CS 880: Complexity of Counting Problems 02/07/2012

Lecture 5: Holant* Dichotomy Part I: Tractability

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1 The Dichotomy

Before stating the dichotomy, let's make a quick note on degenerate signatures. A signature $[x_0, \ldots, x_n]$ is called degenerate iff the matrix $\begin{pmatrix} x_0 & \ldots & x_{n-1} \\ x_1 & \ldots & x_n \end{pmatrix}$ has rank less than 2. Equivalently speaking, f of arity n is degenerate iff there exists a, b such that $f = [a, b]^{\otimes n}$. Thus, if we have a degenerate signature, it's equivalent to replace it with n copies of unary signatures. In the following we only consider non-degenerate signatures.

The Holant* problem is the Holant problem when all unary signatures are free to use. When we say hardness, we are referring to a finite subset of the signature set.

Theorem 1 (Holant* Dichotomy)**.** *Holant*($\mathcal{F}$) is #P-hard unless $\mathcal{F}$ has the following form, in which case it is tractable:*

1. *Every $f \in \mathcal{F}$ is of arity less than 2.*
2. *$\exists (a, b) \neq 0$, s.t. $\forall [x_0, \ldots, x_n] \in \mathcal{F}$, either $ax_k + bx_{k+1} - ax_{k+2} = 0$, or $n = 2$ and $[x_0, x_1, x_2] = [2a\lambda, b\lambda, -2a\lambda]$.*
3. *Every signature $[x_0, \ldots, x_n] \in \mathcal{F}$ satisfies either $x_k + x_{k+2} = 0$ or $n = 2$ and it's of the form $[\lambda, 0, \lambda]$.*

The first case can be easily computed by matrix product and taking the trace. Now we explain why the other two are tractable as well.

2 Case 2

The second case is trivial when $a = 0$. When $a \neq 0$, we have the characteristic polynomial $x^2 = (b/a)x + 1$. Let's say this equation has two roots λ_1, λ_2.

2.1 Case 2.1: $\lambda_1 \neq \lambda_2$

Let $M = \begin{pmatrix} 1 & \lambda_1 \\ 1 & \lambda_2 \end{pmatrix}$, and we know that $\lambda_1 \lambda_2 = -1$. For any signature $f \in \mathcal{F}$, from the recurrence relation, we have that $f = u \begin{pmatrix} 1 \\ \lambda_1 \end{pmatrix}^{\otimes n} + v \begin{pmatrix} 1 \\ \lambda_2 \end{pmatrix}^{\otimes n}$ for some u and v. So we have

$$M^{\otimes n} f = u \begin{pmatrix} 1+\lambda_1^2 \\ 0 \end{pmatrix}^{\otimes n} + v \begin{pmatrix} 0 \\ 1+\lambda_2^2 \end{pmatrix}^{\otimes n}$$

which is a generalized equality function.

In the meantime, if there is a signature g of arity 2 as specified in the condition. we have

$$M^{\otimes 2} g = [0, \frac{4a^2+b^2}{a}, 0].$$

On the other side, the binary equality becomes

$$(=_2)(M^{-1})^{\otimes 2} = [1+\lambda_2^2, 0, 1+\lambda_1^2].$$

The upshot is that unary signatures become unaries, fibonacci equalities, g disequalities, $=_2$ still equalities. When there's only equalities, disequalities, and unaries left, one can just propagate the value and then flip it to compute the Holant quickly.

2.2 Case 2.1: $\lambda_1 \neq \lambda_2$

Since $\lambda_1 \lambda_2 = -1$, we may assume $a = 1$ and $b = \pm 2i$. In this case g is $[1, i]^{\otimes 2}$, then we don't need to care about it. For any $[x_0, \ldots, x_n] \in \mathcal{F}$, we have $x_k = ski^{k-1} + ti^k$.

Let X' as follows:

$$c \begin{pmatrix} 1 \\ 0 \end{pmatrix}^{\otimes n} + d \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix}^{\otimes n-1} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \cdots + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}^{\otimes n-1} \right]$$

where $c = t - \frac{sni}{2}$, and $d = \frac{si}{2}$.

Now we claim that $Z^{\otimes n} X' = X$, where $Z = \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}$. This can be checked by picking one term of each weight since the signature is symmetric.

Then $\text{Holant}(=_2 \cup \mathcal{U} \mid X \cup \mathcal{U}) = \text{Holant}((=_2 Z) \cup \mathcal{U}Z \mid Z^{-1}X \cup Z^{-1}\mathcal{U}) = \text{Holant}((=_2 Z) \cup \mathcal{U} \mid Z^{-1}X \cup \mathcal{U})$.

For a binary signature v, one can write v as a matrix V, then $M^T V M$ is the matrix form of $vM^{\otimes 2}$. Then $(=_2)Z^{\otimes 2}$ becomes $Z^T Z = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$, which is disequality. Thus, on the left side, there has to be exactly half 0 and half 1 assigned. However, on the right each signature has only as most one 1. For any unary signatures, we can do the propagate and reduce the arity of signatures on the right. However, if we want to have exactly half 1s, all signatures must of arity less than or equal to 2, in which case the problem is tractable.

3 Case 3

First we notice that the Holant is 0 unless the graph G is bipartite. Consider any odd cycle. For an arbitrary assignment σ. Change it into σ' by flipping all assignment along the cycle. For any vertex on the cycle, if the two assignments on its incident edges are the same in σ, the evaluation of σ equals the negative of that of σ'. Otherwise it's the same. However it's an odd cycle, there has to be odd number of non-changes along the cycle. Thus, these two assignments has opposite evaluation and the Holant cancels out completely.

The rest details are left as exercise.