CS 880: Complexity of Counting Problems 03/29/2012

Lecture 20: Proof of Lemma 8.8; Claim 8.1 and 8.2

Instructor: Jin-Yi Cai Scribe: Hesam Dashti

Last time we saw

$$\mathbf{R}_{u,v} = Z^2 \sum_{i\in[d]} U_i^P W_i^Q \sum_{T\in\mathcal{T}_i} X_{u,v,T},$$

where Z is independent of u and v, and U and W are functions of μ and finally

$$X_{u,v,T} = \left(\sum_{a\in[h]} \mathbf{D}_{b,a}\overline{\mathbf{D}_{c,a}}\mathbf{H}_{a,u}\overline{\mathbf{H}_{a,v}}\right)\left(\sum_{a\in[h]} \mathbf{D}_{b',a}\overline{\mathbf{D}_{c',a}\mathbf{H}_{a,u}}\mathbf{H}_{a,v}\right), \quad \text{for } T = \begin{pmatrix} b & c \\ b' & c' \end{pmatrix}. \tag{1}$$

One can rewrite the matrix $\mathbf{R}$ as follows:

$$\mathbf{R}^{[n]} = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \text{ and } \mathbf{R}^{[n]}_{(1,u),(1,v)} = (F)(S),$$

where F and S are the sum's as mentioned above. Notice that these sum's are symmetric as the defined gadget was. We also calculated the coefficient of the leading term $U_1^P W_1^Q$ that was

$$X_{u,u,\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}} = \left(\sum_{a\in[h]} |\mathbf{D}_{1,a}|^2\right)^2 = \|\mathbf{D}_{1,*}\|^4.$$

Last time this was discussed that why we assume this value is not zero. Since it is not zero, for sufficiently large n, the coefficient is bigger than zero $\mathbf{R}^{[n]}_{u,u} > 0$. One nice observation from the above statements is that since $X_{u,u}$, Z, U, and W are independent of u, then $\mathbf{R}_{u,u}$ is independent of u. So we have

$$\mathbf{R}_{u,u} = \mathbf{R}_{1,1}.$$

Next let us look at $X_{u,v,T}$ as defined in equation 1. This is a product of the D terms by roots of unity, therefore

$$X_{u,v,\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}} \leq \|\mathbf{D}_{1,*}\|^4 \text{ the maximum possible.}$$

Notice that for the case that it is strictly less than $\|\mathbf{D}_{1,*}\|^4$, since $\mathbf{R}$ is a stratification of X times the leading terms, then $\mathbf{R}$ is strictly less than a maximum possible. Here recall the shape of $\mathbf{R} = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}$ and consider the Bulatov-Grohe, we can conclude that to have a non-#P-hard problem, either $det(\mathbf{R}) = 0$ or the matrix is zero.
If $det(\mathbf{R}) = 0$, for sufficiently large n since the diagonals are equal they must be equal to zero. In either case the polynomial that defines the diagonal of $\mathbf{R}$ must be zero. Hence, by

using the vanishing lemma B, all the coefficients of the polynomial must be zero. Therefore we can conclude that the 'strictly less' is not the case and we can have the following property:
Property 8.1 For every sufficiently large n $|\mathbf{R}_{1,1}| > 0$ and

$$|\mathbf{R}_{u,v}| \in \{0, |\mathbf{R}_{1,1}|\}, \quad \forall u, v \in [h]$$

Following we consider $u = 1$ and define:

$$\mathcal{H}_{*,v} = \mathbf{H}_{*,1} \circ \overline{\mathbf{H}_{*,v}},$$

therefore

1. $X_{1,v,T} = X_{v,T} = \left(\sum_{a\in[h]} \mathbf{D}_{b,a}\overline{\mathbf{D}_{c,a}}\mathcal{H}_{a,v}\right)\left(\sum_{a\in[h]} \mathbf{D}_{b',a}\overline{\mathbf{D}_{c',a}\mathcal{H}_{a,v}}\right).$
2. Since $\mathcal{H}_{a,1}$ are constants, therefore the set $\mathcal{H}_{a,v|v\in[h]}$ is a set of orthogonal basis.
3. $\sum_{a\in[h]} \mathcal{H}_{a,v}\overline{\mathcal{H}_{a,v'}} = \begin{cases} h & v = v' \\ 0 & \text{O/W} \end{cases}.$

We need two more definitions to continue this section, let

$$K = \{i \in [h] | \mathbf{D}_{1,i} \neq 0\} \neq \emptyset,$$
$$A = \{v \in [h] | \forall i, j \in K, \mathcal{H}_{i,v} = \mathcal{H}_{j,v}$$

The following figure is given to visualize the definitions:

$$[h]\left[\begin{array}{l} \qquad\qquad\qquad\quad \overbrace{\qquad}^{[A]} \\ [K]\quad \begin{bmatrix} i & & \\ & & \\ j & & \end{bmatrix} \begin{array}{l} \leftarrow \mathcal{H}_{i,*} \\ \\ \leftarrow \mathcal{H}_{j,*} \end{array} \\ \\ \\ \end{array}\right].$$

Using the definitions we have

1. If $|K| = 1$, then $A = [h]$.
2. If $A = [h]$, then $|K| = 1$.
3. If $\mathcal{H}_{*,1} = 1$, then $1 \in A \neq \emptyset$.
4. If $K = [h]$, then $|A| = 1$.

Let us reconsider the X term with $T = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$:

$$
\begin{aligned}
X_{v,\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}} &= \left(\sum_{a \in [h]} \mathbf{D}_{b,a} \overline{\mathbf{D}_{c,a}} \mathcal{H}_{a,v} \right) \left(\sum_{a \in [h]} \mathbf{D}_{b',a} \overline{\mathbf{D}_{c',a} \mathcal{H}_{a,v}} \right) \\
&= \left(\sum_{a \in [h]} |\mathbf{D}_{1,a}|^2 \mathcal{H}_{a,v} \right) \left(\sum_{a \in [h]} |\mathbf{D}_{1,a}|^2 \overline{\mathcal{H}_{a,v}} \right)
\end{aligned}
$$

In this equation, a is in fact over K, because D is zero elsewhere. Here if we consider the cases where $v \in A$ then

$$
X_{v,\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}} = h \cdot \|\mathbf{D}_{1,*}\|^4.
$$

Notice that the norm is independent of v, so we can conclude the property 8.2 as follows:
Property 8.2 For any $v \in A$ and sufficiently large n

$$
|\mathbf{R}_{1,v}| = |\mathbf{R}_{1,1}|.
$$

Next we want to show that the non-zero values of $\mathbf{D}$ are the same. Recall that $\mathbf{D}_{1,a} \neq 0$ iff $a \in K$.
To do so, let $B = [h] - A$ and $B \neq \emptyset$. Note that if $B = \emptyset$ then $A = [h]$ which means $|K| = 1$ and $\mathbf{D}_{1,a}$ on $a \in K$ is the same as itself and no need to prove it.
Let us consider the magnitude of $X_{v,\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}}$, where $v \in B$. The coefficient is a product of leading terms by $|\mathbf{D}|^2$ which is a product of the $\mathbf{D}$ terms and non-constant roots of unity (because $v \in B$). This makes magnitude of X to be strictly less that the maximum $\|\mathbf{D}_{1,*}\|^4$.
By the same argument as we did before today property 8.3 states:
Property 8.3 For sufficiently large n, $\mathbf{R}_{1,v} = 0$ and by vanishing lemma B

$$
\sum_{T \in \mathcal{T}_i} X_{v,T} = 0 \quad \forall v \in B, \forall i \in [d].
$$

Using this property

$$
X_{v,\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}} = \left(\sum_{a \in [h]} |\mathbf{D}_{1,a}|^2 \mathcal{H}_{a,v} \right) \left(\sum_{a \in [h]} |\mathbf{D}_{1,a}|^2 \overline{\mathcal{H}_{a,v}} \right) = 0.
$$

This means $|\mathbf{D}_{1,*}|^2 \perp \{\mathcal{H}_{*,v}|v \in B\}$. On the other hand $\{\mathcal{H}_{*,v}|v \in B\} \perp \{\mathcal{H}_{*,v}|v \in A\}$, therefore

$$|\mathbf{D}_{1,*}|^2 \in span\{\mathcal{H}_{*,v}|v \in A\}.$$

Therefore

$$|\mathbf{D}_{1,*}|^2 = \sum_{i=1}^{|A|} \lambda_i \mathcal{H}_{*,v_i}.$$

If we recall the properties of A, the value of $|\mathbf{D}_{1,*}|^2$ is a constant, independent of v, for every $a \in K$ and for all $a \notin K$ this value is zero.
So we showed that $\mathbf{D}_{1,a}$ for all $a \in K$ are the same constants. This is claim 8.1:
Claim 8.1 For any $v \in B$, $|\mathbf{D}_{1,*}|^2 \perp \mathcal{H}_{*,v}$ and $|\mathbf{D}_{1,*}|^2$ is a constant on K and zero elsewhere.

Next we want to consider $\mathbf{D}_{2,*}$ and show that on K, $\mathbf{D}_{2,*}$ is a multiple of $\mathbf{D}_{1,*}$.
Again we consider $B \neq 0$ as we discussed before. The ultimate goal is to show $\mathbf{D}_{2,*}$ is a multiple of $\mathbf{D}_{1,*}$ for every a, but first we focus on $a \in K$.
We start with considering new $X_{v,T}$. Let us consider two T matrices:

$$T_1 = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \qquad T_2 = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}.$$

Based on the general definition of the T matrices these matrices belong to a set $\mathcal{T}_g$ such that

$$\mathcal{T}_g = \left\{ T_1 = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, T_2 = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, T_3 = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}, T_4 = \begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix} \right\}.$$

The shape of matrix T_3 and T_4 shows that $X_{v,T_3} = X_{v,T_4} = 0$. Moreover, using property 8.3, we know

$$\sum_{T \in \mathcal{T}_g} X_{v,T} = 0 \quad v \in B. \tag{2}$$

Let call a matrix 'a conjugate pair form' if it is of the form $T = \begin{pmatrix} b & c \\ c & b \end{pmatrix}$. Note that for such a matrix T, we have $X_{v,T} \geq 0$, because the magnitude has a square form. Since the matrices $T_1, T_2 \in \mathcal{T}_g$ are in conjugate form, $X_{v,T_3} = X_{v,T_4} = 0$, and equation 2 is true for the sum on all $T \in \mathcal{T}_g$, therefore $X_{v,T} = 0$ for every $T \in \mathcal{T}_g$.
Therefore,

$$X_{v,T_1} = \left| \sum_{a \in K} \mathbf{D}_{1,a} \overline{\mathbf{D}_{2,a}} \mathcal{H}_{a,v} \right|^2 = 0 \quad \forall v \in B.$$

This means that

$$\mathbf{D}_{1,*} \circ \overline{\mathbf{D}_{2,*}} \perp \{\mathcal{H}_{*,v}\} \quad \forall v \in B.$$

Using the same argument as before, if a set is orthogonal to a part of a orthogonal set then it belongs to span of the rest:

$$\mathbf{D}_{1,*} \circ \overline{\mathbf{D}_{2,*}} = span\{\mathcal{H}_{*,v}\} \text{ for } v \in A.$$

On the other hand, we know that (a) the right hand side has constant values for $a \in K$, (b) the $\mathbf{D}_{1,*} > 0$, and (c) the $\mathbf{D}_{1,*} = \mathbf{D}_{1,1}$ therefore
Claim 8.2 There is a complex number λ such that

$$\mathbf{D}_{2,*} = \lambda \mathbf{D}_{1,*} \quad \forall a \in K.$$