CS 880: Complexity of Counting Problems	03/06/2012
Lecture 13:	
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1 Last Time

Last time we proved the problem ANTICHAIN is #P hard by using the facts that both VC and IDS are #P hard.

2 This Time

Given H and G, recall that the number of graph homomorphism from G to H can be expressed as follows:

$$Z_H(G) = \sum_{\sigma: V_G \to V_H} \prod_{e=\{u,v\} \in E_G} H(\sigma(u), \sigma(v)).$$

We first prove the following lemma.

Lemma 1. *The graph homomorphism is* #P *hard for the following* H*:*

$$H = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

Note that the matrix H specifies that "1" cannot reach "0", as shown in Fig. 1

Proof. Given a partial order $P = (V, E)$ being a directed acyclic graph, we construct P' : $V^{(1)} \cup V^{(2)}$ where $V^{(1)}$ and $V^{(2)}$ are two disjoint copies of P. First, for all p we make $p^{(1)} \to p^{(2)}$. Next for all $p < q$ in the partial order, make $p^{(1)} \to q^{(1)}$, $p^{(1)} \to q^{(2)}$, $p^{(2)} \to q^{(1)}$

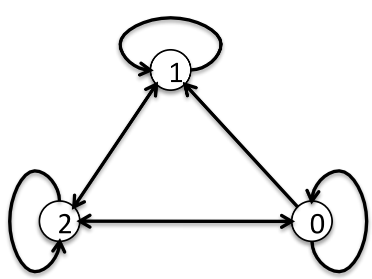

Figure 1: Graphical relationship. Note "1" cannot reach "0"

and $p^{(2)} \to q^{(2)}$ as show in Fig. 2. Define $K = \{p \in P \mid \sigma(p^{(1)}) = \text{"1"} \text{ or } \sigma(p^{(2)}) = \text{"1"}\}$, in order words, "1" $\in \{\sigma(p^{(1)}), \sigma(p^{(2)})\}$. Also define $K' = \{p \in K \mid p \text{ is a minimal in } K\}$.

Since we have $p^{(1)} \to p^{(2)}$, there can only be four cases, i.e., "1" $\to$ "1", "1" $\to$ "2", "0" $\to$ "1" and "2" $\to$ "1". Above K (meaning the filter of K', i.e., those x such that there is some $p \in K'$ such that $p \leq x$) we can only have "1" and "2" but not "0". Below K' (meaning $P - \{\text{filter of } K'\}$) we can only have "0" and "2" but not "1". All these choices are valid. K' is the ANTICHAIN problem and we have $Z_H(P')$ equivalent to the problem of counting ANTICHAIN, namely the value is the number of ANTICHAINs in P times $4^{|V|}$. Every ANTICHAIN in P corresponds to some K' for a certain valid assignment. □

We next prove another lemma in which H takes a more general form.

Lemma 2. *Let H have the following form:*

$$H = \begin{pmatrix} J_{aa} & J_{ab} & J_{ac} \\ O_{ba} & J_{bb} & J_{bc} \\ J_{ca} & J_{cb} & J_{cc} \end{pmatrix},$$

where $J_{ij}(O_{ij})$ denotes the $i \times j$ matrix with all entries being $1(0)$. Let $a, b > 0$ and $c \geq 0$, then Z_H is #P hard.

Proof. We prove by reducing from the problem considered in Lemma 1. Let

$$H_1 = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

We further assume $c > 0$ (the case $c = 0$ is left as an exercise).

Given an input graph G for Z_{H_1} we construct $G^{(k,l)}$ such that each vertex in G grows out l vertices and grows in k vertices (see Fig. 3).

Note that the first row/column in H can be interpreted as the "0", the second row/column in H can be interpreted as the "1" and the third row/column in H can be interpreted as the "2". Therefore, the assignment function σ for $G^{(k,l)}$ can be seen as an extension to the one for G.

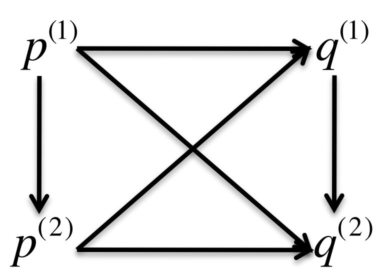

Figure 2:

Let N_{xyz} denote the number assignments with value 1 in $Z_{H_1}(G)$ where x many vertices of $V(G)$ are assigned "0", y many vertices of $V(G)$ are assigned "1" and z many vertices of $V(G)$ are assigned "2". The value for $Z_{H_1}(G)$ is $\sum_{x+y+z=|G|} N_{xyz}$. Then the value for $Z_H(G^{(k,l)})$ is

$$\sum_{x+y+z=|G|} a^x b^y c^z (a+b+c)^{xl}(a+c)^{xk}(a+b+c)^{yk}(b+c)^{yl}(a+b+c)^{z(l+k)}.$$

To see how we obtain this, consider a "0" vertex. There are many different ways for it to be assigned in the domain of H. This gives the factor of a^x for all the "0" verticies. For each "0" vertex, it has l out-grown edges. Since "0" can go to "0", "1", "2", this gives $(a+b+c)^l$ choices for those l out-grown edges and $(a+b+c)^{xl}$ over all "0" vertices. Also, for each "0" vertex, there are k in-grown edges. They must avoid being assigned as "1" as "1" cannot go to "0". Thus we get the factor $(a+c)^{xk}$. These derivation carry over to all y "1"-assigned vertices and z "2"-assigned vertices.

It can be reduced to (by taking out a common constant factor):

$$\sum_{x+y+z=|G|} N_{xyz} a^x b^y c^z \left(\frac{a+c}{a+b+c}\right)^{xk} \left(\frac{b+c}{a+b+c}\right)^{yl}.$$

Setting $l = kN$, $\frac{a+c}{a+b+c} = p$ and $\frac{b+c}{a+b+c} = q$ we have

$$\sum_{\substack{0 \le x,y \le |G| \\ x+y \le |G| \\ z=|G|-x-y}} N_{xyz} a^x b^y c^z (p^x q^{yN})^k.$$

We set $N > \log_{\frac{1}{q}} (\frac{1}{p})^{|G|}$ such that for all x and y with $0 \le x+y \le |G|$, $x, y \ge 0$ the $p^x q^{yN}$ are pair-wise distinct. To derive the lower bound on N, we note that if

$$\left(\frac{1}{p}\right)^n < \left(\frac{1}{q}\right)^N,$$

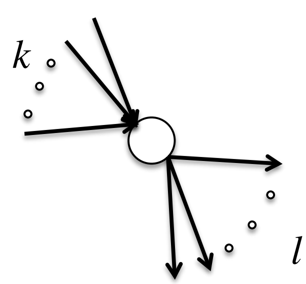

Figure 3:

then for any distinct pairs (x, y) and (x', y') satisfying $0 \leq x + y \leq |G|$, $x, y \geq 0$, we have $p^x q^{yN} \neq p^{x'} q^{y'N}$. This follows from simple algebra. Finally, we note that $[(p^x q^{yN})^k]_{(x,y),k}$ is a Vandermonde system. $\square$

We start with a relational clone $\langle\Gamma\rangle$ that is not rectangular thus have a reflexive, not symmetric and pp-definable domain R of minimum arity. By not symmetric we mean $(a, b) \in R$ but $(b, a) \notin R$ for some a and b. We then have the following properties:

1. $\forall c,\ a \to c \to b.$

2. $\forall c,\ \text{if } c \to a,\ \text{then } \forall d\ c \to d.$

3. $\forall c,\ \text{if } c \nrightarrow a,\ \text{then } \forall d\ d \to c.$

Let $F(x) = \exists y\ \exists z\ (y \to x \wedge C_a(y) \wedge x \to z \wedge C_b(z)$ where $C_a(y)$ denotes the pinning of y on a. Define $R' = R \cap (F \times F)$. For all $a, b \in R'$, we can see that property 1 holds.

To show property 2 still holds, suppose otherwise, i.e., $\exists c, d$ s.t. $c \to a \wedge c \nrightarrow d$. Define the domain shrinkage as $F'(x) = \exists y\ (C_c(y) \wedge y \to x)$. This means that $c \to x$ thus $a, b \in R' \cap (F' \times F')$. This contradicts the fact that R is minimum arity. Thus property 2 holds.

The work to show property 3 holds is left as homework.

Let $D = \{c \in F \mid c \to a\}$, $B = F - D$, $D_d = \{c \in D \mid d \to c\} = \{a_1, a_2, \ldots, a_k\}$ for $d \in B$, $a \in D$, $b \in B$, $a \notin D_c$ and for all $c \in B$. Choose $d \in B$ such that D_d is maximal. Set $B' = \{c \in B \mid D_c = D_d\}$, then $x \in B' \cup D \Leftrightarrow \exists y_1, y_2, \ldots, y_k \left(\bigwedge_{i=1}^{k} (x \to y_i) \wedge C_{a_i}(y_i)\right)$. Then by letting $\sigma' = R \cap (B' \cup D)^2$ we have a graphical relation represented shown in Fig. 1