_\ell}(G) \mid \ell \in I_j\}$ for $1 \le j \le N$ in polynomial time. If all these values are equal, then we know all the values $Z_{A_\ell}(G)$ for $\ell \in I_j$. Otherwise, we perform the second reduction again, for the graph $H_{I_j} = \cup_{\ell\in I_j} H_\ell$. We obtain a tree of polynomial-time reductions, where each internal node has at least two children, and there are at most T leaves. (A leaf is obtained when all values $Z_{A_\ell}(G)$ in the cell of the partition are equal, which will certainly happen when the cell is a singleton set.) There are at most T internal nodes in such a tree. That is, we must perform at most $T + 1$ reductions in all. This guarantees that we can obtain all the values $Z_{A_\ell}(G)$ for $1 \le \ell \le T$ in polynomial time, as required.

We must now consider the case of bipartite components. Let us number the components of H so that H_ℓ is non-bipartite for $1 \le \ell \le \nu$ and H_ℓ is bipartite for $\nu+1 \le \ell \le T$, with $\nu < T$. Suppose that H_ℓ has vertex bipartition $C_\ell^{(1)} \cup C_\ell^{(2)}$ for $\nu + 1 \le \ell \le T$. Let $r > 0$ be an even integer such that $[A^r]_{ij} > 0$ for all $i, j \in C_\ell$ when $1 \le \ell \le \nu$, and $[A^r]_{ij} > 0$ for all $i, j \in C_\ell^{(a)}$ for $a = 1, 2$ and $\nu + 1 \le \ell \le T$. The existence of such an r is guaranteed by applying Lemma 4.2 to each component of H and taking the maximum value of r. Note that r can be found in constant time.

We will assume that G is bipartite. Otherwise $Z_{A_\ell}(G) = 0$ for $\nu + 1 \le \ell \le T$ and the problem reduces to the non-bipartite case. For every ℓ let $x_\ell = Z_{A_\ell}(G)$ as before. Let v be the distinguished vertex of G. For $\nu + 1 \le \ell \le T$ let $x_{\ell,a}$ be the number of H_ℓ-colourings X of G such that $X(v) \in C_\ell^{(a)}$, for $a = 1, 2$. Clearly $Z_{A_\ell}(G) = x_\ell = x_{\ell,1} + x_{\ell,2}$ for $\nu + 1 \le \ell \le T$.

Our first construction is now modified as follows. For $1 \le s \le T$, choose an integer α, where $0 \le \alpha \le s$. We will form a graph $G_s^{(\alpha)}$ with edge bipartition $E' \cup E'' \cup E'''$. Take s copies of G, where vertex v_i is the copy of vertex v in the ith copy of G. All these edges are in E'. Now for $1 \le i < j \le s$ let

$$ \begin{array}{ll} \{v_i, v_j\} \in E'' & \text{if} \quad i, j \le \alpha \quad \text{or} \quad \alpha + 1 \le i, j, \\ \{v_i, v_j\} \in E''' & \text{otherwise.} \end{array} $$

This gives the graph $G_s^{(\alpha)}$. These graphs (for $1 \le s \le T$, $0 \le \alpha \le s$) can be formed from G in polynomial time. Now for $1 \le p \le s^{2k^2}$ replace each edge in E'' by p copies of P_r and replace each edge in E''' by p copies of P_{r+1}, giving the graph $B_{(s,p,\alpha)}$ (which can be formed in polynomial time). We now follow the same argument as that leading to the expression $f_s = \sum_{\ell=1}^{T} {x_\ell}^s$ above, but with $B_{(s,p,\alpha)}$ in place of $(S_rT_p)''G_s$. It is easy to verify that the effect of our construction is to replace ${x_\ell}^s$ in the expression for f_s by ${x_{\ell,1}}^\alpha {x_{\ell,2}}^{s-\alpha} + {x_{\ell,1}}^{s-\alpha} {x_{\ell,2}}^\alpha$, for $\nu + 1 \le \ell \le T$. Thus we can compute all quantities

$$f_s^{(\alpha)} = \sum_{\ell=1}^{\nu} {x_\ell}^s + \sum_{\ell=\nu+1}^{T} \left({x_{\ell,1}}^\alpha \, {x_{\ell,2}}^{s-\alpha} + {x_{\ell,1}}^{s-\alpha} \, {x_{\ell,2}}^\alpha\right)$$

for $0 \le \alpha \le s$ and $1 \le s \le T$, in polynomial time. We now take a weighted sum of these to give

$$f_s = \sum_{\alpha=0}^{s} \binom{s}{\alpha} f_s^{(\alpha)} = \sum_{\ell=1}^{\nu} (2x_\ell)^s + 2 \sum_{\ell=\nu+1}^{T} {x_\ell}^s$$

for $1 \le s \le T$. Proceeding as before we can construct and solve, in polynomial time, a polynomial equation with roots

$$2x_1,\ 2x_2, \ldots, 2x_\nu,\ x_{\nu+1},\ x_{\nu+1},\ x_{\nu+2},\ x_{\nu+2},\ \ldots, x_T,\ x_T.$$

Again we will assume that there are N distinct roots. If $N = 1$, with root x, then we set $x_\ell = x/2$ for $1 \le \ell \le \nu$ and $x_\ell = x$ for $\nu + 1 \le \ell \le T$, and we are done.

If $N > 1$ then we modify the second construction. We search for the smallest *bipartite* graph Γ such that there are N distinct values among the set

$$\{2Z_{A_\ell}(\Gamma) \mid 1 \le \ell \le \nu\} \cup \{Z_{A_\ell}(\Gamma) \mid \nu + 1 \le \ell \le T\}.$$

As before, such a graph Γ can be found in constant time. Let $y_\ell = Z_{A_\ell}(\Gamma)$ for $1 \le \ell \le T$. Define the equivalence relation $\sim$ on $\{1, \ldots, T\}$ by $i \sim j$ if and only if $\widehat{y}_i = \widehat{y}_j$, where $\widehat{y}_\ell = 2y_\ell$ if $1 \le \ell \le \nu$ and $\widehat{y}_\ell = y_\ell$ otherwise. Form the partition $\pi = I_1 \cup \cdots \cup I_N$ of $\{1, \ldots, T\}$ corresponding to this equivalence relation, as before. Let Γ have vertex bipartition $\Lambda_1 \cup \Lambda_2$ such that the distinguished vertex $w \in \Lambda_1$. For $\nu + 1 \le \ell \le T$ and $a = 1, 2$, define the quantity $y_{\ell,a}$ for Γ similarly to $x_{\ell,a}$ for G. Let μ be the maximum of μ_j for $1 \le j \le N$, where

$$\mu_j = |I_j \cap \{1, \ldots \nu\}| + 2|I_j \cap \{\nu + 1, \ldots, T\}|.$$

For $1 \le s \le \mu$ and $1 \le t \le N$, choose integers α, β with $0 \le \alpha \le s$ and $0 \le \beta \le t$. Form the graph $G_{(s,t)}^{(\alpha,\beta)}$ with edge bipartition $E' \cup E'' \cup E'''$, as follows. Start with the graph $G_s^{(\alpha)}$ as described above. Then take t copies of Γ, adding all these edges into E' (and expanding the vertex set as well). Let v_i be the copy of v in the ith copy of G and let w_j be the copy of w in the jth copy of Γ. Finally, add extra edges to the sets E'', E''' as follows: for $1 \le i < j \le t$ let

$$\begin{aligned} &\{w_i, w_j\} \in E'' \qquad \text{if} \quad i, j \le \beta \quad \text{or} \quad \beta + 1 \le i, j, \\ &\{w_i, w_j\} \in E''' \qquad \text{otherwise,} \end{aligned}$$

and for $1 \le i \le s$, $1 \le j \le t$ let

$$\begin{array}{ll} \{v_i, w_j\} \in E'' & \text{if} \quad i \le \alpha \quad \text{and} \quad j \le \beta, \quad \text{or} \quad \alpha+1 \le i \quad \text{and} \quad \beta+1 \le j, \\ \{v_i, w_j\} \in E''' & \text{otherwise.} \end{array}$$

This describes the graph $G^{(\alpha,\beta)}_{(s,t)}$. These graphs (for $1 \le s \le \mu$, $1 \le t \le N$, $0 \le \alpha \le s$, $0 \le \beta \le t$) can be formed from G in polynomial time. Now for $1 \le p \le (s+t)^{2k^2}$, replace each edge in E'' by p copies of P_r and replace each edge in E''' by p copies of P_{r+1} to produce the graph $B^{(\alpha,\beta)}_{(s,t,p)}$ (in polynomial time). Now argue as before, but with $B^{(\alpha,\beta)}_{(s,t,p)}$ in place of $(S_rT_p)''G_{(s,t)}$, to find that we can compute all the quantities

$$f^{(\alpha,\beta)}_{(s,t)} = \sum_{\ell=1}^{\nu} {x_\ell}^s {y_\ell}^t + \sum_{\ell=\nu+1}^{T} ({x_{\ell,1}}^\alpha\, {x_{\ell,2}}^{s-\alpha}\, {y_{\ell,1}}^\beta\, {y_{\ell,2}}^{t-\beta} + {x_{\ell,1}}^{s-\alpha}\, {x_{\ell,2}}^{\alpha}\, {y_{\ell,1}}^{t-\beta}\, {y_{\ell,2}}^{\beta})$$

for $0 \le \alpha \le s$, $0 \le \beta \le t$, in polynomial time. Taking a weighted sum of these quantities gives

$$f_{(s,t)} = \sum_{\alpha=0}^{s} \sum_{\beta=0}^{t} \binom{s}{\alpha} \binom{t}{\beta} f^{(\alpha,\beta)}_{(s,t)} = \sum_{\ell=1}^{\nu} (2x_\ell)^s (2y_\ell)^t + 2 \sum_{\ell=\nu+1}^{T} {x_\ell}^s {y_\ell}^t$$

for $1 \le s \le \mu$ and $1 \le t \le N$. We now proceed as we did with (14), refining the partition in a constant number of steps until we are done. To give a bit more detail: the equation above can be rewritten as

$$f_{(s,t)} = \sum_{j=1}^{N} {c^{(s)}}_j {z_j}^t$$

where z_j is the common value of $\widehat{y}_\ell$ for all $\ell \in I_j$ and

$${c^{(s)}}_j = \sum_{\ell \in I_j \cap \{1,\ldots,\nu\}} (2x_\ell)^s + 2 \left(\sum_{\ell \in I_j \cap \{\nu+1,\ldots,T\}} {x_\ell}^s \right).$$

So using Lemma 3.2 we can find ${c^{(s)}}_j$ for $1 \le j \le N$, $1 \le s \le \mu$, in polynomial time. Then using Newton's theorem we can find the set

$$\{2Z_{A_\ell}(G) \mid \ell \in I_j \cap \{1, \ldots, \nu\}\} \cup \{Z_{A_\ell}(G) \mid \ell \in I_j \cap \{\nu+1, \ldots, T\}\}$$

in polynomial time. If all of these values are equal to some integer x then we know that $Z_{A_\ell}(G) = x/2$ for $1 \le \ell \le \nu$ and $Z_{A_\ell}(G) = x$ for $\nu+1 \le \ell \le T$, and we are done. If they are not all equal then we repeat the argument with $H_{I_j} = \cup_{\ell \in I_j} H_\ell$ to continue to refine the partition. □

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