CS 880: Complexity of Counting Problems	02/09/2012

Lecture 6: Holant* Dichotomy Part II: Hardness for a single ternary signature

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1 When is a ternary signature hard?

Assume we have $f = [x_0, x_1, x_2, x_3]$. The Holant$^*(f)$ is hard unless one of the following holds:

1. $\exists (a, b) \neq 0$, $ax_0 + bx_1 - ax_2 = 0$, and $ax_0 + bx_1 - ax_2 = 0$.
2. $x_0 + x_2 = 0$, and $x_1 + x_3 = 0$.

Since $\begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ has a 1-dimension solution, we can get a recurrence relation going normally or backwards. Thus depending on if it has double roots, we have three cases

1. $x_k = \alpha_1^{3-k}\alpha_2^k + \beta_1^{3-k}\beta_2^k$, where $\begin{pmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{pmatrix}$ is non-singular.
2. $x_k = ck\alpha^{k-1} + d\alpha^k$, where $c \neq 0$ because it's non-degenerate.
3. The reversal of case 2.

1.1 Case 1

In this case, we have $X = [x_0, x_1, x_2, x_3] = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}^{\otimes 3} + \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}^{\otimes 3}$. Let $M = \begin{pmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{pmatrix}$, then $X = M^{\otimes 3}(=_3)$. Thus Holant$(=_2 \cup \mathcal{U} \mid M^{\otimes 3}(=_3) \cup \mathcal{U})$ = Holant$((=_2 M^{\otimes 2}) \cup \mathcal{U} \mid (=_3) \cup \mathcal{U})$, and we kind of stuck.

Now let's check the two tractable conditions, not satisfying condition 1 implies $\alpha_1\beta_1 + \alpha_2\beta_2 \neq 0$ and condition 2 implies $\alpha_1^2 + \alpha_2^2 \neq 0$. Then we the holographic transformation under a complex-orthogonal matrix

$$T = \frac{1}{\sqrt{\alpha_1^2 + \alpha_2^2}} \begin{pmatrix} \alpha_1 & \alpha_2 \\ -\alpha_2 & \alpha_1 \end{pmatrix}$$

Thus Holant$^*(=_2 \mid X)$ = Holant$^*\left(=_2 \mid \begin{pmatrix} 1 \\ 0 \end{pmatrix}^{\otimes 3} + \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}^{\otimes 3}\right)$, and $\beta_1\beta_2 \neq 2$. Here we are renaming the β's.

The Vertex-Cover problem is hard even on 3-regular graphs. In the Holant language, it is Holant([0, 1, 1] | [1, 0, 0, 1]. Now let $M = \begin{pmatrix} 1 & \beta_1 \\ 0 & \beta_2 \end{pmatrix}$, we have

$$\text{Holant}([0,1,1] \mid [1,0,0,1]) = \text{Holant}\left([0,1,1](M^{-1})^{\otimes 2} \mid \begin{pmatrix} 1 \\ 0 \end{pmatrix}^{\otimes 3} + \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}^{\otimes 3} \right)$$

On the left is in fact $[0, 1, 1](M^{-1})^{\otimes 2} = \frac{1}{\beta^2}[0, \beta_2, 1 - 2\beta_1]$.

Thus, if we can construct the binary signature on the left, the problem is hard. Now we construct the gadget by linearly connecting three arity-3 nodes and each connected with a unary signature. The two unary on either side is the same, call it $[t_0, t_1]$ and the middle one different, $[s_0, s_1]$. We want this gadget to be $[0, \beta_2, 1 - 2\beta_1]$. This is not hard to compute and we in fact have two degrees of freedom here. Thus it will succeed on most of the cases.

The exception cases are as follows:

1. $\beta_1 = 1, \beta_2 = \pm i$.
2. $\beta_1 = -\frac{1}{2}, \beta_2 = \pm\frac{i}{\sqrt{2}}$.
3. $\beta_1 = \frac{1}{2}, \beta_2 = \pm\frac{i}{2}$.
4. $\beta_1 = \frac{1}{2}, \beta_2 = \pm\frac{i}{\sqrt{2}}$.

Now we consider another hard problem to start with, that is, the Independent-Set problem on 3-regular graphs. In the Holant language it is Holant([1, 1, 0] | [1, 0, 0, 1]). By the same argument, we need to construct $[\beta_2, 1 - \beta_1, \beta_1^2 - 2\beta_1]$. In case 1, it is equality and is for free. In case 2, we want to construct a different gadget with only two nodes and one unary signature used. It will cover case 2. For case 3 and 4 we apply a transformation of $M = \begin{pmatrix} -i & -\sqrt{2} \\ \sqrt{2} & -i \end{pmatrix}$ or $\begin{pmatrix} i & -\sqrt{2} \\ \sqrt{2} & i \end{pmatrix}$, and it can be covered as well.

1.2 Case 2

Similarly, for case 2, not satisfying condition 1 implies $\alpha \neq \pm i$, and condition 2 gives nothing other than $c \neq 0$.

Now let $T = \begin{pmatrix} 1 & \frac{d-1}{3} \\ \alpha & c + \frac{d-1}{3}\alpha \end{pmatrix}$. Claim $T^{\otimes 3}\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} = X$. This expression is from previous works about matchgates.

Then we do the QR decomposition of T. So $T = QR$ where Q is symmetric and orthogonal. $Q = \frac{1}{\sqrt{1+\alpha^2}}\begin{pmatrix} 1 & \alpha \\ \alpha & -1 \end{pmatrix}$. $QT = R = \begin{pmatrix} r_1 & r_2 \\ 0 & r_3 \end{pmatrix}$. So Holant$^*(X)$ = Holant$^*(Q^{\otimes 3}X)$. But

$$Q^{\otimes 3}X = R^{\otimes 3}\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}^{\otimes 3} + \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix}^{\otimes 2} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \cdots + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}^{\otimes 2} \right] \right)$$

It is easily verified that the last two entries of this signature are 0. The expression is in fact $[*, r_1^2 r_3, 0, 0]$. We can normalize it and get, say, $[v, 1, 0, 0]$. Now we do one more $\begin{pmatrix} 1 & \frac{v-1}{3} \\ 0 & 1 \end{pmatrix}$ and we are done.