Progress on the Complexity of Counting Problems

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“P versus NP — a gift to Mathematics from Computer Science”

Steve Smale

The P vs. NP Question

It is generally conjectured that many combinatorial problems in the class NP are not computable in polynomial time.

Conjecture: P $\neq$ NP.

P $=^?$ NP is:

Can "clever guesses" be systematically eliminated?

#P

Counting problems:

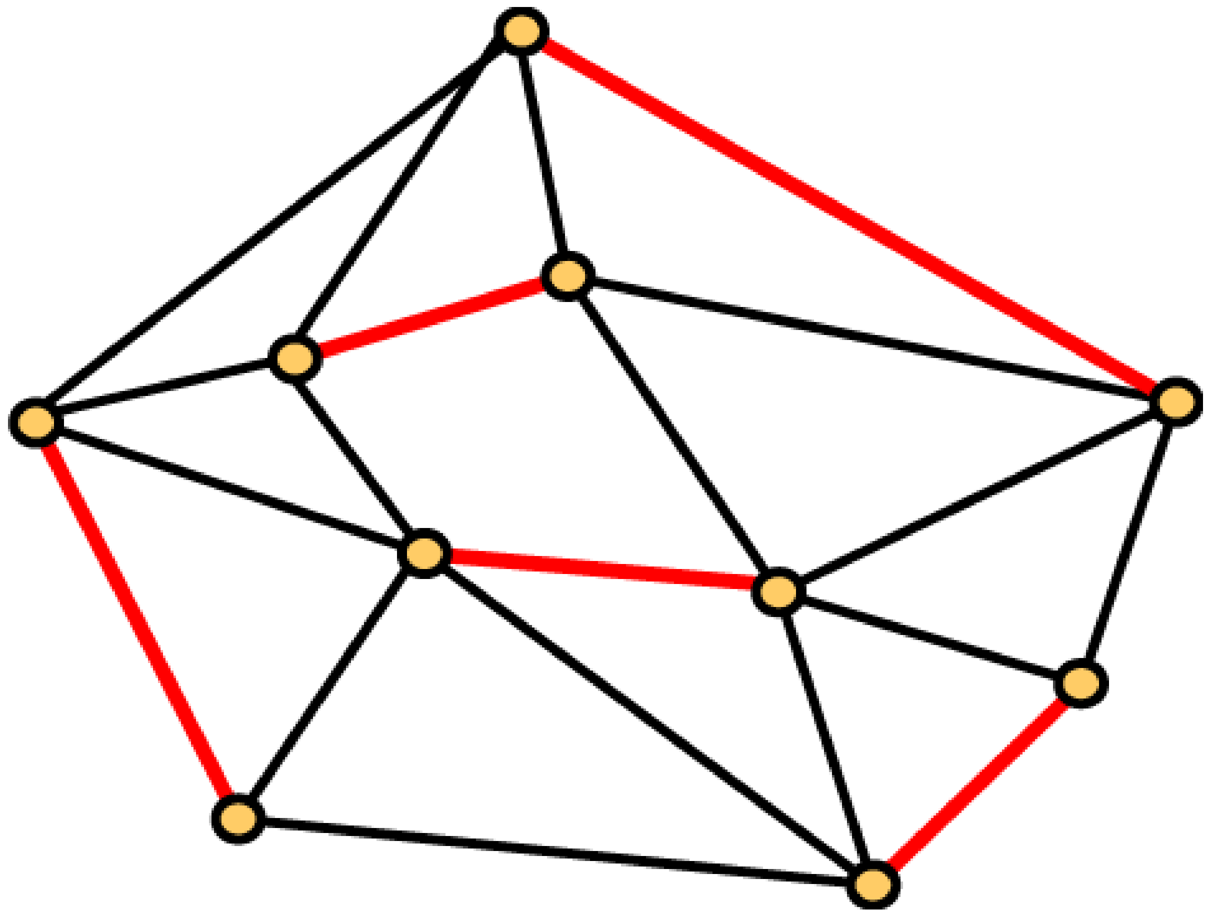
#SAT: How many satisfying assignments are there in a Boolean formula?

#PerfMatch: How many perfect matchings are there in a graph?

#P is at least as powerful as NP, and in fact subsumes the entire polynomial time hierarchy $\cup_i \Sigma_i^p$ [Toda].

#P-completeness: #SAT, #PerfMatch, Permanent, etc.

Perfect Matching



Graph Homomorphisms

Graph Homomorphisms or H-Coloring was defined by Lovász (1967).

Let

$$H = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

be a Triangle.

A graph homomorphism from G to H, is a mapping ξ from $V(G)$ to $V(H)$ such that

$$(u, v) \in E(G) \implies (\xi(u), \xi(v)) \in E(H).$$

I.e., ξ is a Three-Coloring **of G.**

Partition Function

Let $\mathbf{A} = (A_{i,j}) \in \mathbb{C}^{m \times m}$ be a symmetric complex matrix.

The graph homomorphism problem $\mathsf{EVAL}(\mathbf{A})$ **is:**

INPUT: **An undirected graph $G = (V, E)$.**

OUTPUT:

$$Z_{\mathbf{A}}(G) = \sum_{\xi : V \to [m]} \prod_{(u,v) \in E} A_{\xi(u), \xi(v)}.$$

ξ is an assignment to the vertices of G and

$$\mathbf{wt}_{\mathbf{A}}(\xi) = \prod_{(u,v) \in E} A_{\xi(u), \xi(v)}$$

is called the weight of ξ.

Some Examples

Let

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

then $\mathsf{EVAL}(\mathbf{A})$ **counts the number of** Vertex Covers **in** G**.**

Some More Examples

Let

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & \cdots & 1 \\ 1 & 0 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 0 \end{pmatrix}$$

then $\mathsf{EVAL}(\mathbf{A})$ **counts the number of** k-Colorings **in** G**.**

Let

$$\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

then $\mathsf{EVAL}(\mathbf{A})$ **is equivalent to counting the number of induced subgraphs of** G **with an even number of edges.**

Dichotomy Theorems for Counting

Creignou and Hermann proved a dichotomy theorem for counting SAT problems: Either solvable in P or #P-complete.

Creignou, Khanna and Sudan:
Complexity Classifications of Boolean Constraint Satisfaction Problems.
SIAM Monographs on Discrete Math and Applications. 2001.

Graph homomorphism

Lovász first studied Graph homomorphisms.

L. Lovász: Operations with structures, Acta Math. Hung. 18 (1967), 321-328.

http://www.cs.elte.hu/~lovasz/hom-paper.html

Dichotomy Theorems for Graph Homomorphisms

Theorem (Hell and Nešetřil)

Dichotomy Theorem for the decision Graph Homomorphism problem: Either in P or NP-complete.

Theorem (Dyer and Greenhill)

Dichotomy Theorem for $Z_H(G)$, for all 0-1 H: Either in P or #P-hard.

Theorem (Bulatov and Grohe)

Dichotomy Theorem for $Z_H(G)$, for all non-negative H.

Theorem (Dyer, Goldberg and Paterson)

Dichotomy Theorem for all directed and acyclic H.

Some definitions

A graph homomorphism is a map f from $V(G)$ to $V(H)$ such that if $\{u, v\} \in E(G)$, then $\{f(u), f(v)\} \in E(H)$.

A symmetric 0-1 matrix is identified with its underlying (undirected) graph.

A general symmetric matrix gives a weighted (undirected) graph.

- **Connected components.**
- **Bipartite graphs.**

Non-negative Matrices

Theorem (Bulatov and Grohe)
Let $\mathbf{A} \in \mathbb{R}^{m\times m}$ be a symmetric and connected matrix with non-negative entries:

- **If $\mathbf{A}$ is bipartite, then $\mathsf{EVAL}(\mathbf{A})$ is in polynomial time if the rank of $\mathbf{A}$ is at most 2; otherwise $\mathsf{EVAL}(\mathbf{A})$ is #P-complete.**
- **If $\mathbf{A}$ is not bipartite, then $\mathsf{EVAL}(\mathbf{A})$ is in polynomial time if the rank of $\mathbf{A}$ is at most 1; otherwise $\mathsf{EVAL}(\mathbf{A})$ is #P-complete.**

Real Matrices

Theorem (Goldberg, Jerrum, Grohe and Thurley)
There is a complexity dichotomy theorem for $\mathsf{EVAL}(\mathbf{A})$.

For any symmetric real matrix $\mathbf{A} \in \mathbb{R}^{m \times m}$, the problem of computing $Z_{\mathbf{A}}(G)$, for any input G, is either in P or #P-hard.

A complexity dichotomy for partition functions with mixed signs

arXiv:0804.1932v2 [cs.CC]

`http://arxiv.org/abs/0804.1932`

A monumental achievement.

Main Dichotomy Theorem

Theorem (C, Chen and Lu)
There is a complexity dichotomy theorem for $\mathsf{EVAL}(\mathbf{A})$.

For any symmetric complex vlaued matrix $\mathbf{A} \in \mathbb{C}^{m \times m}$, the problem of computing $Z_{\mathbf{A}}(G)$, for any input G, is either in P or #P-hard.

(111 pages)

Main Dichotomy Theorem

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(**111 pages** . . . *not in binary* — **Lipton's P = NP blog**)

Reduction to Connected Components

Lemma

Let $\mathbf{A} \in \mathbb{C}^{m\times m}$ be a symmetric matrix with components $\mathbf{A}_1, \mathbf{A}_2, ..., \mathbf{A}_t$. Then

- **If $\mathsf{EVAL}(\mathbf{A}_i)$ is #P-hard for some $i \in [t]$ then $\mathsf{EVAL}(\mathbf{A})$ is #P-hard;**
- **Otherwise, $\mathsf{EVAL}(\mathbf{A})$ is polynomial-time computable.**

Pinning Lemma

A Pinning Lemma gives a reduction of the problem $\mathsf{EVAL}(\mathbf{A})$ to the restriction of the problem where a distinguished vertex of G is pinned to a particular value.

This is used to prove a reduction from $\mathsf{EVAL}(\mathbf{A})$ to $\mathsf{EVAL}(\mathbf{A}')$, where $\mathbf{A}'$ are connected components of $\mathbf{A}$.

We prove a Pinning Lemma for complex matrices.

The proof uses Interpolation and Vandermonde matrices.

Bipartite and Non-bipartite

The proof of the main Dichotomy Theorem is first reduced to Connected Components, and then further divided into the cases of Bipartite and Non-bipartite connected graphs.

Overview of Bipartite Case

The proof consists of two parts: the hardness part and the tractability part.

The hardness part is further divided into three steps, in which we gradually "simplify" the problem $\mathsf{EVAL}(\mathbf{A})$ being considered.

One can view the three steps as three filters which remove hard $\mathsf{EVAL}(\mathbf{A})$ problems using different arguments.

In the tractability part, we show that all the EVAL problems that survive the three filters are indeed polynomial-time solvable.

General Structure of a Filter

In each of the three filters in the hardness proof, we consider an EVAL problem that is passed down by the previous step (Step 1 starts with EVAL($\mathbf{A}$) itself) and show that

- **either the problem is #P-hard; or**
- **the matrix that defines the problem satisfies certain structural properties; or**
- **the problem is polynomial-time equivalent to a new EVAL problem and the matrix that defines the new problem satisfies certain structural properties.**

A Purified Matrix

A is purified bipartite, if there exists an $k \times (m-k)$ matrix B of the form

$$\mathbf{B} = \begin{pmatrix} c_1 & & & \\ & c_2 & & \\ & & \ddots & \\ & & & c_k \end{pmatrix} \begin{pmatrix} \zeta_{1,1} & \zeta_{1,2} & \cdots & \zeta_{1,m-k} \\ \zeta_{2,1} & \zeta_{2,2} & \cdots & \zeta_{2,m-k} \\ \vdots & \vdots & \ddots & \vdots \\ \zeta_{k,1} & \zeta_{k,2} & \cdots & \zeta_{k,m-k} \end{pmatrix} \begin{pmatrix} c_{k+1} & & & \\ & c_{k+2} & & \\ & & \ddots & \\ & & & c_m \end{pmatrix}$$

where every $c_i > 0$, every $\zeta_{i,j}$ is a root of unity, and A is the bipartisation of B:

$$\mathbf{A} = \begin{pmatrix} \mathbf{0} & \mathbf{B} \\ \mathbf{B}^{\mathtt{T}} & \mathbf{0} \end{pmatrix}.$$

Step 1: Purification of Matrix A

Start with problem $\mathsf{EVAL}(\mathbf{A})$ **in which** $\mathbf{A} \in \mathbb{C}^{m \times m}$ **is a symmetric, connected and bipartite matrix.**

Theorem

Let $\mathbf{A} \in \mathbb{C}^{m \times m}$ **be a symmetric, connected, and bipartite matrix. Then either** $\mathsf{EVAL}(\mathbf{A})$ **is #P-hard or there exists an** $m \times m$ **purified bipartite matrix** $\mathbf{A}'$ **such that** $\mathsf{EVAL}(\mathbf{A}) \equiv \mathsf{EVAL}(\mathbf{A}')$**.**

Step 2: Reduction to Discrete Unitary Matrix

Now let $\mathbf{A} \in \mathbb{C}^{m \times m}$ denote a purified bipartite matrix.

To study EVAL($\mathbf{A}$)**, we define a new and larger class of** EVAL **problems.**

These EVAL **problems have edge weights as well as vertex weights. Moreover the vertex weights are partitioned into modular classes according to the** $\deg(v)$**.**

Definition

Let $\mathbf{C} \in \mathbb{C}^{m\times m}$ be a symmetric matrix, and

$$\mathfrak{D} = \{\mathbf{D}^{[0]}, \mathbf{D}^{[1]}, ..., \mathbf{D}^{[N-1]}\}$$

be a sequence of diagonal matrices in $\mathbb{C}^{m\times m}$ for some $N \geq 1$ (we use $D_i^{[t]}$ to denote the $(i,i)^{th}$ entry of $\mathbf{D}^{[t]}$). We define the following problem $\mathsf{EVAL}(\mathbf{C}, \mathfrak{D})$: Given an undirected graph $G = (V, E)$, compute $Z_{\mathbf{C},\mathfrak{D}}(G)$

$$\sum_{\xi: V \to [m]} \left(\prod_{(u,v)\in E} A_{\xi(u),\xi(v)} \right) \left(\prod_{i=0}^{N-1} \left(\prod_{v\in V,\ \deg(v)\equiv i \bmod N} D_{\xi(v)}^{[i]} \right) \right)$$

Discrete Unitary Matrix

We prove that $\mathsf{EVAL}(\mathbf{A})$ **is either** $\#$**P-hard or polynomial-time equivalent to** $\mathsf{EVAL}(\mathbf{C}, \mathfrak{D})$ **in which C is a discrete unitary matrix.**

Definition

Let $\mathbf{F} \in \mathbb{C}^{m\times m}$ **be a (not necessarily symmetric) matrix with entries** $(F_{i,j})$**. We say F is an** M**-discrete unitary matrix, for some positive integer** M**, if it satisfies the following conditions:**

1. **Every entry** $F_{i,j}$ **is a power of** $\omega_M = e^{2\pi\sqrt{-1}/M}$ **(the** M**th root of unity);**
2. M **= lcm of the orders of** $F_{i,j}$**;**
3. $F_{1,i} = F_{i,1} = 1$ **for all** $i \in [m]$**;**
4. **For all** $i \neq j \in [m]$**,** $\langle \mathbf{F}_{i,*}, \mathbf{F}_{j,*}\rangle = 0$ **and** $\langle \mathbf{F}_{*,i}, \mathbf{F}_{*,j}\rangle = 0$**.**

Some Simple Examples of Discrete Unitary Matrices

$$\mathbf{H_2} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \qquad \mathbf{H_4} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix},$$

$$\boldsymbol{\mathcal{F}}_3 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix}, \quad \boldsymbol{\mathcal{F}}_5 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & \zeta & \zeta^{-1} & \zeta^2 & \zeta^{-2} \\ 1 & \zeta^2 & \zeta^{-2} & \zeta^{-1} & \zeta \\ 1 & \zeta^{-1} & \zeta & \zeta^{-2} & \zeta^2 \\ 1 & \zeta^{-2} & \zeta^2 & \zeta & \zeta^{-1} \end{pmatrix},$$

where $\omega = e^{2\pi i/3}$ **and** $\zeta = e^{2\pi i/5}$**.**

Theorem

Let $\mathbf{A} \in \mathbb{C}^{m\times m}$ be a purified bipartite matrix, then either problem $\mathsf{EVAL}(\mathbf{A})$ is #P-hard or there exists a triple $((M,N),\mathbf{C},\mathfrak{D})$ such that $\mathsf{EVAL}(\mathbf{A}) \equiv \mathsf{EVAL}(\mathbf{C},\mathfrak{D})$ and $((M,N),\mathbf{C},\mathfrak{D})$ satisfies the following condition $(\mathcal{U})$:

$(\mathcal{U}_1)$ M **and** N **are positive integers that satisfy** $M \mid N$**.** $\mathbf{C}$ **is a** $2n \times 2n$ **complex matrix for some** $n \geq 1$ **and** $\mathfrak{D} = \{\mathbf{D}^{[0]}, \mathbf{D}^{[1]}, ..., \mathbf{D}^{[N-1]}\}$ **is a sequence of** N $2n \times 2n$ **diagonal matrices;**

$(\mathcal{U}_2)$ $\mathbf{C}$ **is the bipartisation of an** M**-discrete unitary matrix** $\mathbf{F} \in \mathbb{C}^{n\times n}$**;**

$(\mathcal{U}_3)$ **For all** $i \in [2n]$**,** $D_i^{[0]} = 1$**, and for all** r **and** $i \in [2n]$**,** $D_i^{[r]}$ **is either zero or a power of** ω_N**.**

Step 3: Canonical Form of C, F and D

After the first two steps, the original problem $\mathsf{EVAL}(\mathbf{A})$ **is either shown to be #P-hard or reduced to a new problem** $\mathsf{EVAL}(\mathbf{C}, \mathfrak{D})$**. We also know there exist positive integers** M, N **such that** $((M, N), \mathbf{C}, \mathfrak{D})$ **satisfies condition** $(\mathcal{U})$**.**

Now we number rows and columns from $\{0, 1, \ldots, m-1\}$**.**

We also denote the upper-right $m \times m$ **block of C by F.**

If $M = 1$**, then since F is** M**-discrete unitary,** m **has to be** 1**. In this case, it is easy to check that problem** $\mathsf{EVAL}(\mathbf{C}, \mathfrak{D})$ **is tractable.**

Now assume $M > 1$**.**

Step 3.1

First, we show that either $\mathsf{EVAL}(\mathbf{C}, \mathfrak{D})$ is hard or we can permute the rows and columns of F so that the new F is the tensor product of a collection of *Fourier matrices*.

Definition

Let $q > 1$ be a prime power. We call the following $q \times q$ matrix $\boldsymbol{\mathcal{F}}_q$ a q-Fourier matrix : The $(x, y)^{th}$ entry, where $x, y \in [0 : q-1]$, is

$$\omega_q^{xy} = e^{2\pi i \left(xy/q \right)}.$$

Theorem

Suppose $((M, N), \mathbf{C}, \mathfrak{D})$ **satisfies condition** $(\mathcal{U})$ **and** $M > 1$**.** **Then either** $\mathsf{EVAL}(\mathbf{C}, \mathfrak{D})$ **is #P-hard or there exist**

1. **two permutations** Σ **and** Π **from** $[0 : m-1]$ **to** $[0 : m-1]$**; and**

2. **a sequence** $q_1, q_2, ..., q_k$ **of** k **prime powers, for some** $k \geq 1$**,**

such that

$$\mathbf{F}_{\Sigma,\Pi} = \bigotimes_{i \in [k]} \boldsymbol{\mathcal{F}}_{q_i}. \qquad (1)$$

Suppose there do exist Σ, Π, q_i such that F satisfies (1), then we let $\mathbf{C}_{\Sigma,\Pi}$ denote the bipartisation of $\mathbf{F}_{\Sigma,\Pi}$, and $\mathfrak{D}_{\Sigma,\Pi}$ denote a sequence of N $2m \times 2m$ diagonal matrices in which the r^{th} matrix is

$$\begin{pmatrix} D^{[r]}_{\Sigma(0)} & & & & & \\ & \ddots & & & & \\ & & D^{[r]}_{\Sigma(m-1)} & & & \\ & & & D^{[r]}_{\Pi(0)+m} & & \\ & & & & \ddots & \\ & & & & & D^{[r]}_{\Pi(m-1)+m} \end{pmatrix}.$$

It is clear that permuting the rows and columns of matrices $\mathbf{C}$ and $\mathfrak{D}$ does not affect the complexity of $\mathsf{EVAL}(\mathbf{C}, \mathfrak{D})$, so it is polynomial-time equivalent to $\mathsf{EVAL}(\mathbf{C}_{\Sigma,\Pi}, \mathfrak{D}_{\Sigma,\Pi})$. From now on, we let $\mathbf{F}$, $\mathbf{C}$ and $\mathfrak{D}$ denote $\mathbf{F}_{\Sigma,\Pi}$, $\mathbf{C}_{\Sigma,\Pi}$ and $\mathfrak{D}_{\Sigma,\Pi}$, respectively. By (1), the new $\mathbf{F}$ satisfies

$$\mathbf{F} = \bigotimes_{i \in [k]} \boldsymbol{\mathcal{F}}_{q_i}. \tag{2}$$

Before moving forward we rearrange the prime powers $q_1, ..., q_k$ and divide them into groups according to different primes. We need the following notation.

Let $\mathbf{p} = (p_1, ..., p_s)$ **be a sequence of primes such that** $p_1 < ... < p_s$ **and** $\mathbf{t} = (t_1, ..., t_s)$ **be a sequence of positive integers. Let** $\mathfrak{q} = \{\mathbf{q}_i, i \in [s]\}$ **be a collection of** s **sequences in which every** $\mathbf{q}_i$ **is a sequence** $(q_{i,1}, ..., q_{i,t_i})$ **of powers of** p_i **such that** $q_{i,1} \geq ... \geq q_{i,t_i}$**. We use** q_i **to denote** $q_{i,1}$ **for all** $i \in [s]$**. We let**

$$\mathbb{Z}_{\mathfrak{q}} \equiv \prod_{i \in [s], j \in [t_i]} \mathbb{Z}_{q_{i,j}} \quad \textbf{and} \quad \mathbb{Z}_{\mathbf{q}_i} \equiv \prod_{j \in [t_i]} \mathbb{Z}_{q_{i,j}}, \quad \textbf{for all } i \in [s].$$

$$\mathbb{Z}_{\mathbf{q}_i} \equiv \prod_{j \in [t_i]} \mathbb{Z}_{q_{i,j}} = \mathbb{Z}_{q_{i,1}} \times \cdots \times \mathbb{Z}_{q_{i,t_i}}, \quad \textbf{for all } i \in [s]$$

and

$$\mathbb{Z}_{\mathfrak{q}} \equiv \prod_{i \in [s], j \in [t_i]} \mathbb{Z}_{q_{i,j}} \equiv \mathbb{Z}_{q_{1,1}} \times \cdots \times \mathbb{Z}_{q_{1,t_1}} \times$$

$$\vdots$$

$$\mathbb{Z}_{q_{s,1}} \times \cdots \times \mathbb{Z}_{q_{s,t_s}}$$

When we use x to denote a vector in $\mathbb{Z}_{\mathfrak{q}}$, we denote its $(i,j)^{th}$ entry by $x_{i,j} \in \mathbb{Z}_{q_{i,j}}$. We also use $\mathbf{x}_i$ to denote vector $(x_{i,j}, j \in [t_i]) \in \mathbb{Z}_{\mathbf{q}_i}$. Finally, given $\mathbf{x}, \mathbf{y} \in \mathbb{Z}_{\mathfrak{q}}$ and $k, l \in \mathbb{Z}$, we use $k\mathbf{x} \pm l\mathbf{y}$ to denote the vector in $\mathbb{Z}_{\mathfrak{q}}$ whose $(i,j)^{th}$ entry is

$$kx_{i,j} \pm ly_{i,j} \pmod{q_{i,j}}.$$

Similarly, for every $i \in [s]$, we can define $k\mathbf{x} \pm l\mathbf{y}$ for vectors $\mathbf{x}, \mathbf{y} \in \mathbb{Z}_{\mathbf{q}_i}$. It is easy to check that both $\mathbb{Z}_{\mathfrak{q}}$ and $\mathbb{Z}_{\mathbf{q}_i}$ are finite Abelian groups under these operations.

The tensor product decomposition

$$\mathbf{F} = \bigotimes_{i \in [k]} \boldsymbol{\mathcal{F}}_{q_i}.$$

gives $\mathbf{p}, \mathbf{t}, \mathfrak{q}$ **such that** $((M,N), \mathbf{C}, \mathfrak{D}, (\mathbf{p}, \mathbf{t}, \mathfrak{q}))$ **satisfies the following condition** $(\mathcal{R})$**:**

($\mathcal{R}_1$) **$\mathbf{p} = (p_1, ..., p_s)$ is a sequence of primes such that $p_1 < p_2 < ... < p_s$; $\mathbf{t} = (t_1, ..., t_s)$ is a sequence of positive integers; $\mathfrak{q} = (\mathbf{q}_i, i \in [s])$ is a collection of s sequences in which every $\mathbf{q}_i$ is a sequence $(q_{i,1}, ..., q_{i,t_i})$ of powers of p_i such that $q_{i,1} \geq ... \geq q_{i,t_i}$;**

($\mathcal{R}_2$) **$\mathbf{C} \in \mathbb{C}^{2m\times 2m}$ is the bipartisation of $\mathbf{F}$, $m = \prod_{i\in[s],j\in[t_i]} q_{i,j}$, and $((M,N), \mathbf{C}, \mathfrak{D})$ satisfies ($\mathcal{U}$);**

($\mathcal{R}_3$) **There is a one-to-one correspondence ρ from $[0 : m-1]$ to $\mathbb{Z}_\mathfrak{q}$ such that**

$$F_{a,b} = \prod_{i\in[s],j\in[t_i]} \omega_{q_{i,j}}^{x_{i,j}\, y_{i,j}}, \quad \textbf{for all } a, b \in [0 : m-1],$$

where $(x_{i,j}, i \in [s], j \in [t_i]) = \mathbf{x} = \rho(a)$ and $(y_{i,j}, i \in [s], j \in [t_i]) = \mathbf{y} = \rho(b)$.

Step 3.2

Now we have a 4-tuple that satisfies condition $(\mathcal{R})$. In this step, we show for every $r \in [N-1]$ (recall that $\mathbf{D}^{[0]}$ is already known to be the identity matrix), the nonzero entries of the r^{th} matrix $\mathbf{D}^{[r]}$ in $\mathfrak{D}$ must have a very nice "*group*" structure, otherwise $\mathsf{EVAL}(\mathbf{C}, \mathfrak{D})$ is #P-hard.

For every $r \in [N-1]$, we define Λ_r and $\Gamma_r \subset \mathbb{Z}_\mathfrak{q}$ as

$$\Lambda_r = \left\{\mathbf{x} \in \mathbb{Z}_\mathfrak{q}, D^{[r]}_{(0,\mathbf{x})} \neq 0\right\} \quad \textbf{and} \quad \Gamma_r = \left\{\mathbf{x} \in \mathbb{Z}_\mathfrak{q}, D^{[r]}_{(1,\mathbf{x})} \neq 0\right\}.$$

Theorem

Let $((M,N),\mathbf{C},\mathfrak{D},(\mathbf{p},\mathbf{t},\mathfrak{q}))$ **be a** 4**-tuple that satisfies** $(\mathcal{R})$**.** **Then either** $\mathsf{EVAL}(\mathbf{C},\mathfrak{D})$ **is #P-hard or sets** $\Lambda_r \subset \mathbb{Z}_{\mathfrak{q}}$ **and** $\Gamma_r \subset \mathbb{Z}_{\mathfrak{q}}$ **satisfy the following condition** $(\mathcal{L})$**:**

($\mathcal{L}_1$) **For every** $r \in \mathcal{S}$**,** $\Lambda_r = \prod_{i=1}^{s} \Lambda_{r,i}$**, where for every** $i \in [s]$**,** $\Lambda_{r,i}$ **is a coset in** $\mathbb{Z}_{\mathbf{q}_i}$**; and**

($\mathcal{L}_2$) **For every** $r \in \mathcal{T}$**,** $\Gamma_r = \prod_{i=1}^{s} \Gamma_{r,i}$**, where for every** $i \in [s]$**,** $\Gamma_{r,i}$ **is a coset in** $\mathbb{Z}_{\mathbf{q}_i}$**.**

Step 3.3

In the final step, we show that, for every $r \in [N-1]$, the nonzero entries of $\mathbf{D}^{[r]}$ must have a quadratic structure, otherwise $\mathsf{EVAL}(\mathbf{C}, \mathfrak{D})$ is #P-hard.

This is the most difficult part of the proof for the bipartite case.

Tractability

Theorem

Let $((M,N),\mathbf{C},\mathfrak{D},(\mathbf{p},\mathbf{t},\mathfrak{q}))$ be a 4-tuple that satisfies all the three conditions $(\mathcal{R}),(\mathcal{L})$ and $(\mathcal{D})$, then problem $\mathsf{EVAL}(\mathbf{C},\mathfrak{D})$ can be solved in polynomial time.

Non-trivial algorithm, $\ldots$ mainly character sums $\ldots$

Back to Discrete Unitary

Definition

Let $\mathbf{A} = (A_{i,j}) \in \mathbb{C}^{m\times m}$. We say $\mathbf{A}$ is an M-discrete unitary matrix, for some positive integer M, if

1. **Every entry $A_{i,j}$ is a power of $\omega_M = e^{2\pi\sqrt{-1}/M}$;**
2. **M = lcm of the orders of $F_{i,j}$;**
3. **$A_{1,i} = A_{i,1} = 1$ for all $i \in [m]$;**
4. **For all $i \neq j \in [m]$, $\langle \mathbf{A}_{i,*}, \mathbf{A}_{j,*}\rangle = 0$ and $\langle \mathbf{A}_{*,i}, \mathbf{A}_{*,j}\rangle = 0$.**

Inner product $\langle \mathbf{A}_{i,*}, \mathbf{A}_{j,*}\rangle = \sum_{k=1}^{m} \mathbf{A}_{i,k}\overline{\mathbf{A}_{j,k}}$.

A Group Condition

Theorem

Let A be a symmetric M-discrete unitary matrix. Then

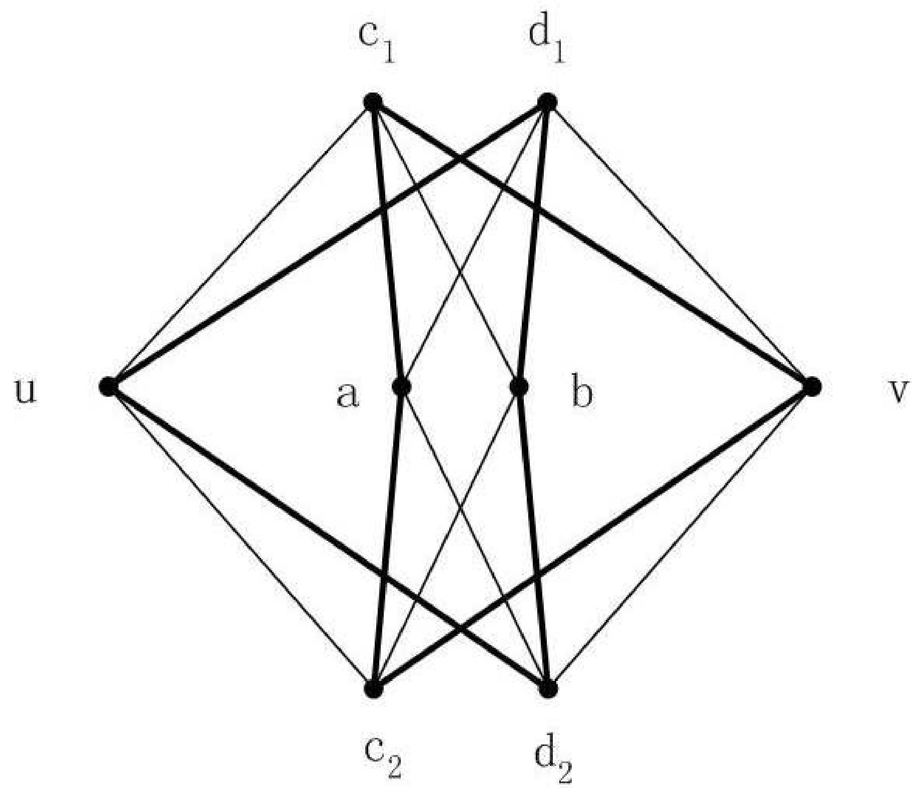
- **either $Z_{\mathbf{A}}(\cdot)$ is #P-hard,**
- **or A must satisfy the following Group-Condition *(GC)*:**

  $\forall\ i, j \in [0 : m-1]$**,** $\exists\ k \in [0 : m-1]$ **such that**

  $\mathbf{A}_{k,*} = \mathbf{A}_{i,*} \circ \mathbf{A}_{j,*}$**.**

$\mathbf{v} = \mathbf{A}_{i,*} \circ \mathbf{A}_{j,*}$ is the Hadamard product with $v_\ell = \mathbf{A}_{i,\ell} \cdot \mathbf{A}_{j,\ell}$.

A Gadget Construction



Special case $p = 2$. Thick edges denote $M-1$ parallel edges.

An Edge Gets Replaced

Replacing every edge e by the gadget ...

$$G \quad \Longrightarrow \quad G^{[p]}.$$

Define $G^{[p]} = (V^{[p]}, E^{[p]})$ **as**

$$V^{[p]} = V \cup \{a_e, b_e, c_{e,1}, \ldots, c_{e,p}, d_{e,1}, \ldots, d_{e,p} \mid e \in E\}$$

and $E^{[p]}$ **contains exactly the following edges:** $\forall e = uv \in E$, **and** $\forall 1 \le i \le p$,

1. **One edge between** $(u, c_{e,i})$, $(c_{e,i}, b_e)$, $(d_{e,i}, a_e)$, **and** $(d_{e,i}, v)$;
2. $M - 1$ **edges between** $(c_{e,i}, v)$, $(c_{e,i}, a_e)$, $(d_{e,i}, b_e)$, **and** $(d_{e,i}, u)$.

A Reduction

$\forall p \geq 1$, **there is a symmetric matrix $\mathbf{A}^{[p]} \in \mathbb{C}^{2m \times 2m}$ which only depends on $\mathbf{A}$, such that**

$$Z_{\mathbf{A}^{[p]}}(G) = Z_{\mathbf{A}}(G^{[p]}), \quad \textbf{for all } G.$$

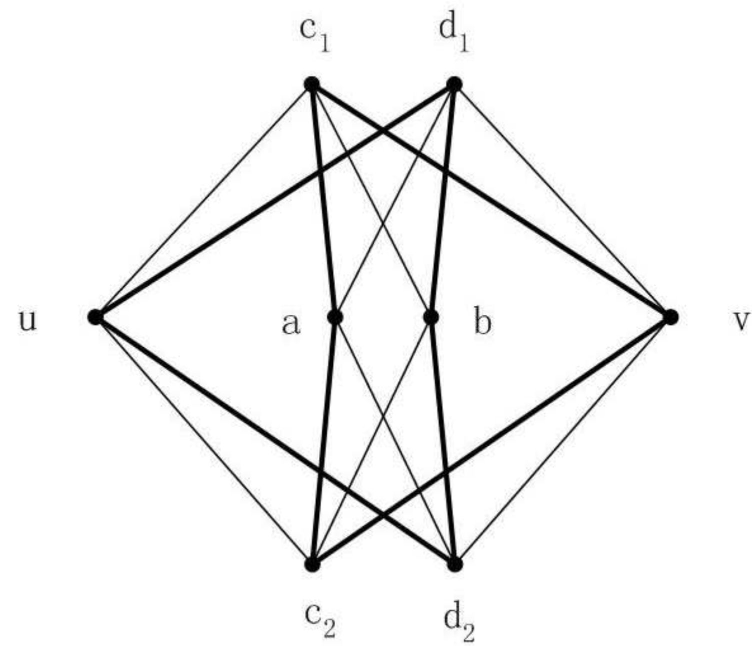
Thus $Z_{\mathbf{A}^{[p]}}(\cdot)$ is reducible to $Z_{\mathbf{A}}(\cdot)$, and therefore

$Z_{\mathbf{A}}(\cdot)$ **is not #P-hard**

$$\Longrightarrow$$

$Z_{\mathbf{A}^{[p]}}(\cdot)$ **is not #P-hard for all $p \geq 1$.**

Expression for $\mathbf{A}^{[p]}$



The $(i,j)^{th}$ **entry of** $\mathbf{A}^{[p]}$**, where** $i,j \in [0:m-1]$**, is**

$$A_{i,j}^{[p]} = \sum_{a=0}^{m-1} \sum_{b=0}^{m-1} \left(\sum_{c=0}^{m-1} A_{i,c} \overline{A_{a,c}} A_{b,c} \overline{A_{j,c}} \right)^p \left(\sum_{d=0}^{m-1} \overline{A_{i,d}} A_{a,d} \overline{A_{b,d}} A_{j,d} \right)^p .$$

Note $(A_{a,c})^{M-1} = \overline{A_{a,c}}$**, etc.**

Properties of $\mathbf{A}^{[p]}$

$$A_{i,j}^{[p]} = \sum_{a=0}^{m-1}\sum_{b=0}^{m-1}\left|\sum_{c=0}^{m-1} A_{i,c}\overline{A_{a,c}}A_{b,c}\overline{A_{j,c}}\right|^{2p}$$

$$= \sum_{a=0}^{m-1}\sum_{b=0}^{m-1}\left|\langle \mathbf{A}_{i,*}\circ\overline{\mathbf{A}_{j,*}}, \mathbf{A}_{a,*}\circ\overline{\mathbf{A}_{b,*}}\rangle\right|^{2p},$$

$\mathbf{A}^{[p]}$ is symmetric and non-negative.

In fact $A_{i,j}^{[p]} > 0$. (By taking $a = i$ and $b = j$).

Diagonal and Off-Diagonal

$$A^{[p]}_{i,i} = \sum_{a=0}^{m-1}\sum_{b=0}^{m-1} \left|\langle \mathbf{1}, \mathbf{A}_{a,*} \circ \overline{\mathbf{A}_{b,*}}\rangle\right|^{2p} = \sum_{a=0}^{m-1}\sum_{b=0}^{m-1} \left|\langle \mathbf{A}_{a,*}, \mathbf{A}_{b,*}\rangle\right|^{2p}.$$

As A is a discrete unitary matrix, we have $A^{[p]}_{i,i} = m \cdot m^{2p}$**.**

$Z_{\mathbf{A}}(\cdot)$ **is not #P-hard**

$\Longrightarrow$ **(by a known result for non-negative matrices)**

$$\det \begin{pmatrix} A^{[p]}_{i,i} & A^{[p]}_{i,j} \\ A^{[p]}_{j,i} & A^{[p]}_{j,j} \end{pmatrix} = 0.$$

and thus $A^{[p]}_{i,j} = m^{2p+1}$ **for all** $i, j \in [0 : m-1]$**.**

Another Way to Sum $A_{i,j}^{[p]}$

$$
\begin{aligned}
A_{i,j}^{[p]} &= \sum_{a=0}^{m-1} \sum_{b=0}^{m-1} \left| \langle \mathbf{A}_{i,*} \circ \overline{\mathbf{A}_{j,*}}, \mathbf{A}_{a,*} \circ \overline{\mathbf{A}_{b,*}} \rangle \right|^{2p} \\
&= \sum_{x \in X_{i,j}} s_{i,j}^{[x]} \cdot x^{2p},
\end{aligned}
$$

where $s_{i,j}^{[x]}$ is the number of pairs (a, b) such that

$$
x = |\langle \mathbf{A}_{i,*} \circ \overline{\mathbf{A}_{j,*}}, \mathbf{A}_{a,*} \circ \overline{\mathbf{A}_{b,*}} \rangle|.
$$

Note that $s_{i,j}^{[x]}$, for all x, do not depend on p.

A Linear System

So

$$A_{i,j}^{[p]} = \sum_{x \in X_{i,j}} s_{i,j}^{[x]} \cdot x^{2p}.$$

Meanwhile, it is also known that for all $p \geq 1$,

$$A_{i,j}^{[p]} = m^{2p+1}.$$

We can view, for each i and j fixed,

$$\sum_{x \in X_{i,j}} s_{i,j}^{[x]} \cdot x^{2p} = m^{2p+1}$$

as a linear system ($p = 1, 2, 3, \ldots$) in the unknowns $s_{i,j}^{[x]}$.

A Vandermonde System

It is a Vandermonde system.

We can "solve" it, and get $X_{i,j} = \{0, m\}$**,**

$$s_{i,j}^{[m]} = m \quad \textbf{and} \quad s_{i,j}^{[0]} = m^2 - m, \quad \textbf{for all } i, j \in [0 : m-1].$$

This implies that for all $i, j, a, b \in [0 : m-1]$**,**

$$|\langle \mathbf{A}_{i,*} \circ \overline{\mathbf{A}_{j,*}}, \mathbf{A}_{a,*} \circ \overline{\mathbf{A}_{b,*}} \rangle| \textbf{ is either } m \textbf{ or } 0.$$

Toward GC

Set $j = 0$. Because $\mathbf{A}_{0,*} = \mathbf{1}$, we have

$$|\langle \mathbf{A}_{i,*} \circ \mathbf{1}, \mathbf{A}_{a,*} \circ \overline{\mathbf{A}_{b,*}} \rangle| = |\langle \mathbf{A}_{i,*} \circ \mathbf{A}_{b,*}, \mathbf{A}_{a,*} \rangle|,$$

which is either m or 0, for all $i, a, b \in [0 : m-1]$.

Meanwhile, as $\{\mathbf{A}_{a,*}, a \in [0 : m-1]\}$ is an orthogonal basis, where each $||\mathbf{A}_{a,*}||^2 = m$, by Parseval's Equality, we have

$$\sum_a |\langle \mathbf{A}_{i,*} \circ \mathbf{A}_{b,*}, \mathbf{A}_{a,*} \rangle|^2 = m \|\mathbf{A}_{i,*} \circ \mathbf{A}_{b,*}\|^2.$$

Consequence of Parseval

Since every entry of $\mathbf{A}_{i,*} \circ \mathbf{A}_{b,*}$ is a root of unity, $\|\mathbf{A}_{i,*} \circ \mathbf{A}_{b,*}\|^2 = m$. Hence

$$\sum_a |\langle \mathbf{A}_{i,*} \circ \mathbf{A}_{b,*}, \mathbf{A}_{a,*} \rangle|^2 = m^2.$$

Recall

$$|\langle \mathbf{A}_{i,*} \circ \mathbf{A}_{b,*}, \mathbf{A}_{a,*} \rangle| \textbf{ is either } m \textbf{ or } 0.$$

As a result, for all $i, b \in [0 : m-1]$, there exists a unique a such that $|\langle \mathbf{A}_{i,*} \circ \mathbf{A}_{b,*}, \mathbf{A}_{a,*} \rangle| = m$.

A Sum of Roots of Unity

Every entry of $\mathbf{A}_{i,*}, \mathbf{A}_{b,*}$ and $\mathbf{A}_{a,*}$ is a root of unity.

Denote the inner product of rows $\langle \mathbf{A}_{i,*} \circ \mathbf{A}_{b,*}, \mathbf{A}_{a,*} \rangle$ is a sum of m terms each of complex norm 1. To sum to a compelx number of norm m, they must be all aligned exactly the same.

Thus,

$$\mathbf{A}_{i,*} \circ \mathbf{A}_{b,*} = e^{i\theta} \mathbf{A}_{a,*}.$$

But $\mathbf{A}_{i,1} = \mathbf{A}_{a,1} = \mathbf{A}_{b,1} = 1$. Hence

$$\mathbf{A}_{i,*} \circ \mathbf{A}_{b,*} = \mathbf{A}_{a,*}.$$

When A is not Bipartite . . .

. . . more proofs . . .

Some References

Some papers can be found on my web site

`http://www.cs.wisc.edu/~jyc`

THANK YOU!