CS520: Homework 1 Due date: Thursday, Sept 14, 2023

Reading: Chapter 0. Chapter 1, Section 1. and Section 2.

Please note that your reading assignment is an integral part of your homework.

1. Suppose $P(n)$ is a logical statement which is either true or false, for every integer $n \geq 0$.

   1.1 Suppose $P(0), P(1)$ and $P(2)$ are all true. And for all integers $n \geq 0$, if $P(n), P(n+1)$ and $P(n+2)$ are true then so is $P(n+3)$.

   For what n, can you be certain that $P(n)$ is true?

   1.2. Suppose $P(1)$ is true. And for all integer n, if $P(n)$ is true, then so is $P(2n)$.

   For what n, can you be certain that $P(n)$ is true?

2. There are n straight lines on a plane (a straight line is infinite in both directions). We assume they are "in general position" which means that no two lines are parallel (this means that every two lines intersect at a unqiue distinct point), and no three lines intersect at a single point.

   How many triangles are formed by these lines? (Note that triangles need not be disjoint, indeed one can be totally contained in another.)

3. This problem is an example of inductive thinking.

   3.1. Suppose you have n distinct points on a straight line. In how many segments do these n points divide the straight line?

   3.2. Now you have a plane, on which there are n infinite straight lines, and we assume they are "in general position".

   In how many connected "regions" (here they are not necessarily triangles, try draw a few and see for yourself) do the n straight lines cut the plane into?

   Hint Surely, if $n = 1$, we have two "regions" which are each "half" planes. If $n = 2$, we have 4 "regions". Now try $n = 3$. Draw some pictures.

   The key to solving this problem in general, starts here at $n = 3$. You should imagine yourself as a "point" (a little bug if you will) crawling along the 3rd line. The "world" (the plane) has already been cut up into 4 "regions" by the other 2 lines. But what can the bug see? How many "intersection points" will the bug see? Those are precisely the intersection points the 3rd line intersects with the other 2 lines.

   From how many intersection points the bug sees, the clever bug should be able to infer how many of the existing 4 regions (cut by the other 2 lines) have now been further divided by this 3rd line. Then it should be able to infer how many new regions there are now on the plane. (You now should realize that the problem you did in 3.1. is helpful here.)

   3.3 Now we are more ambitious. We consider 3-dimensional space, and we have n planes. We also assume that the planes are in general position: Every two planes intersect at a unique line, every three planes intersect at a unique point, and no four planes intersect at a single point.

   Now imagine yourself as a two-dimensional (flat) being moving around on the n-th plane. What can you discover? You can not see beyond the two-dimensional plane, but you can discover how many two-dimensional "regions" the other $n-1$ planes have cut this n-th plane into. (Do you think the work you did in 3.2. might be helpful?)

From this knowledge, you should be able to infer how many new 3-dimensional "regions" this n-th plane has added.

3.4.(extra credit) Can you think of this problem in four-dimensions?

Can you give a formula for the number of four-dimensional connected regions formed by n 3-dimensional flats floating aroung in 4-space in general position? (What does it mean by "in general position". What kind of bug?)

4. A domino consists of two adjacent squares. Each square is 1 by 1, and a domino is 1 by 2. We have a rectangle R which is 2 by n, consisting of $2n$ squares.

   (The given 2 by n rectangle has $2n$ squares that are labeled with integers 1 to $2n$. Imagine R is placed horizontally in front of you, with two rows of squares, the northwest corner is numbered 1, the square just below it on the first column is numbered 2, etc. The southeast corner square is numbered $2n$. A tiling is a partition of the integer set $\{1, 2, \ldots, 2n\}$ into n subsets, such that each subset corresponds to two adjacent squares.)

   We want to count how many ways $W(n)$ we can tile the rectangle R by n dominos. Clearly when $n = 1$ there is exactly one way, $W(1) = 1$. When $n = 2$ there are exactly two ways $W(2) = 2$. Find a recursive relation for $W(n)$.

   Compute the first few numbers of $W(n)$. (Does this sequence look familiar to you?)

5. Design a DFA accepting all strings over $\{a, b\}$ such that the number of a's plus twice the number of b's is even.

6. (extra credit) You are given 12 coins, and you are told that exactly 11 of which are real and one is fake. The only way to tell which is real and which is fake is by weighing them. All real coins weigh exactly the same, but the fake one weighs either heavier or lighter than the real one.

   You are given a scale on which you can weigh any two disjoint subsets of the 12 coins. This is the basic operation: To weigh any chosen subset against another disjoint subset. For example, you can decide to weigh any 2 coins against another 2 coins. Upon the weighing result (either the first subset weighs more or equal or less than the second one), you can decide what to weigh next.

   Devise a strategy that identifies the fake coin in no more than 3 sequential weighings. Can your method tell whether the fake coin weighs more or less than the real one?

Note:

Please be concise. You are strongly advised to start thinking about the problems right away, and not wait till the due days.