Holographic Algorithms and Complexity of Counting Problems

Jin-Yi Cai
University of Wisconsin, Madison

Complexity of Counting Problems

Valiant introduced the class #P.

#P is Powerful

#P is at least as powerful as NP, and in fact subsumes the entire polynomial time hierarchy $\cup_i \Sigma_i^p$ [Toda].

#P-completeness and #P-hardness: #SAT, #PerfMatch, Permanent, etc.

Some Major Advances on Complexity of Counting Problems

Bulatov proved a sweeping dichotomy theorem for #CSP(Γ), for any constraint langauge Γ.

The method uses deep structural theory in Universal Algebra.

Dyer and Richerby gave an alternative proof, and also proves the decidability of the dichotomy criterion.

Also major advances on Graph Homomorphisms.

Graph Homomorphisms

Theorem (Hell and Nešetřil)
Dichotomy Theorem for the decision problem of Graph Homomorphism:
Either in P or NP-complete.

Theorem (Dyer and Greenhill)
Dichotomy Theorem for $Z_H(G)$, for all 0-1 H:
Either in P or #P-complete.

Theorem (Bulatov and Grohe)
Dichotomy Theorem for $Z_H(G)$, for all non-negative (algebraic) H.

Theorem (Dyer, Goldberg and Paterson)
Dichotomy Theorem for all directed and acyclic H.

Symmetric Real Matrices

Theorem (Goldberg, Jerrum, Grohe and Thurley)
There is a complexity dichotomy theorem for $Z_H(G)$, for all symmetric real matrix (algebraic) H.

A complexity dichotomy for partition functions with mixed signs

arXiv:0804.1932v2 [cs.CC]

Symmetric Complex Matrices

Theorem (C, Chen and Lu)

There is a complexity dichotomy theorem for $Z_H(G)$, for all symmetric complex matrix (algebraic) H.

Graph Homomorphisms with Complex Values: A Dichotomy Theorem

arXiv:0903.4728v1 [cs.CC]

Hermitian Matrices

Theorem (Thurley)

There is a complexity dichotomy theorem for $Z_H(G)$, for all Hermit matrix (algebraic) H.

The Complexity of Partition Functions on Hermitian Matrices

arXiv:1004.0992v1 [cs.CC]

Counting Problems when Cancelations Occur

It seems that there is a major difference when cancelations can occur in a counting problem.

Additional tractable problems appear, and to carve out exactly those tractable problems from the intractable ones presents additional difficulties.

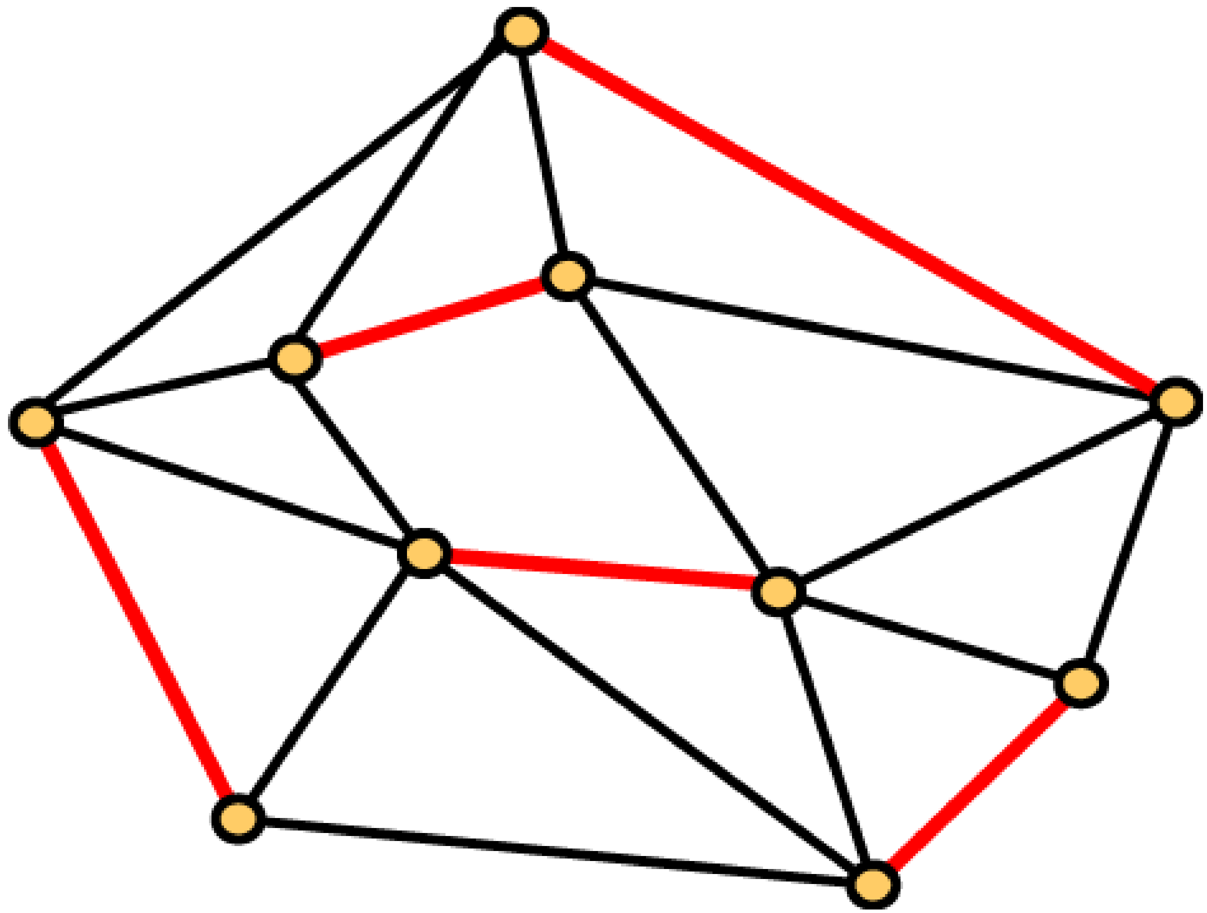
Think of the paradigmic example of Determinant and Permanent.

Plan of the Talk

- **Holographic Algorithms based on Matchgates.**
- **Holographic Reductions as hardness proofs.**
- **Holographic Algorithms based on Fibonacci gates.**
- **Holant Problems.**
- **Dichotomy Theorems for Holant Problems.**

Holographic Algorithms based on Matchgates

Perfect Matching



Some Surprising Tractability with Matching

The following problems are solvable in P:

- **Whether there exists a Perfect Matching in a general graph (Edmonds).**
- **Count the number of Perfect Matchings in a planar graph (Fisher, Temperley, Kasteleyn)**

Note that the problem of counting the number of (not necessarily perfect) matchings in a planar graph is still #P-complete [Jerrum].

Valiant's Matchgates-based Holographic Algorithms

Let us consider the following special case of #3SAT.

We represent a 3SAT instance $\Phi(x_1, x_2, \ldots, x_n)$ **as a bipartite graph** G_Φ**, where RHS are labeled with variables** x_i**, LHS are labeled by the** $\textsc{Or}$ **function.**

Suppose each variable x_i **appears positively, and in exactly 2 clauses—**G_Φ **is a 2-3 regular bipartite graph.**

We can write down the truth table for the $\textsc{Or}$ **function**

$x \vee y \vee z$

$(0, 1, 1, 1, 1, 1, 1, 1)$**.**

Restricted #3SAT problem continued

Now instead of thinking the variables x_i fanning out truth values, think equivalently the edges taking on values $\{0, 1\}$, subject to the requirement that both incident edges at each x_i take consistent values.

In other words, we assign a binary EQUALITY **function** $(=_2)$ **at each** x_i**.**

The truth table for $(=_2)$ **is**

$(1, 0, 0, 1)$**.**

Tensor Products

We have assigned $(1,0,0,1)$ **to each** x_i **on RHS, and** $(0,1,1,1,1,1,1,1)$ **to each** Or **function on LHS.**

Now take the tensor product $(1,0,0,1)^{\otimes n}$**.**

This forms a vector (tensor) of dimension 2^{2n} **indexed by the** $2n$ **edges in the 2-3 regular bipartite graph** G_Φ**.**

Similarly take the tensor product $(0,1,1,1,1,1,1,1)^{\otimes m}$ **for the** m **clauses. It has dimension** 2^{3m}**.**

(Being 2-3 regular, $2n = 3m$**.)**

Go Slowly ...

Suppose we have three variables x, y, z.

The tensor product $(1,0,0,1)^{\otimes 3}$ has dimension 2^6.

It is indexed by $b_1b_2b_3b_4b_5b_6 \in \{0,1\}^6$, corresponding to a truth assignment to the 6 edges (two edges from each variable).

On the clause side, we have two clauses, and the tensor product $(0,1,1,1,1,1,1,1)^{\otimes 2}$ also has dimension 2^6 and takes value one iff both Or **evaluates to True.**

Then the contraction (inner product) of the two tensors $(1,0,0,1)^{\otimes 3}$ and $(0,1,1,1,1,1,1,1)^{\otimes 2}$ gives the number of satisfying assignments.

Basic Idea of Matchgate Computation

Define $\mathrm{PerfMatch}(G) = \sum_M \prod_{(i,j) \in M} w_{ij}$**, where the sum is over all perfect matchings** M**.**

For planar graphs this quantity is computable in polynomial time.

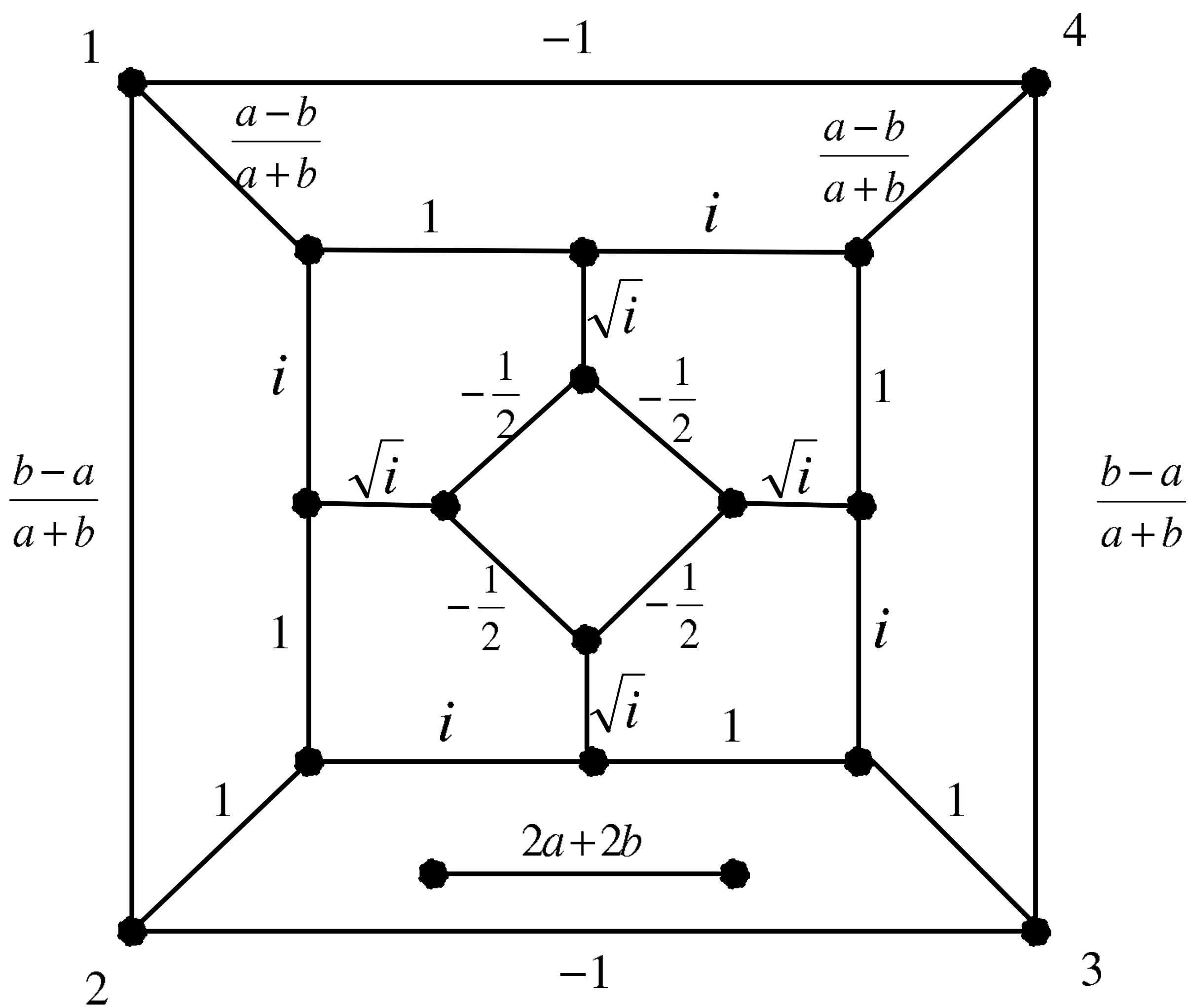
Matchgate

A planar matchgate $\Gamma = (G, X)$ is a weighted graph $G = (V, E, W)$ with a planar embedding, having external nodes, placed on the outer face.

Matchgates with only output nodes are called generators.

Matchgates with only input nodes are called recognizers.

A Matchgate



Standard Signatures

A matchgate Γ is assigned a Standard Signature

$$G = (G^S) \text{ and } R = (R_S),$$

for generators and recognizers respectively.

$$G^S = \text{PerfMatch}(G - S).$$

$$R_S = \text{PerfMatch}(G' - S).$$

Each entry is indexed by a subset S of external nodes.

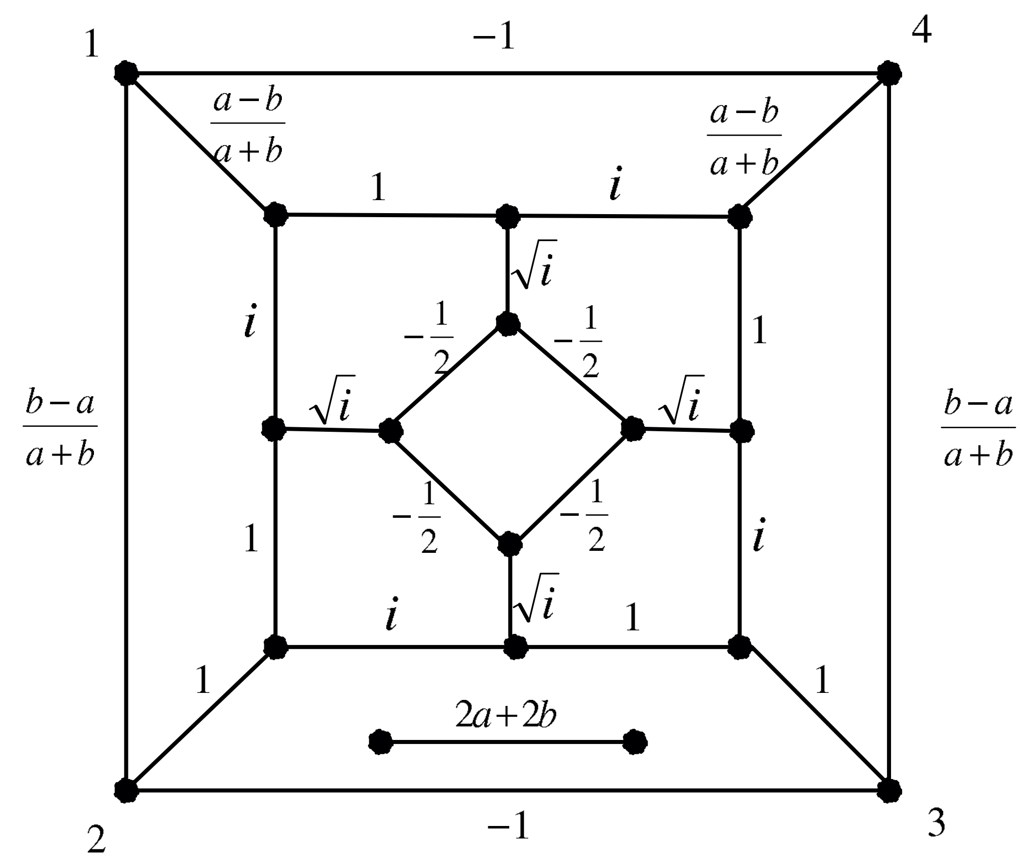


Figure 1: This planar matchgate has standard signature $(2a + 2b, 0, 0, -2a + 2b, 0, 2a - 2b, -2a - 2b, 0, 0, -2a - 2b, 2a - 2b, 0, -2a + 2b, 0, 0, 2a + 2b)^T$**.**

Restricted #3SAT Problem

Now suppose for some miraculous design we have two graph fragments:

Γ_1 **has two external nodes. As a generator its Standard Signature is**

$$G^S = \text{PerfMatch}(G_1 - S) = (1, 0, 0, 1)^{\mathtt{T}},$$

when $S = 00, 01, 10, 11$**.**

Γ_2 **has three external nodes. As a recognizer its Standard Signature is**

$$R_S = \text{PerfMatch}(G_2 - S) = (0, 1, 1, 1, 1, 1, 1, 1),$$

when $S = 000, 001, 010, 011, 100, 101, 110, 111$**.**

An expression of Number of Solutions

Then $\mathrm{PerfMatch}(\Omega)$ **is the number of satisfying assignments, where** Ω **is obtained from the 2-3 regular graph replacing RHS vertices (**x_i**) by** Γ_1 **and LHS vertices (clauses) by** Γ_2**.**

Note that this restricted version of #3SAT Problem is still #P-complete.

Holographic Algorithm

But things are not so simple.

While

$$G^S = \mathrm{PerfMatch}(G_1 - S) = (1,0,0,1)^{\mathtt{T}},$$

is realizable by a matchgate,

$$R_S = \mathrm{PerfMatch}(G_2 - S) = (0,1,1,1,1,1,1,1),$$

is not realizable.

The idea of Holographic Algorithm is to find a basis change for the tensors, so that they become realizable.

A Non-Obvious Realizability

Let $\boldsymbol{b}$ denote the standard basis,

$$\boldsymbol{b} = [e_0, e_1] = \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right].$$

Consider another basis

$$\boldsymbol{\beta} = [n, p] = \left[\begin{pmatrix} n_0 \\ n_1 \end{pmatrix}, \begin{pmatrix} p_0 \\ p_1 \end{pmatrix} \right].$$

Let $\boldsymbol{\beta} = \mathrm{b}T$. Denote $T = (t^i_j)$ and $T^{-1} = (\tilde{t}^i_j)$.

(Upper index is for row and lower index is for column.)

Contravariant and Covariant Tensors

Each generator Γ is assigned a contravariant tensor $\boldsymbol{G} = (G^{\alpha})$.

Under a basis transformation,

$$(G')^{i'_1 i'_2 \ldots i'_n} = \sum G^{i_1 i_2 \ldots i_n} \tilde{t}_{i_1}^{i'_1} \tilde{t}_{i_2}^{i'_2} \cdots \tilde{t}_{i_n}^{i'_n} \tag{1}$$

Correspondingly, each recognizer Γ gets a covariant tensor $\boldsymbol{R} = (R_{\alpha})$.

$$(R')_{i'_1 i'_2 \ldots i'_n} = \sum R_{i_1 i_2 \ldots i_n} t_{i'_1}^{i_1} t_{i'_2}^{i_2} \cdots t_{i'_n}^{i_n} \tag{2}$$

The contraction

$$\text{Holant} = \langle \otimes \boldsymbol{R}, \otimes \boldsymbol{G} \rangle = \sum_{x \in \boldsymbol{\beta}^{\otimes |E|}} \left\{ [\prod_j R_j(x|_{R_j})] \cdot [\prod_i G_i(x|_{G_i})] \right\}$$

is invariant under a basis change.

Realization for the OR gate

So we want the following

$$(0, 1, 1, 1, 1, 1, 1, 1)$$

as a (non-standard) signature under some basis.

Let

$$\left[\begin{pmatrix} 1+\omega \\ 1-\omega \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right],$$

where $\omega = e^{2\pi i/3}$ **is a primitive third root of unity.**

The Transformation Matrix from R' to R

$$\left(\begin{pmatrix} 1+\omega & 1 \\ 1-\omega & 1 \end{pmatrix}^{-1}\right)^{\otimes 3}$$ **is** $\frac{1}{8}$ **times**

$$\begin{pmatrix}
1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \\
-1+\omega & 1+\omega & 1-\omega & -1-\omega & 1-\omega & -1-\omega & -1+\omega & 1+\omega \\
-1+\omega & 1-\omega & 1+\omega & -1-\omega & 1-\omega & -1+\omega & -1-\omega & 1+\omega \\
-3\omega & -2-\omega & -2-\omega & \omega & 3\omega & 2+\omega & 2+\omega & -\omega \\
-1+\omega & 1-\omega & 1-\omega & -1+\omega & 1+\omega & -1-\omega & -1-\omega & 1+\omega \\
-3\omega & -2-\omega & 3\omega & 2+\omega & -2-\omega & \omega & 2+\omega & -\omega \\
-3\omega & 3\omega & -2-\omega & 2+\omega & -2-\omega & 2+\omega & \omega & -\omega \\
3+6\omega & 3 & 3 & -1-2\omega & 3 & -1-2\omega & -1-2\omega & -1
\end{pmatrix}$$

Back to Standard Signature

By covariant transformation, (adding the last 7 rows),

$$(R_{i_1 i_2 i_3}) = \frac{1}{4}(0,1,1,0,1,0,0,1).$$

There indeed exists a matchgate with three external nodes with the standard signature $= \frac{1}{4}(0,1,1,0,1,0,0,1)$**. Thus,**

$$R'_C = (0,1,1,1,1,1,1,1) = \frac{1}{4}(0,1,1,0,1,0,0,1)\left(\begin{pmatrix} 1+\omega & 1 \\ 1-\omega & 1 \end{pmatrix}\right)^{\otimes 3}.$$

Fundamental Questions for a Holographic Algorithm

Can the desired local constraint functions be realized as a matchgate (standard) signatures?

If not, can they be realized as non-standard signatures by a basis transformation?

Can the generators and recognizers be simultaneously realized under some basis transformation?

A More Systematic Approach

In *Holographic algorithms: From art to science* **with Pinyan Lu**

(STOC 2007, Journal version to appear in

Journal of Computer and System Sciences Volume 77, Issue 1, January 2011, Pages 41-61)

We make some progress on these problems.

Parity Requirements

Standard signatures (of either generators or recognizers) are characterized by the following two sets of conditions. i

(1) The parity requirements: either all even weight entries are 0 or all odd weight entries are 0.

This is due to perfect matchings.

Matchgate Identities

(2) A set of Matchgate Identities (MGI): Let $\underline{G}$ be a standard signature of arity n (Same for $\underline{R}$).

A pattern α is an n-bit string, i.e., $\alpha \in \{0,1\}^n$. A position vector $P = \{p_i\}, i \in [l]$, is a subsequence of $\{1, 2, \ldots, n\}$, i.e., $p_i \in [n]$ and $p_1 < p_2 < \cdots < p_l$. We also use p to denote the pattern, whose $(p_1, p_2, \ldots, p_l)$-th bits are 1 and others are 0. Let $e_i \in \{0,1\}^n$ be the pattern with 1 in the i-th bit and 0 elsewhere.

Let $\alpha \oplus \beta$ be the pattern obtained from bitwise XOR of the patterns α and β. Then for any pattern $\alpha \in \{0,1\}^n$ and any position vector $P = \{p_i\}, i \in [l]$, we have the following identity:

$$\sum_{i=1}^{l} (-1)^i \underline{G}^{\alpha \oplus e_{p_i}} \underline{G}^{\alpha \oplus p \oplus e_{p_i}} = 0. \qquad (3)$$

Realizability under a basis change using MGI

A signature is symmetric if the value of an entry only depends on the Hamming weight of the index bits.

e.g. Boolean Or **on 3 bits is**

$$(0, 1, 1, 1, 1, 1, 1, 1)$$

We denote it as $[0, 1, 1, 1]$**.**

Using MGI we can give a closed form expression for all realizable symmetric signatures.

A characterization Theorem for Symmetric Signatures

Theorem

A symmetric signature $[x_0, x_1, \cdots, x_n]$ **is realizable on some basis iff there exist three constants** a, b, c **(not all zero), such that for all** k**,** $0 \leq k \leq n-2$**,**

$$a x_k + b x_{k+1} + c x_{k+2} = 0. \tag{4}$$

Basis Manifold $\mathcal{M}$

We will identify the set of 2-dimensional bases $\left[\begin{pmatrix} n_0 \\ n_1 \end{pmatrix}, \begin{pmatrix} p_0 \\ p_1 \end{pmatrix}\right]$ **with** $\mathrm{GL}_2(\mathbf{F})$**. Over the complex field** $\mathbf{F} = \mathbf{C}$**, it has dimension 4. However, by a simple proposition of Valiant, the essential underlying structure has only dimension 2.**

Proposition (Valiant)
If there is a generator (recognizer) with certain signature for basis $\{(n_0, n_1), (p_0, p_1)\}$ then there is a generator (recognizer) with the same signature for basis $\{(xn_0, yn_1), (xp_0, yp_1)\}$ or $\{(xn_1, yn_0), (xp_1, yp_0)$ for any $x, y \in \mathbf{F}$ and $xy \neq 0$.

In other words, one can multiply any non-zero constants to each row, and permuting the rows, we get equivalent basis.

$$\mathcal{M} = \mathrm{GL}_2(\mathbf{F})/\sim .$$

Simultaneous Realizability

Definition

Let $B_{rec}([x_0, x_1, \ldots, x_n])$ (resp. $B_{gen}([x_0, x_1, \ldots, x_n])$) be the set of all possible bases in $\mathcal{M}$ for which a symmetric signature $[x_0, x_1, \ldots, x_n]$ for a recognizer (resp. a generator) is realizable. We also use $B_{rec}(R)$ and $B_{gen}(G)$ to denote the realizability subvarieties for general (unsymmetric) signatures R and G.

A complete and mutually exclusive list of realizable symmetric signatures for recognizers follows.

Simultaneous realizability is obtained by taking intersections.

List of Realizable Symmetric Signatures

Lemma

$$B_{rec}(\lambda[a^n, a^{n-1}b, \ldots, b^n]) = \left\{ \left[\begin{pmatrix} a \\ n_1 \end{pmatrix}, \begin{pmatrix} b \\ p_1 \end{pmatrix} \right] \in \mathcal{M} \middle| n_1, p_1 \in \mathbf{F} \right\}.$$

Lemma

$$B_{rec}([x_0, x_1, x_2])$$

$$= \left\{ \left[\begin{pmatrix} n_0 \\ n_1 \end{pmatrix}, \begin{pmatrix} p_0 \\ p_1 \end{pmatrix} \right] \in \mathcal{M} \middle| \begin{matrix} x_0 p_1^2 - 2x_1 p_1 n_1 + x_2 n_1^2 = 0, x_0 p_0^2 - 2x_1 p_0 n_0 + x_2 n_0^2 = 0 \\ \text{or} \quad x_0 p_0 p_1 - x_1(n_0 p_1 + n_1 p_0) + x_2 n_0 n_1 = 0 \end{matrix} \right\}.$$

Lemma

Let $\lambda_1 \neq 0$**. Let** $p = \text{char}.\mathbf{F}$**. Suppose** $p = 0$**, or** $p \nmid n$**,**

$$B_{rec}([0, 0, \ldots, 0, \lambda_1, \lambda_2]) = \left\{ \left[\begin{pmatrix} 0 \\ n\lambda_1 \end{pmatrix}, \begin{pmatrix} 1 \\ \lambda_2 \end{pmatrix} \right] \right\}.$$

For $p|n$ **and** $\lambda_2 = 0$**,**

$$B_{rec}([0, 0, \ldots, 0, \lambda_1, 0]) = \left\{ \left[\begin{pmatrix} 0 \\ n_1 \end{pmatrix}, \begin{pmatrix} 1 \\ p_1 \end{pmatrix} \right] \in \mathcal{M} \middle| n_1, p_1 \in \mathbf{F} \right\}.$$

For $p|n$ **and** $\lambda_2 \neq 0$**, the signature** $[0, 0, \ldots, 0, \lambda_1, \lambda_2]$ **is not realizable.**

Lemma

For $AB \neq 0$,

$$B_{rec}([A, A\alpha, A\alpha^2, \ldots, A\alpha^n + B]) = \left\{ \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} \alpha + \omega \\ \alpha - \omega \end{pmatrix} \right] \middle| \omega^n = \pm \frac{B}{A} \right\}$$

Lemma

For $AB \neq 0$ and $\alpha \neq \beta$,

$$B_{rec}([A\alpha^i + B\beta^i | i = 0, 1, \ldots, n]) = \left\{ \left[\begin{pmatrix} 1 + \omega \\ 1 - \omega \end{pmatrix}, \begin{pmatrix} \alpha + \beta\omega \\ \alpha - \beta\omega \end{pmatrix} \right] \middle| \omega^n = \pm \frac{B}{A} \right\}.$$

Lemma

Let $p = \mathrm{char}.\mathbf{F}$ **and let** $A \neq 0$**.**

Case 1: $p = 0$ **or** $p \nmid n$**.**

$$B_{rec}([Ai\alpha^{i-1} + B\alpha^i | i = 0, 1, \ldots, n]) = \left\{ \left[\begin{pmatrix} 1 \\ B \end{pmatrix}, \begin{pmatrix} \alpha \\ nA + B\alpha \end{pmatrix} \right] \right\}.$$

Case 2: $p | n$ **and** $x_0 = 0$**. In this case, the signature has entries** $x_i = Ai\alpha^{i-1}$**, with** $B = 0$ **in the above form.**

$$B_{rec}([Ai\alpha^{i-1} | i = 0, 1, \ldots, n]) = \left\{ \left[\begin{pmatrix} 1 \\ n_1 \end{pmatrix}, \begin{pmatrix} \alpha \\ p_1 \end{pmatrix} \right] \in \mathcal{M} \;\middle|\; n_1, p_1 \in \mathbf{F} \right\}.$$

Case 3: $p | n$ **and** $x_0 \neq 0$**. In this case the signature** $[Ai\alpha^{i-1} + B\alpha^i | i = 0, 1, \ldots, n]$ **is not realizable.**

Simultaneous Realizability

Definition

The Simultaneous Realizability Problem:

Input: A set of symmetric signatures for generators and/or recognizers.

Output: A common basis of these signatures if any exists; "NO" if they are not simultaneously realizable.

This can be solved in polynomial time.

An Example using the Machinery

$\#_7$**Pl-Rtw-Mon-3CNF**

$\#_{2^k-1}$**Pl-Rtw-Mon-kCNF**

Given a planar formula in kCNF form, where each variable appears positively, and read twice.

(2-k-regular planar bipartite graph.)

Replace each variable by a generator with the signature $[1, 0, 1]$,

Replace each clause by a recognizer with the signature $[0, 1, 1, \cdots, 1]$ (with k 1's).

Holographic Transformation

Then the question boils down to whether there is a basis in $\mathcal{M}$ **where** $[1,0,1]$ **for a generator and** $[0,1,1,\cdots,1]$ **(with** k **1's) for a recognizer can be simultaneously realized.**

From Lemma, with $A=1, B=-1, \alpha=1, \beta=0$**, we have**

$$B_{rec}([0,1,1,\cdots,1]) = \left\{ \left[\begin{pmatrix} 1+\omega \\ 1-\omega \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right] \middle| \omega^k = \pm 1 \right\}.$$

We look for some $\omega^k = \pm 1$**, such that** $\left[\begin{pmatrix} 1+\omega \\ 1-\omega \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right] \in B_{gen}([1,0,1])$. **According to Lemma, we want** $(1+\omega)^2+1=(1-\omega)^2+1=0$ **or** $(1+\omega)(1-\omega)+1=0$.

The first case is impossible, and in the second case we require $\omega^2=2$. **Together with the condition** $\omega^k=\pm1$**, we**

have $2^k - 1 = 0$. From this we can already see that for every prime $p | 2^k - 1$, $\#_p$Pl-Rtw-Mon-kCNF is computable in polynomial time. In particular this is true for every Mersenne prime $2^q - 1$. (Note that $\omega^2 = 2$ means that 2 is a quadratic residue.)

Holographic Reductions as Hardness Proofs

Going Beyond Matchgates: Holant Problems

It turns out that Holographic Reduction is also a powerful tool to prove hardness.

A *signature grid* $\Omega = (G, \mathcal{F})$ **is a tuple, where** $G = (V, E)$ **is a graph, and each** $v \in V(G)$ **is assigned a function** $F_v \in \mathcal{F}$**.**

$$\text{Holant}_\Omega = \sum_{\sigma} \prod_{v \in V} F_v(\sigma \mid_{E(v)}).$$

Symmetric signatures: $[f_0, f_1, \ldots, f_n]$

2-3 Regular Graphs

Consider 2-3 Regular Bipartite Graphs.

$$G = (U, V, E), \qquad \deg(u) = 3 \quad \forall u \in U, \quad \textbf{and} \quad \deg(v) = 2 \quad \forall v \in V.$$

Each $v \in V$ **is assigned a constraint function** F_v**.**

We aim to prove a Complexity Dichotomy for

$$\text{Holant}(\Omega) = \sum_{\sigma} \prod_{v \in V} F_v(\sigma \mid_{E(v)}),$$

according to the type of functions F_v**.**

A Dichotomy Theorem

We use the notation $\text{Holant}([x_0, x_1, x_2] \,|\, [y_0, y_1, y_2, y_3])$**.**

This includes VERTEX COVER $\text{Holant}([0, 1, 1] \,|\, [1, 0, 0, 1])$**.**

and PERFECT MATCHING $\text{Holant}([1, 0, 1] \,|\, [0, 1, 1, 1])$**.**

Theorem

Every counting problem $\text{Holant}([x_0, x_1, x_2] \,|\, [y_0, y_1, y_2, y_3])$**, where** $[x_0, x_1, x_2]$ **and** $[y_0, y_1, y_2, y_3]$ **are Boolean signatures, is either**

- **in P; or**
- **#P-complete but solvable in P for planar graphs; or**
- **#P-complete even for planar graphs.**

Holographic Reductions for Both Directions

We use Holographic Reductions to prove both tractability as well as hardness.

For Tractability we introduce a new class of Holographic Algorithms, called Fibonacci Gates.

For hardness proofs, we apply Holographic Reductions from a known #P-hard problem.

A Glance at a Dichotomy Theorem

$f_2 \mid g_3$	$[0,1,0]$	$[1,0,1]$	$[1,1,0]$
$[0,0,1,0]$	**T**	**P**	**T**
$[0,0,1,1]$	**T**	**H**	**T**
$[0,1,0,0]$	**T**	**P**	**H**
$[0,1,0,1]$	**F**	**F**	**H**
$[0,1,1,0]$	**P**	**P**	**H**
$[0,1,1,1]$	**H**	**H**	**H**
$[1,0,0,1]$	**T**	**T**	**H**
$[1,0,1,0]$	**F**	**F**	**H**
$[1,0,1,1]$	**H**	**F**	**H**
$[1,1,0,0]$	**T**	**H**	**H**
$[1,1,0,1]$	**H**	**F**	**F**
$[1,1,1,0]$	**H**	**H**	**H**

Non-degeneracy

Definition

For $n \geq 2$, a signature $[x_0, x_1, \ldots, x_n]$ is called non-degenerate if

$$\operatorname{rank} \begin{bmatrix} x_0 & \ldots & x_{n-1} \\ x_1 & \ldots & x_n \end{bmatrix} = 2.$$

Two Ideas in Hardness Proof

The First Step: Holographic reductions. To show $\text{Holant}([x_0, x_1, x_2] \big| [y_0, y_1, y_2, y_3])$ **is #P-Complete, we use holographic reductions to reduce either**

$$[0, 1, 1] \big| [1, 0, 0, 1]$$

or

$$[1, 0, 1] \big| [1, 1, 0, 0]$$

to

$$[z_0, z_1, z_2] \big| [y_0, y_1, y_2, y_3]$$

for some z_0, z_1 **and** z_2**.**

The first is Vertex Cover, the second is Matching.

Holographic Reductions

For every *non-degenerate* **signature** $[y_0, y_1, y_2, y_3]$**, there exists a symmetric signature** $[z_0, z_1, z_2]$ **of arity two, such that under a suitable holographic reduction,**

$$\#[z_0, z_1, z_2]\Big|[y_0, y_1, y_2, y_3] \equiv \textbf{A known \#P-Complete problem.}$$

Thus,

$\#[z_0, z_1, z_2]\Big|[y_0, y_1, y_2, y_3]$ **is #P-Complete.**

Note that now z_0, z_1, z_2 **may take complex values.**

Second Step

Second, to show that $\text{Holant}([x_0, x_1, x_2] \big| [y_0, y_1, y_2, y_3])$ **is #P-Complete, we show how the pair**

$$[x_0, x_1, x_2] \Big| [y_0, y_1, y_2, y_3]$$

can "simulate" (or "interpolate")

$$[z_0, z_1, z_2] \Big| [y_0, y_1, y_2, y_3]$$

In fact, we show how to "simulate" $[x, y, z] \Big| [y_0, y_1, y_2, y_3]$ **for all** $[x, y, z]$**.**

Interpolation Method

The second idea is also due to Valiant: Interpolation.

This has been further developed by

- **Vadhan**
- **Dyer**
- **Greenhill**
- **Bulatov**
- **Dalmau**
- **Grohe**
- **Creignou**
- **Hermann**
- **Goldberg**

- **Jerrum**
- **Xia-Zhang-Zhao**
- **Goldberg-Grohe-Jerrum-Thurley, . . .**

Interpolation Method

Given $\Omega = (G, [x, y, z] \Big| [y_0, y_1, y_2, y_3])$**. Let**

$$f = [x, y, z].$$

$f(00) = x$**,** $f(01) = f(10) = y$ **and** $f(11) = z$**.**

$V_f =$ **the subset of** V **assigned** f **in** Ω**.**

$|V_f| = n$**.**

An Expression for Holant

$$\text{Holant}(\Omega) = \sum_{i+j+k=n} c_{i,j,k} x^i y^j z^k,$$

$c_{i,j,k} =$ **is the sum over all edge assignments** σ**, of products of evaluations at all** $v \in V(G) - V_f$**, where** σ **satisfies the property that the number of vertices in** V_f **having exactly 0 or 1 or 2 incident edges assigned 1 is** i **or** j **or** k**, respectively.**

Holant_{Ω_s}

Let $\{f_s\} = \{[x_s, y_s, z_s]\}$**, for** $s = 0, 1, \ldots$**.**

Replace f **by** f_s **in** Ω

$$\text{Holant}_{\Omega_s} = \sum_{i+j+k=n} c_{i,j,k} x_s^i y_s^j z_s^k. \tag{5}$$

Note that the same set of values $c_{i,j,k}$ **occur.**

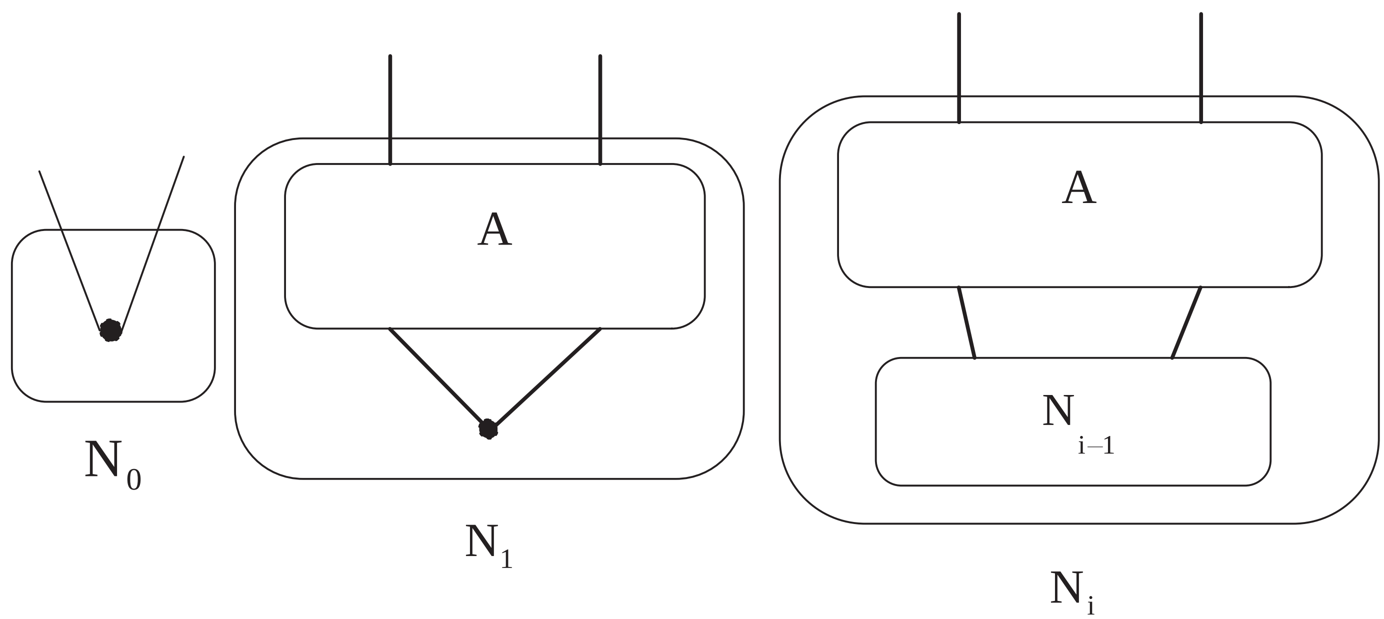
$c_{i,j,k}$ **is independent of** s**.**

Now consider (5) as a linear system in the unknowns $c_{i,j,k}$**.**

Recursive Construction

A sequence of gadgets N_s will be recursively constructed, producing $f_s = [x_s, y_s, z_s]$.

$$\begin{bmatrix} x_s \\ y_s \\ z_s \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_{s-1} \\ y_{s-1} \\ z_{s-1} \end{bmatrix}. \tag{6}$$



A
A
N
i−1
N0
N1
Ni

An example

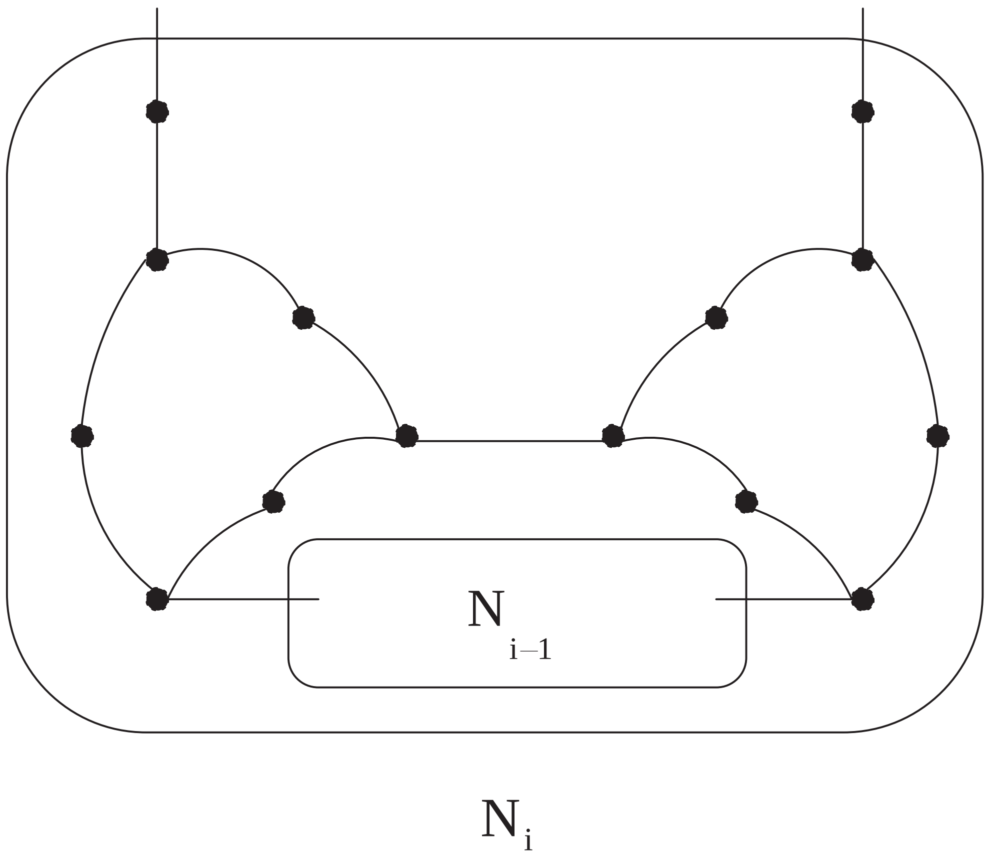


Figure 2: Gadget 1.

Interpolation Theorem

Theorem

Suppose the recurrence matrix A of the construction N_s satisfies

1. $\det(A) \neq 0$**,**
2. **The initial signature $[x_0, y_0, z_0]$ is not orthogonal to any row eigenvector of A, and**
3. **For all $(i, j, k) \in \mathbf{Z}^3 - \{(0, 0, 0)\}$ with $i + j + k = 0$, $\alpha^i \beta^j \gamma^k \neq 1$.**

Then all $c_{i,j,k}$ can be computed in polynomial time.

An Algebraic Condition via Galois Theory

The key condition is the lattice condition:

For all $(i, j, k) \in \mathbf{Z}^3 - \{(0, 0, 0)\}$ **with** $i + j + k = 0$**,**

$$\alpha^i \beta^j \gamma^k \neq 1.$$

Lemma

Let $f(x) = x^3 + c_2 x^2 + c_1 x + c_0 \in \mathbf{Q}[x]$**, with roots** α**,** β **and** γ**.**

It is decidable in P whether the lattice condition holds.

If f **is irreducible, except of the form** $x^3 + c$ **for some** $c \in \mathbf{Q}$**, the condition holds.**

An example

The counting problem $\text{Holant}([1,1,0]\big|[1,1,1,0])$**.**

A recursive construction gives the following recursive relation:

$$\begin{bmatrix} a_i \\ b_i \\ c_i \end{bmatrix} = \begin{bmatrix} 7191 & 12618 & 5535 \\ 3816 & 6723 & 2961 \\ 2025 & 3582 & 1584 \end{bmatrix} \begin{bmatrix} a_{i-1} \\ b_{i-1} \\ c_{i-1} \end{bmatrix}.$$

Characteristic polynomial

$$\chi(x) = x^3 - 15498x^2 + 419904x - 19683.$$

$$\Longrightarrow$$

#P-complete

Fibonacci Gates

Definition and Tractability

Let $\{f_k\}_{k=0}^{n}$ **be a sequence, satisfying** $f_{k+2} = f_{k+1} + f_k$ **for all** $k = 0, 1, \ldots, n-2$**. For any initial values** f_0 **and** f_1**, the sequence defines a Fibonacci gate** $[f_0, f_1, \ldots, f_n]$**.**

Theorem

For any Ω**, the holant problem on** Ω **where all vertex constraint functions are from Fibonacci Gates** $\mathcal{F}$ **can be computed in polynomial time.**

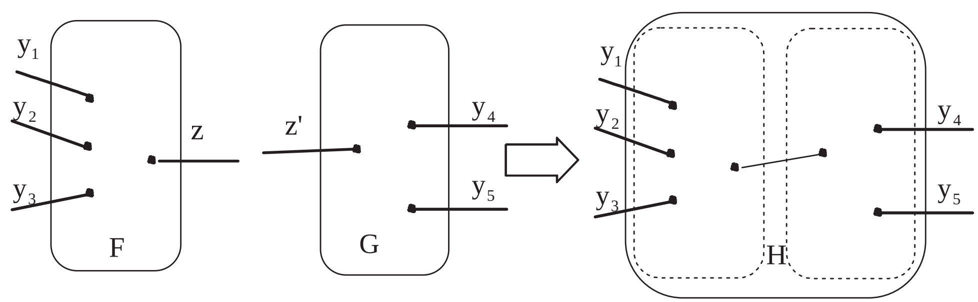


Figure 3: First operation.

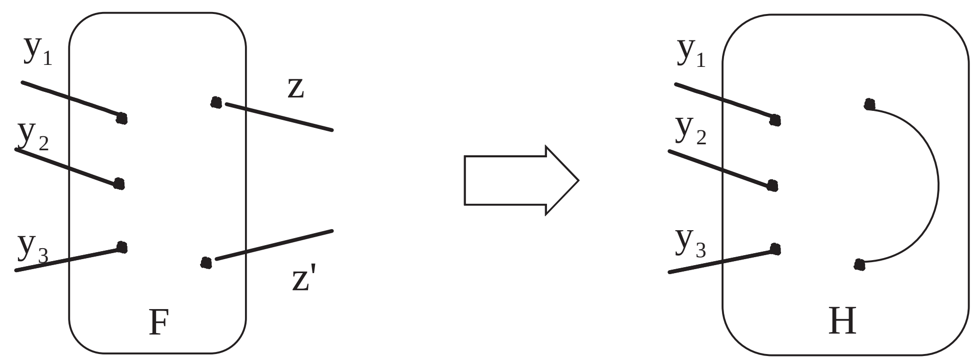


Figure 4: Second operation.

Realizability

Theorem

A set of symmetric generators $G_1, G_2, \ldots, G_s$ and recognizers $R_1, R_2, \ldots, R_t$ are simultaneously realizable as Fibonacci gates on some basis of size 1 iff there exist three constants a, b and c, such that $b^2 - 4ac \neq 0$ and the following two conditions are satisfied:

1. For any recognizer $R_i = [x_1^{(i)}, x_2^{(i)}, \ldots, x_{n_i}^{(i)}]$ and any $k = 0, 1, \ldots, n_i - 2$, $ax_k^{(i)} + bx_{k+1}^{(i)} + cx_{k+2}^{(i)} = 0$.

2. For any generator $G_j = [y_1^{(j)}, y_2^{(j)}, \ldots, y_{m_j}^{(j)}]$ and any $k = 0, 1, \ldots, m_j - 2$, $cy_k^{(j)} - by_{k+1}^{(j)} + ay_{k+2}^{(j)} = 0$.

Generalized Fibonacci Gates

For any fixed parameter m, we consider any sequence $[f_0, f_1, \ldots, f_n]$, where it satisfies the following recurrence, for $i = 0, \ldots, n-2$,

$$f_{i+2} = m \cdot f_{i+1} + f_i.$$

All results on Fibonacci gates can be extended to generalized Fibonacci gates which also admit polynomial time algorithms.

Holant Problems

A *signature grid* $\Omega = (G, \mathcal{F}, \pi)$ **consists of a graph** $G = (V, E)$**, and a labeling** π **of each vertex** $v \in V$ **with a function** $f_v \in \mathcal{F}$**. The Holant problem on instance** Ω **is to compute** $\text{Holant}_\Omega = \sum_{\sigma: E \to \{0,1\}} \prod_{v \in V} f_v(\sigma \mid_{E(v)})$**.**

A Holant problem is parameterized by a set of signatures.

Definition

Given a set of signatures $\mathcal{F}$, we define a counting problem $\text{Holant}(\mathcal{F})$**:**

Input: A *signature grid* $\Omega = (G, \mathcal{F}, \pi)$**;**

Output: Holant_Ω**.**

Holant* Problems

Definition

Let $\mathcal{U}$ denote the set of all unary signatures. Then $\text{Holant}^*(\mathcal{F}) = \text{Holant}(\mathcal{F} \cup \mathcal{U})$.

A degenerate signature is a tensor product of unary signatures.

Theorem

Let $\mathcal{F}$ be a set of non-degenerate *symmetric* signatures over $\mathbb{C}$. Then $\text{Holant}^*(\mathcal{F})$ is computable in polynomial time in the following three Classes**. In all other cases,** $\text{Holant}^*(\mathcal{F})$ **is #P-hard.**

1. **Every signature in $\mathcal{F}$ is of arity no more than two;**
2. **There exist two constants a and b (not both zero, depending only on $\mathcal{F}$), such that for all signatures $[x_0, x_1, \ldots, x_n] \in \mathcal{F}$ one of the two conditions is satisfied: (1) for every $k = 0, 1, \ldots, n-2$, we have $ax_k + bx_{k+1} - ax_{k+2} = 0$; (2) $n = 2$ and the signature $[x_0, x_1, x_2]$ is of the form $[2a\lambda, b\lambda, -2a\lambda]$.**
3. **For every signature $[x_0, x_1, \ldots, x_n] \in \mathcal{F}$ one of the two conditions is satisfied: (1) For every $k = 0, 1, \ldots, n-2$, we have $x_k + x_{k+2} = 0$; (2) $n = 2$ and the signature**

$[x_0, x_1, x_2]$ **is of the form** $[\lambda, 0, \lambda]$**.**

The dichotomy is still valid even if the inputs are restricted to planar graphs.

Orthogonal Transformation

Suppose $\mathcal{F}$ is a function set and M is a 2×2 matrix. We use $M \circ \mathcal{F}$ to denote the set consisting of all functions in $\mathcal{F}$ transformed by a matrix M,

$$M \circ \mathcal{F} = \{M^{\otimes r_F} F | F \in \mathcal{F}, r_F = \text{arity}(F)\}.$$

Suppose $M = H$ is orthogonal. Let $\mathcal{E}$ be the class of all EQUALITY **gates.**

We prove the tractability of Holant$^*(H \circ \mathcal{E})$.

We first reformulate it as a bipartite Holant problem Holant$(=_2 | H \circ \mathcal{E})$**.**

Here the edges are replaced by the binary EQUALITY **function** $(=_2) = [1, 0, 1]$**. Now we perform a holographic reduction by the basis transformation** H^{-1} **on the RHS. This (contravariant) transformation on the RHS is accompanied by the (covariant) transformation** $[1, 0, 1] \mapsto [1, 0, 1]H^{\otimes 2}$**. One can verify that an orthogonal** H **keeps** $[1, 0, 1]$ **invariant, namely** $[1, 0, 1]H^{\otimes 2} = [1, 0, 1]$**.**

To wit: let $H = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$**, then**

$$\begin{aligned} [1,0,1]H^{\otimes 2} &= \left((1,0)^{\otimes 2} + (0,1)^{\otimes 2}\right) H^{\otimes 2} \\ &= ((1,0)H)^{\otimes 2} + ((0,1)H))^{\otimes 2} \\ &= (a,b)^{\otimes 2} + (c,d)^{\otimes 2} \\ &= (a^2+c^2, ab+cd, ab+cd, b^2+d^2) \\ &= (1,0,0,1) = [1,0,1] \end{aligned}$$

The EQUALITY Signature Being Factored

Holographic transformations guide the discovery and formulation of our dichotomy theorems.

The EQUALITY **function** $=_2$ **can be "factored" by an orthogonal** H**, and thus "contributes" an orthogonal** H **to the RHS in this holographic transformation:**

$$\text{Holant}(=_2 | H \circ \mathcal{F}) \longleftrightarrow \text{Holant}(=_2 | \mathcal{F}),$$

DisEquality Being Factored

The binary DisEquality **function $\neq_2$ can also be "factored".**

Define

$$Z_1 = \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \quad \textbf{and} \quad Z_2 = \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}.$$

A Factorization via Z

The DisEquality **function $\neq_2$ can be "factored" as**

$$(\neq_2) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cong Z_1^{\mathsf{T}} Z_1 = \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}$$

and thus "contributes" a Z to the RHS in the following holographic transformation:

$$\text{Holant}(=_2 |Z \circ \mathcal{F}) \longleftrightarrow \text{Holant}(\neq_2 |\mathcal{F}).$$

Holant* Dichotomy

Theorem

The following classes of Holant* problems are polynomial time computable.

- **Holant**$^*(\langle \mathcal{T} \rangle)$
- **Holant**$^*(\langle H \circ \mathcal{E} \rangle)$**;**
- **Holant**$^*(\langle Z \circ \mathcal{E} \rangle)$**; and**
- **Holant**$^*(\langle Z \circ \mathcal{M} \rangle)$

Everything else is #P-hard.

THANK YOU!

http://www.cs.wisc.edu/~jyc