Recall from the previous lecture that $D_r = \sum_{b=0}^{N-1} \alpha_{a,b} \omega_r^b N$, and also $D_0 = \sum_{b=0}^{N-1} \alpha_{a,b}$ with positive diagonal entries, such that

$$D_0 = \left( \begin{array}{cc} D_0^{(0,*)} & D_0^{(1,*)} \\ \end{array} \right) = \left( \begin{array}{cc} K_0^{[0]} \otimes L_0^{[0]} & K_0^{[0]} \otimes L_0^{[1]} \\ \end{array} \right).$$

Since $D = \{D_0, \ldots, D_{N-1}\}$ satisfy the $T_3$ properties, then we have:

$$D_r = D_{N-r}.$$ 

In other words, $D_0$ is consists of an upper half and a lower half and there are blocks of a constant times the identity matrix in some size (section 8.4). Here we want to show the same is correct for $D_r$.

$$\exists K_r^{(0,*)} \times (s+t)$$(Shape) and $L_r^{[r]}_{2h \times 2h}$ such that:

$$D_r = \left( \begin{array}{cc} D_r^{(0,*)} & D_r^{(1,*)} \\ \end{array} \right) = \left( \begin{array}{cc} K_r^{[0]} \otimes L_r^{[0]} & K_r^{[0]} \otimes L_r^{[1]} \\ \end{array} \right).$$

To prove this we need to assume (lemma 8.8) for any fixed $1 \leq r \leq N-1$

Let $D_{i,j} = D_r^{(0,(i,j))}$ $\forall i \in [s]$ and $j \in [h]$ 

$$D = (D_{i,j}),$$

with rank$(D) \leq 1$ and if $D_{i,j} \neq 0$ and $D_{i,j'} \neq 0$ then $|D_{i,j}| = |D_{i,j'}|$. Having this assumption in our deposit, every non-zero $D_r^{(0,*)}$ is of rank 1, therefore can be written as $D_r^{(0,*)} = (D_{i,b})/D_{(a,b)}^{(0,*)}$ for any $i \in [s]$. Therefore the obvious choice for the corresponding $K_r^{[r]}$ and $L_r^{[r]}$ are as follows

$$K_r^{(0,i)} = D_{i,b} \text{ and } L_r^{(0,j)} = \frac{D_{a,j}}{D_{a,b}} \forall i \in [s], j \in [h].$$

Hence,

$$D_r^{(0,(i,j))} = D_{i,j} = K_r^{(0,i)} \cdot L_r^{(0,j)}, \forall i \in [s] \text{ and } j \in [h].$$

One can follow the same process for $D_r^{(1,*)}$ and show equation 1 is correct.

To prove the above assumption (lemma 8.8), one needs to to consider the vanishing lemma (8.4.1) as follows.
(8.4.1) The Vanishing Lemma

For a positive integer \( k \) and \( 1 \leq i \leq k \), let \( \{x_{i,n}\}_{n \geq 1} \) be \( k \) infinite sequences of non-zero real numbers. In addition, let \( \{x_{0,n}\}_{n \geq 1} \) be a sequence with \( \{x_{0,n}\}_{n \geq 1} = 1 \). The following is correct for all \( 0 \leq i < k \)

\[
\lim_{n \to \infty} \frac{x_{i+1,n}}{x_{i,n}} = 0.
\]

**Part A** Let \( a_i \) and \( b_i \) be complex coefficients of \( x_{i,n} \). Suppose

\[
\exists 1 \leq l \leq k, \text{ such that } a_i = b_i, \quad \forall 0 \leq i < l.
\]

\[
a_0 = b_0 = 1
\]

\[
\text{Im}(a_l) = \text{Im}(b_l).
\]

For infinity many \( n \), \( |\sum_{i=0}^{k} a_i x_{i,n}| = |\sum_{i=0}^{k} b_i x_{i,n}| \), then \( a_l = b_l \).

**Part B** Let \( a_i \in \mathbb{C} \), for \( 0 \leq i < k \). For infinity many \( n \), \( |\sum_{i=0}^{k} a_i x_{i,n}| = 0 \) then \( a_i = 0 \), for all \( 0 \leq i \leq k \).

We note that in both parts \( x_{i,n} \) are real and cannot be extended to the complex numbers. This is only allowed to choose the coefficients from \( \mathbb{C} \).

**Proof.** The proof of Part A starts with multiplying the equation with the conjugate terms:

\[
\left( \sum_{i=0}^{k} a_i x_{i,n} \right) \left( \sum_{j=0}^{k} \overline{a}_j x_{j,n} \right) = \left( \sum_{i=0}^{k} b_i x_{i,n} \right) \left( \sum_{j=0}^{k} \overline{b}_j x_{j,n} \right)
\]

Next consider the following conditions:

1. \( \max\{i, j\} < l \). Recall that in this case \( a_i = b_i \), then \( a_i \overline{a}_j x_{i,n} x_{j,n} = b_i \overline{b}_j x_{i,n} x_{j,n} \) and the equality is satisfied.

2. \( (\max\{i, j\} > l) \text{ or } (\max\{i, j\} = l \text{ and } \min i, j > 0) \). As \( n \to \infty \) then both side of the equality become of order \( o(|x_{l,n}|) \) and cancel each other, hence the equality is satisfied.

3. \( \max\{i, j\} = l \text{ and } \min\{i, j\} = 0 \). After canceling the identical terms and sending \( n \) to infinity \( n \to \infty \) the equation becomes

\[
(a_l + \overline{a_l})x_{l,n} + o(|x_{l,n}|) = (b_l + \overline{b_l})x_{l,n} + o(|x_{l,n}|).
\]

After dividing the equation by \( x_{l,n} \) the remaining terms show \( \text{Re}(a_l) = \text{Re}(b_l) \). Recall \( \text{Im}(a_l) = \text{Im}(b_l) \), therefore \( a_l = b_l \) and the equation is satisfied.
Note that the equality of the imaginary parts plays an important role in the proof and the lemma is not correct without this condition. A supporting example is

\[ a_1 = 3 + \sqrt{3}i, \quad a_2 = 3(\frac{1}{2} + \frac{\sqrt{3}}{2}i), \quad \text{and} \quad b_1 = b_2 = 3. \]

Then \( |1 + a_1x + a_2x^2| = |1 + b_1x + b_2x^2| \) for all real \( x \), particularly when \( x \to 0 \).

To prove Part B, one needs to consider that \( \{x_{i,n}\} \) is non-zero. Since \( x_{0,n} = 1 \), then \( a_0 = 0 \). Next, for simplicity, let's normalize the sequence by \( x_{1,n} \), then the sum becomes

\[ \left| \sum_{i=1}^{k} a_{i} x_{i,n} / x_{1,n} \right| = |a_1 + a_2 x_{2,n} / x_{1,n} + \ldots| = 0. \]

therefore \( a_1=0 \). By induction it is clear that \( a_i = 0 \).

Here we have enough tools to prove lemma 8.8.

**Lemma 1.** If we are not dealing with \( \text{EVAL}(C, \Xi) \#P \)-hard, then \( \text{rank}(D) \) is at most 1, and if \( D_{i,j} \neq 0 \) and \( D_{i,j'} \neq 0 \) then \( |D_{i,j}| = |D_{i,j'}| \forall i \in [s] \) and \( j, j' \in [h] \).

**Proof.** To proof this lemma consider an undirected graph \( G = (V, E) \) where we substitute every edge in \( uv \in E \), by the gadget shown in the following figure (Figure 4 of the paper).

![Figure 1: The gadget for constructing \( G^{[n]} \). Note that the node \( a \) is connected to \( 2r \) nodes in top and the nodes \( u \) and \( v \) in bottom. These nodes are connected to the conjugate of \( a \) as well. Hence swapping \( u \) and \( v \) does not change the gadget; a symmetric graph.](image)

Some properties of the graph \( G \):

The node \( b \) in this graph is connected to \( x_{1,r} \) via the function \( p_n = n^2N + 1 \equiv 1(N) \) and the node \( c \) is connected to \( y_{1,r} \) via \( g_n = Nn - 1 \equiv -1(N) \) (\( \forall r \geq 1 \)). One observation could be comparing these functions as \( n \to \infty \), which yields to \( p_n > q_n \). Later we will use the fact that
\( \deg(b) \equiv r(N) \) and \( \deg(c) \equiv -r(N) \), which means they are conjugate of each other. When we pass this graph to our partition function \( Z \), one needs to consider all the assignments to \( u \) and \( v \) and moreover computes the collaborations of the virtual vertices. By taking advantage of the symmetry of \( G \), we can construct a matrix \( R \) such that

\[
Z_{R^{[n]}, \mathcal{D}^*}(G) = Z_{C, \mathcal{D}}(G^{[n]})
\]

\( \Rightarrow \text{EVAL}(R^{[n]}, \mathcal{D}^*) \leq \text{EVAL}(C, \mathcal{D}) \) so \( \text{EVAL}(R^{[n]}, \mathcal{D}^*) \) is not \( \#P \)-hard.

Note that the matrix \( R \) has the same dimension as \( C \) but with opposite shape, therefore

\[
R^{[0]} = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}.
\]

Because of the shape of \( R \) we have

\[
R_{(0,u), (1,v)}^{[n]} = R_{(1,u), (0,v)}^{[n]} = 0, \quad \forall u \in I, v \in J.
\]

The entries of \( R \) are as follows:

\[
R_{(1,u), (1,v)}^{[n]} = \left( \sum_{a,b,c \in I \times [h]} (I)^r(II)^r F_{a,u} F_{a,v}^{-1} D^{[0]}_{(0,a)} D^{[r]}_{(0,b)} D^{[N-r]}_{(0,c)} \right)
\]

\[
\times \left( \sum_{a,b,c \in I \times [h]} (I)^r(II)^r F_{a,u} F_{a,v}^{-1} D^{[0]}_{(0,a)} D^{[r]}_{(0,b)} D^{[N-r]}_{(0,c)} \right), \quad \forall u, v \in J \times [h]
\]

Where

\[
I = \sum_{x \in J} F_{(0,x)}^{N-1} F_{(0,x)}^{p_n} D^{[0]}_{(1,x)} = \mu_a^{N-1} \mu_b^{p_n} \sum_{x \in J} \nu_{(x_1)}^{N-1} \nu_{(x_1)}^{p_n} H_{a_2,x_2}^{N-1} H_{b_2,x_2}^{p_n} D^{[0]}_{(1,(x_1,1))}
\]

Note that \( H_{a_2,x_2} \) are roots of unity so we can get the conjugate \( H_{a_2,x_2}^{N-1} = H_{a_2,x_2} \). And also \( p_n \equiv 1(N) \) then \( H_{b_2,x_2}^{p_n} = H_{b_2,x_2} \). Therefore

\[
I = \mu_a^{N-1} \mu_b^{p_n} \sum_{x_1 \in [t]} \nu_{(x_1)}^{N-1} \nu_{(x_1)}^{p_n} D^{[0]}_{(1,(x_1,1))} < H_{a_2,*}, H_{b_2,*} >
\]

\[
< H_{a_2,*}, H_{b_2,*} > = \begin{cases} h & a_2 = b_2 \\ 0 & \text{O/W} \end{cases}
\]

Let \( L = h \cdot \sum_{x_1 \in [t]} \nu_{(x_1)}^{N-1} \nu_{(x_1)}^{p_n} D^{[0]}_{(1,(x_1,1))} \)

\[
I = \mu_a^{N-1} \mu_b^{p_n} L, \text{ where } L \text{ is independent of } u, v, a, b, c, \text{ and also } L > 0.
\]

And the the \( II \) term can be written as

\[
II : \sum_{y \in J} F_{a,y} F_{c,y}^{q_n} D^{[0]}_{(1,y)} = L' \mu_a^{q_n} \mu_{c_1} \quad \forall a_1 = c_1.
\]
The $L'$ term is defined like $L$ as a sum on $h, \nu$, and $D$ and is a positive term.

Next, to simplify $R$, we need to define some notations and keep in mind that (from Shape$_3$)

$$D_{(0,c)}^{N-\gamma} = \overline{D_{(0,c)}^\gamma} = D_{c,1,2}.$$  

In addition, from the $H$'s dot products in the definitions of $I$ and $II$, we only consider $R$ entries in which $u_1 = v_1 = 1$, hence we can denote $R_{u,v} = R_{(1,1,u),(1,1,v)}$. Let $P_n = rP_n, Q_n = rq_n$, and

$$Z = \sum_{a_1 \in [s]} (L \cdot \mu_{a_1}^{N-1}) \cdot \mu_{a_1} \cdot \mu_{a_1} \cdot D_{(0,(a_1,1))}^{[0]}.$$

Therefore (note that $\nu = 1$):

$$R_{u,v} = Z^2 \left( \sum_{b,c \in [s]} \mu_{b}^{P} \mu_{c}^{Q} \sum_{a \in [h]} D_{b,a} \cdot D_{c,a} \cdot H_{a,u} \cdot H_{a,v} \right) \left( \sum_{b',c' \in [s]} \mu_{b'}^{P} \mu_{c'}^{Q} \sum_{a \in [h]} D_{b',a} \cdot D_{c',a} \cdot H_{a,u} \cdot H_{a,v} \right).$$

Note that we normalize $\nu$'s and $\mu$'s based on $\nu_1$ and $\mu_1$ respectively:

$$\nu_1 = 1 > \ldots > \nu_{[s]} \text{ and } \mu_1 = 1 > \ldots > \mu_{[s]}.$$  

Next we want to work on the magnitude of $\mu$ and stratify the sum based on these orders. Notice that $\mu_{b}^{P} \mu_{c}^{Q} \mu_{b'}^{P} \mu_{c'}^{Q} = (\mu_{b} \mu_{b'})^{P} (\mu_{c} \mu_{c'})^{Q}$. These terms are first ordered based on $\mu_{b} \mu_{b'}$ and then $\mu_{c} \mu_{c'}$ (recall that as $n \to \infty P > Q$). Therefore we define the order over a set $T$,

$$T = \left\{ T = \left( \begin{array}{cc} b & c \\ b' & c' \end{array} \right) \mid b, b', c, c' \in [s] \right\}.$$  

Where

$$T_1 \equiv \mu \ T_2 \text{ iff } (\mu_{b_1} \mu_{b_1'} = \mu_{b_2} \mu_{b_2'}) \text{ and } (\mu_{c_1} \mu_{c_1'} = \mu_{c_2} \mu_{c_2'})$$

$$T_1 \leq \mu \ T_2 \text{ if either } (\mu_{b_1} \mu_{b_1'} < \mu_{b_2} \mu_{b_2'}) \text{ or } (\mu_{b_1} \mu_{b_1'} = \mu_{b_2} \mu_{b_2'} \text{ and } \mu_{c_1} \mu_{c_1'} \leq \mu_{c_2} \mu_{c_2'}).$$

Based on this definition we can see

$$T_1 = \left\{ T_1 = \left( \begin{array}{cc} 1 & 1 \\ 1 & 1 \end{array} \right) \right\}$$

$$T_2 = \left\{ T_1 = \left( \begin{array}{cc} 1 & 1 \\ 1 & 2 \end{array} \right), T_2 = \left( \begin{array}{cc} 1 & 1 \\ 2 & 1 \end{array} \right) \right\}$$

$$\ldots,$$

therefore we can divide $T$ into classes $T_1, T_2, \ldots T_d$, from the largest to the smallest order.

Having this definition, we can rewrite the $R$ as follows:

$$R_{u,v} = Z^2 \sum_{i \in [d]} \sum_{T \in T_i} \sum_{T \in T_i} X_{u,v,T}.$$
where

\[ U_i = \mu_b \mu_{b'} \]
\[ W_i = \mu_c \mu_{c'} \]

\[ X_{u,v,T} = \left( \sum_{a \in [h]} D_{b,a} D_{c,a} H_{a,u} H_{a,v} \right) \left( \sum_{a \in [h]} D_{b',a} D_{c',a} H_{a,u} H_{a,v} \right). \]

Therefore, when \( u = v \) the \( H \) terms cancel each other out and one can get

\[ \frac{R_{u,u}}{Z^2} = | \sum_{b,c \in [s]} \sum_{a \in [h]} D_{1,a} \overline{D_{1,a}} |^2 U_1^P W_1^Q. \]

This means the coefficient of the leading term \( U_1^P W_1^Q \) is \( \|D_{1,*}\|^4 \). Next a question arises that whether \( D_{a,*} = 0 \) or not? Consider the case where the equality is correct, then \( \mu_1 \) must be 0 and we can start the process from \( \mu_2 \). On the other hand, if \( D_{1,*} \neq 0 \) and its coefficient is sufficiently big, then \( \mu_2, \ldots \) do not matter. Hence, the other terms become zero for sufficiently large \( n \).

Next we are going to use the vanishing lemma on the matrix \( R \) and show that all the non-zero terms of \( D \) must have the same norm.

(The proof will be continued).