# U. Wisconsin CS/ECE 552
# Introduction to Computer Architecture

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## Arithmetic Part 1 (Chapter 3.1-3.5, B.5-B.6)

www.cs.wisc.edu/~karu/courses/cs552/

Slides combined and enhanced by Karu Sankaralingam from work by Falsafi, Hill, Marculescu, Nagle, Patterson, Roth, Rutenbar,Schmidt, Shen, Sohi, Sorin, Thottethodi, Vijaykumar, & Wood

# Outline

- Representing Integers
  - Unsigned, 2’s Complement
- Addition and subtraction
- Add/Sub ALU
  - full adder, ripple carry, subtraction,
- Carry lookahead
- Overflow
- Barrel shifter
- Defer: multiplication, division, floating-point

# Unsigned Integers

- Recall:
  - n bits give rise to $2^n$ combinations
  - let us call a string of 32 bits as "$b_{31}$ $b_{30}$ . . . $b_3$ $b_2$ $b_1$ $b_0$"
- $f(b_{31} \ldots b_0) = b_{31} \times 2^{31} + \ldots + b_1 \times 2 + b_0 \times 2^0$
- Treat as normal binary number
  - e.g., 0 . . .011010101

  $= 1 \times 2^7+1 \times 2^6+0 \times 2^5+1 \times 2^4+0 \times 2^3+1 \times 2^2+0 \times 2^1+1 \times 2^0$

  $= 128 + 64 + 16 + 4 + 1 = 213$
- max f (111 . . . 11) = $2^{32}$ - 1 = 4, 294, 967, 295
- min f(000 . . . 00) = 0
- range [0, $2^{32}$-1] => # values ($2^{32}$ -1) - 0 + 1 = $2^{32}$

# More Generally

- Bits have no inherent meaning
- Conventions define meaning
  - E.g., represent negative integers?
  - Seek circuit simplicity & speed
- n bits can represent finite possibilities: $2^n$
  - Integers countably infinite → overflow
  - Reals uncountably infinite → overflow, underflow, imprecise

# Integer Representation

- Sign Magnitude: One's Complement Two's Complement

| Sign Magnitude | One's Complement | Two's Complement |
|---|---|---|
| 000 = +0 | 000 = +0 | 000 = +0 |
| 001 = +1 | 001 = +1 | 001 = +1 |
| 010 = +2 | 010 = +2 | 010 = +2 |
| 011 = +3 | 011 = +3 | 011 = +3 |
| 100 = -0 | 100 = -3 | 100 = -4 |
| 101 = -1 | 101 = -2 | 101 = -3 |
| 110 = -2 | 110 = -1 | 110 = -2 |
| 111 = -3 | 111 = -0 | 111 = -1 |

- Balance, number of zeros, ease of arithmetic

# Two's Complement Integers

- $f(b_{31}\ b_{30}\ .\ .\ .\ b_1\ b_0) = -b_{31} \times 2^{31} + .\ .\ . + b_1 \times 2 + b_0 \times 2^0$
  - max $f(0111\ .\ .\ .\ 11) = 2^{31} - 1 = 2147483647$
  - min $f(100\ .\ .\ .\ 00) = -2^{31} = -2147483648$ (asymmetric)
- range $[-2^{31}, 2^{31}-1]$ => #values $(2^{31}-1 - -2^{31} + 1) = 2^{32}$
- E.g., -6
- 000 . . . 0110 --> 111 . . 1001 + 1 --> 111 . . .1010

# Two's Complement Operations

- Negating a two's complement number: invert all bits and add 1
  - 1010 -> 0101 + 1 = 0110
  - 0110 -> 1001 + 1 = 1010

- Converting n bit numbers into numbers with more than n bits:
  - copy the most significant bit (the sign bit)

    ```
    0010  -> 0000 0010
    1010  -> 1111 1010
    ```
  - Called "sign extension"

# Sign extension

- Consider representation of -2:

| 3bit (decimal) | 2-bit (decimal) |
| --- | --- |
| 011 (+3) | |
| 010 (+2) | |
| 001 (+1) | 01 (+1) |
| 000 (0) | 00 (0) |
| 111 (-1) | 11 (-1) |
| 110 (-2) | 10 (-2) |
| 101 (-3) | |
| 100 (-4) | |

# Addition and Subtraction

- Similar to decimal (carry/borrow twos instead of tens)
- Identical operation for signed and unsigned
  - E.g. Unsigned vs Signed

| | | |
|---|---|---|
| 0011 | 3 | 3 |
| 1010 | 10 | -6 |
| 1101 | 13 | -3 |

# Interesting cases

- Show computation in 4-bit 2's complement representation

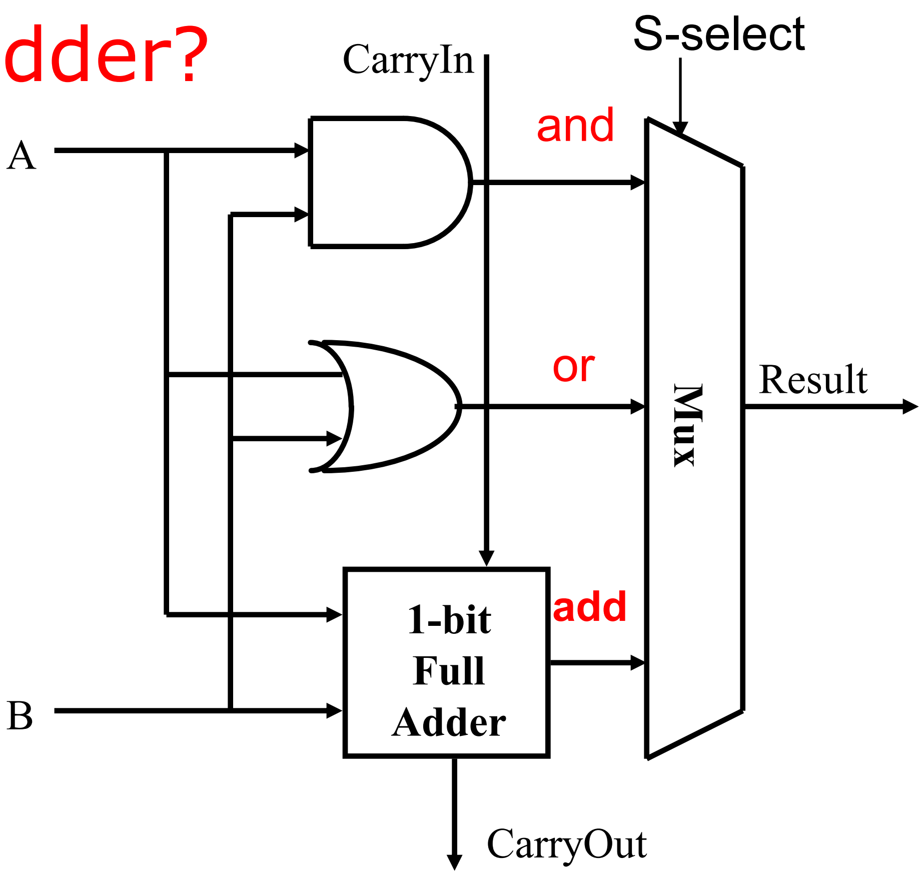
  4+4

  (-4) + (-4)

- Overflow: later

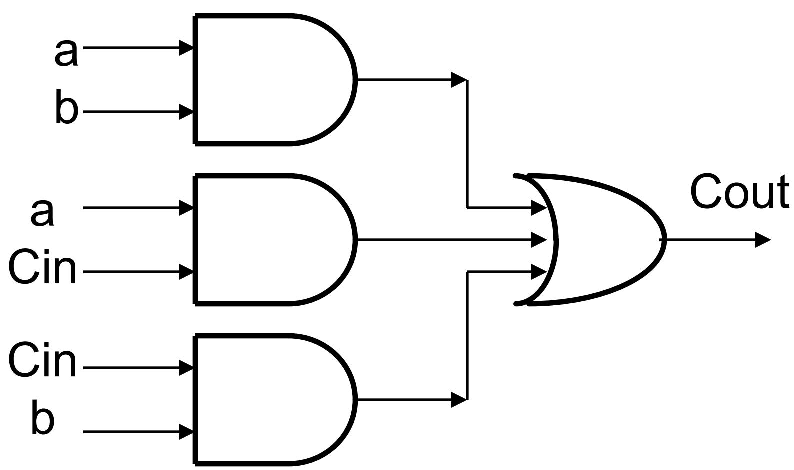
# ALU bit-slice

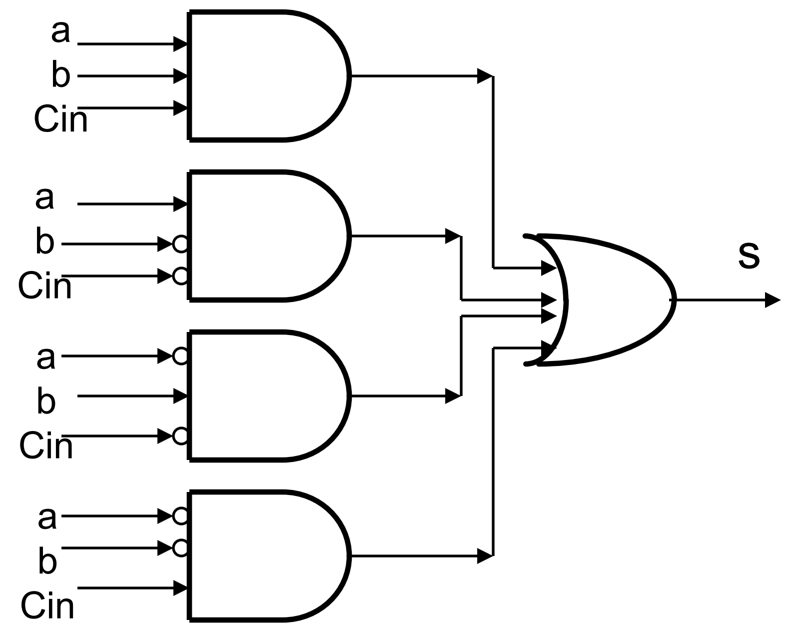
- Bit-wise operation
  - and, or, add
  - Full adder?
  - Sub?



# Full adder

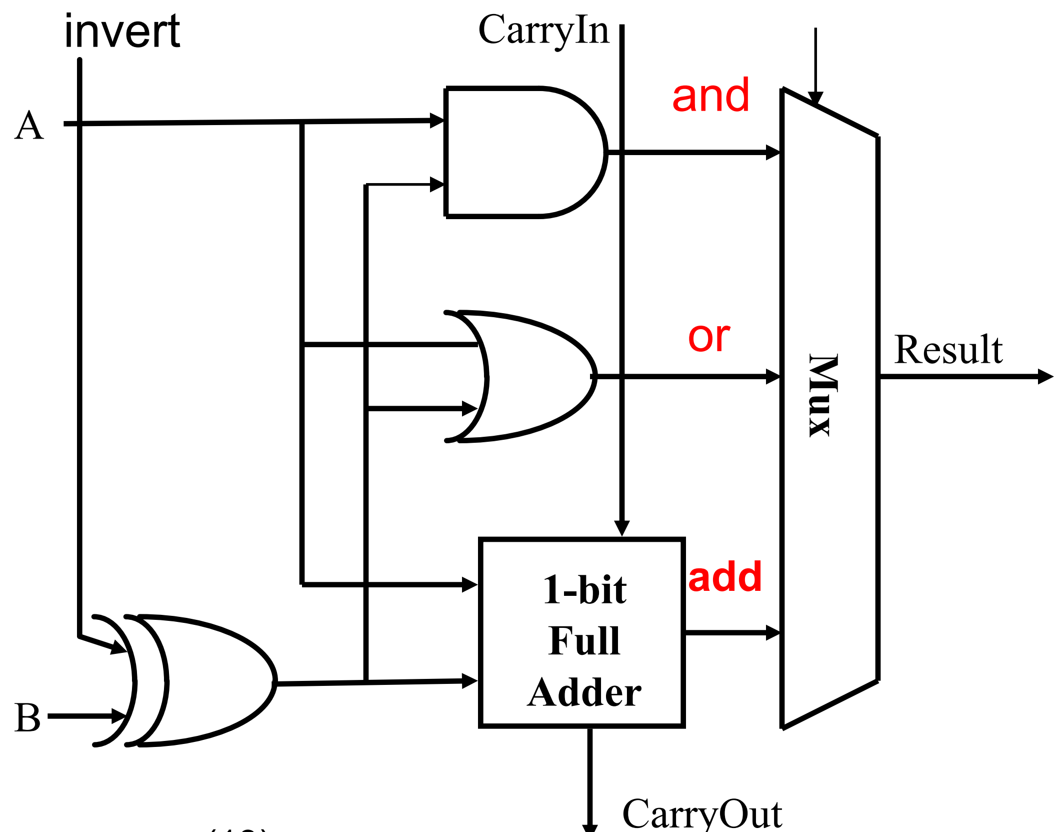
- Three inputs and two outputs
- Cout, s = F(a,b,Cin)
  - Cout : only if at least two inputs are set
  - S : only if exactly one input or all three inputs are set
- Logic?



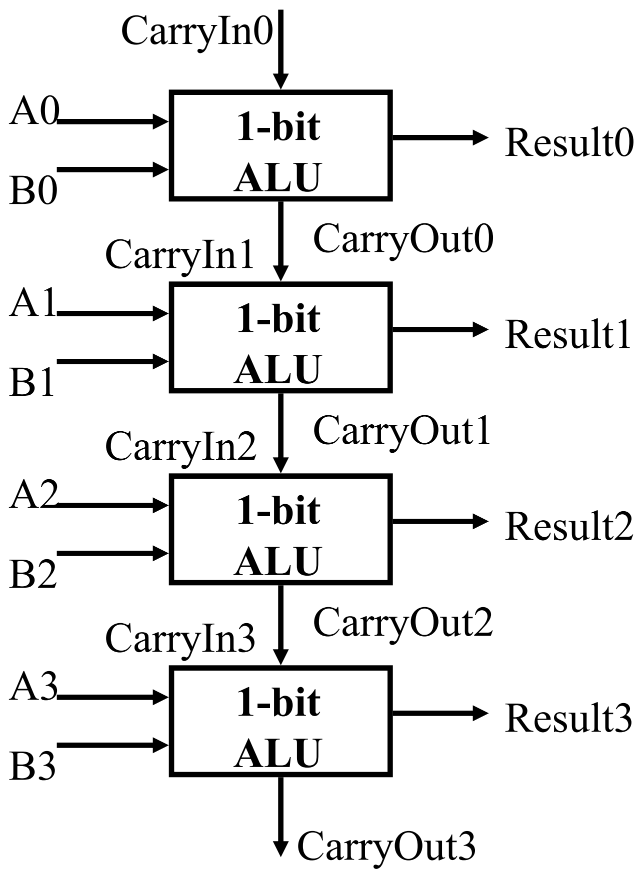



# Subtract

- A - B = A + (– B)
  - form two complement by invert and add one



# Ripple-carry adder



# Problem : Slow

- Is a 32-bit ALU as fast as a 1-bit ALU?
  - Delay = 32x CP(Fast adder) + XOR

- Is there more than one way to do addition?
  - two extremes: ripple carry and sum-of-products
  - Flatten expressions to two levels

Can you see the ripple? How could you get rid of it?

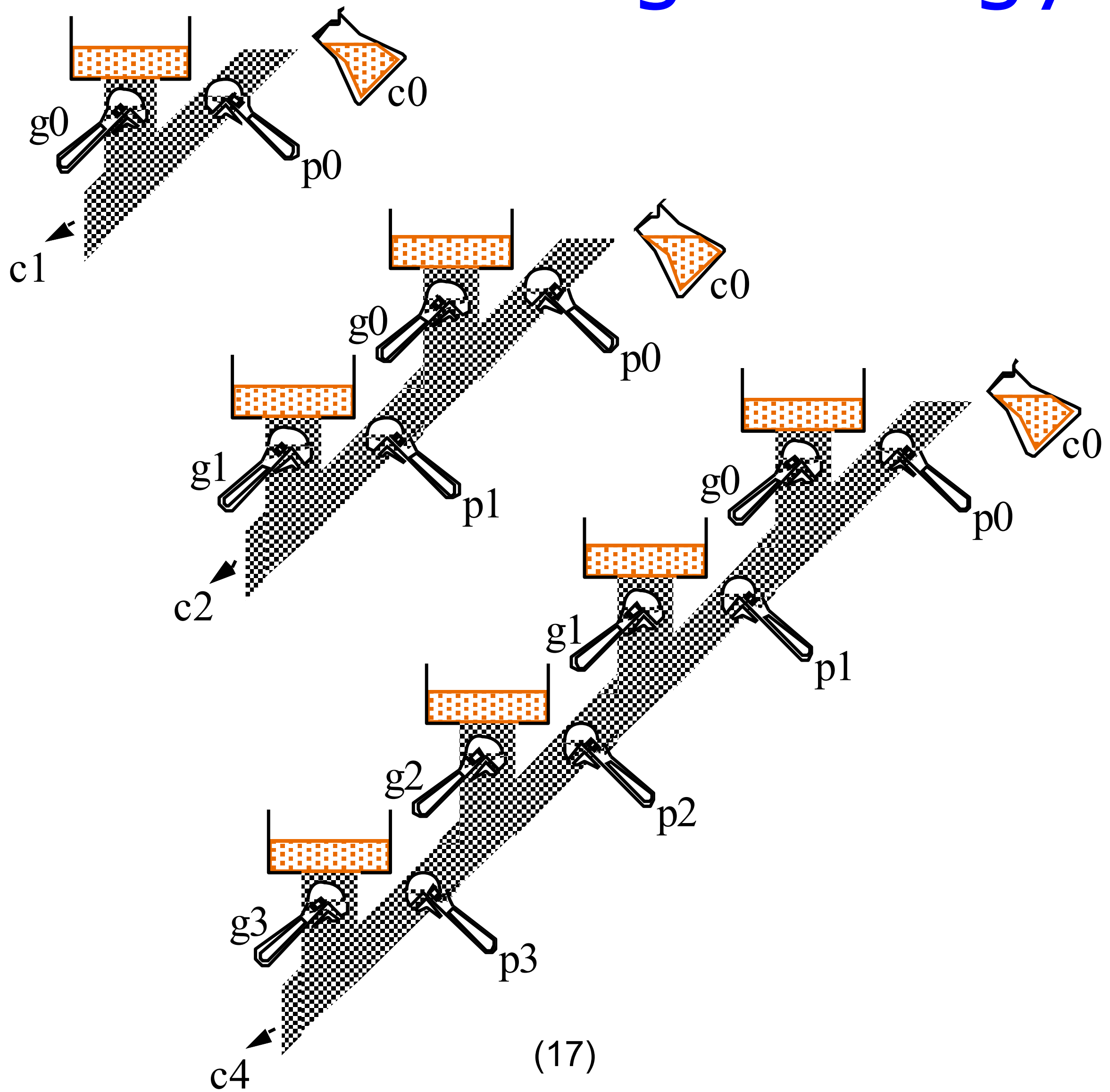
$c_1 = b_0c_0 + a_0c_0 + a_0b_0$
$c_2 = b_1c_1 + a_1c_1 + a_1b_1$
$c_3 = b_2c_2 + a_2c_2 + a_2b_2$
$c_4 = b_3c_3 + a_3c_3 + a_3b_3$

$c_2 = b_1(b_0c_0 + a_0c_0 + a_0b_0) + a_1(b_0c_0 + a_0c_0 + a_0b_0) + a_1b_1$
$c_2 = b_1b_0c_0 + b_1a_0c_0 + b_1a_0b_0 + a_1b_0c_0 + a_1a_0c_0 + a_1a_0b_0 + a_1b_1$
$c_3 = ?$
$c_{31} = ?$ Not feasible! Why? Exponential fanin

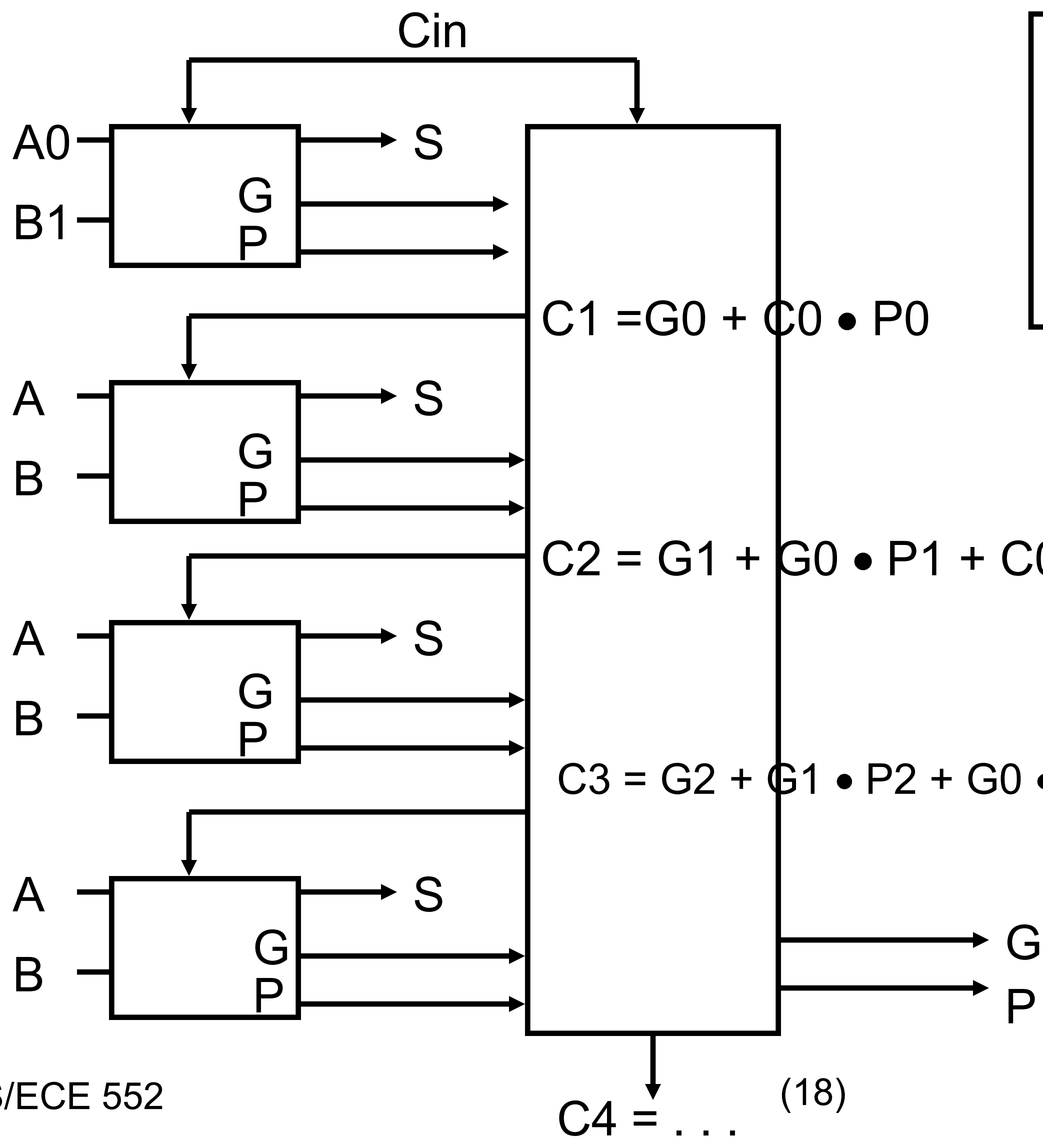
# Carry look-ahead

- An approach in-between our two extremes
- Motivation:
  - If we didn't know the value of carry-in, what could we do?
  - When would we always generate a carry?
    - $g_i = a_i b_i$
  - When would we propagate the carry?
    - $p_i = a_i + b_i$
- Did we get rid of the ripple?

# CLA: Plumbing Analogy



# Carry-lookahead adder



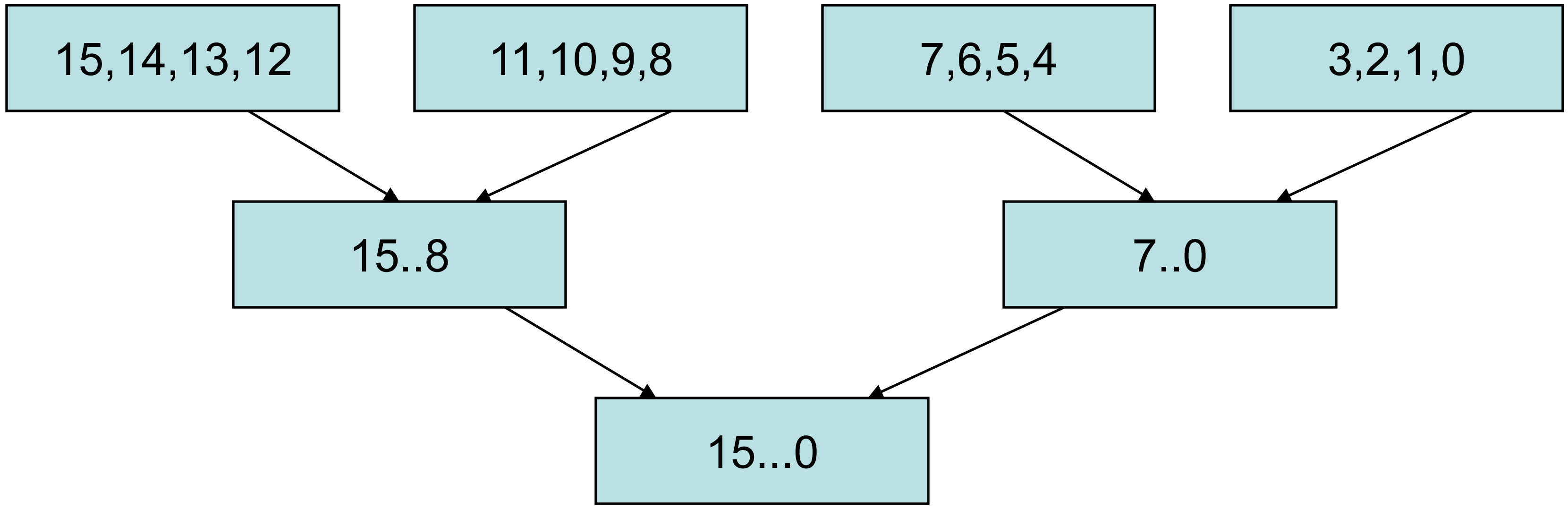

| A | B | C-out | |
|---|---|---|---|
| 0 | 0 | 0 | "kill" |
| 0 | 1 | C-in | "propagate" |
| 1 | 0 | C-in | "propagate" |
| 1 | 1 | 1 | "generate" |

**G = A and B**
**P = A xor B**

# Carry-Lookahead Adder

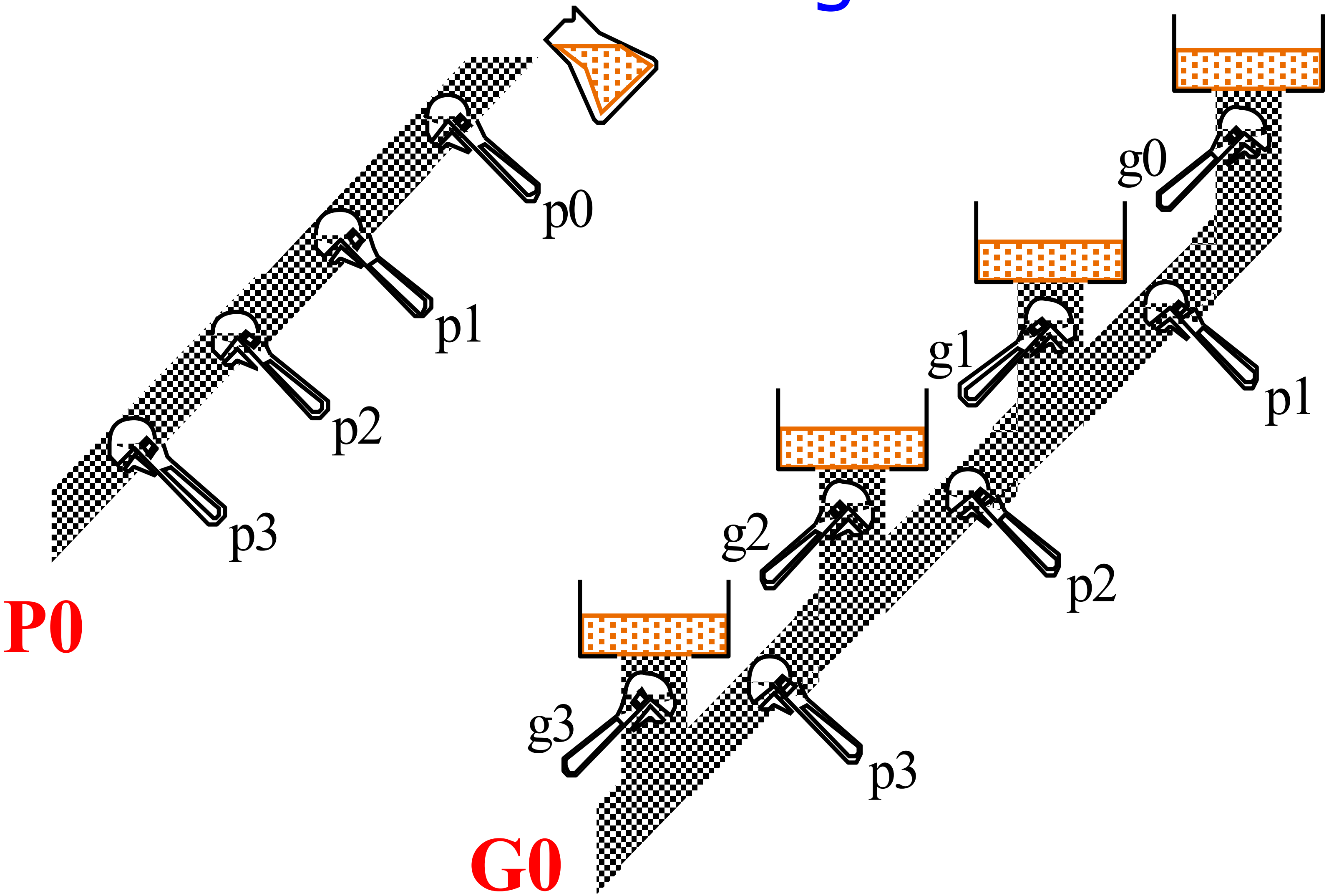
- Waitaminute!
  - Nothing has changed
  - Fanin problems if you flatten!
    - Linear fanin, not exponential
  - Ripple problem if you don't!
- Enables divide-and-conquer
- Figure out Generate and Propagate for 4-bits <u>together</u>
- <u>Compute hierarchically</u>

# Carry Lookahead adder

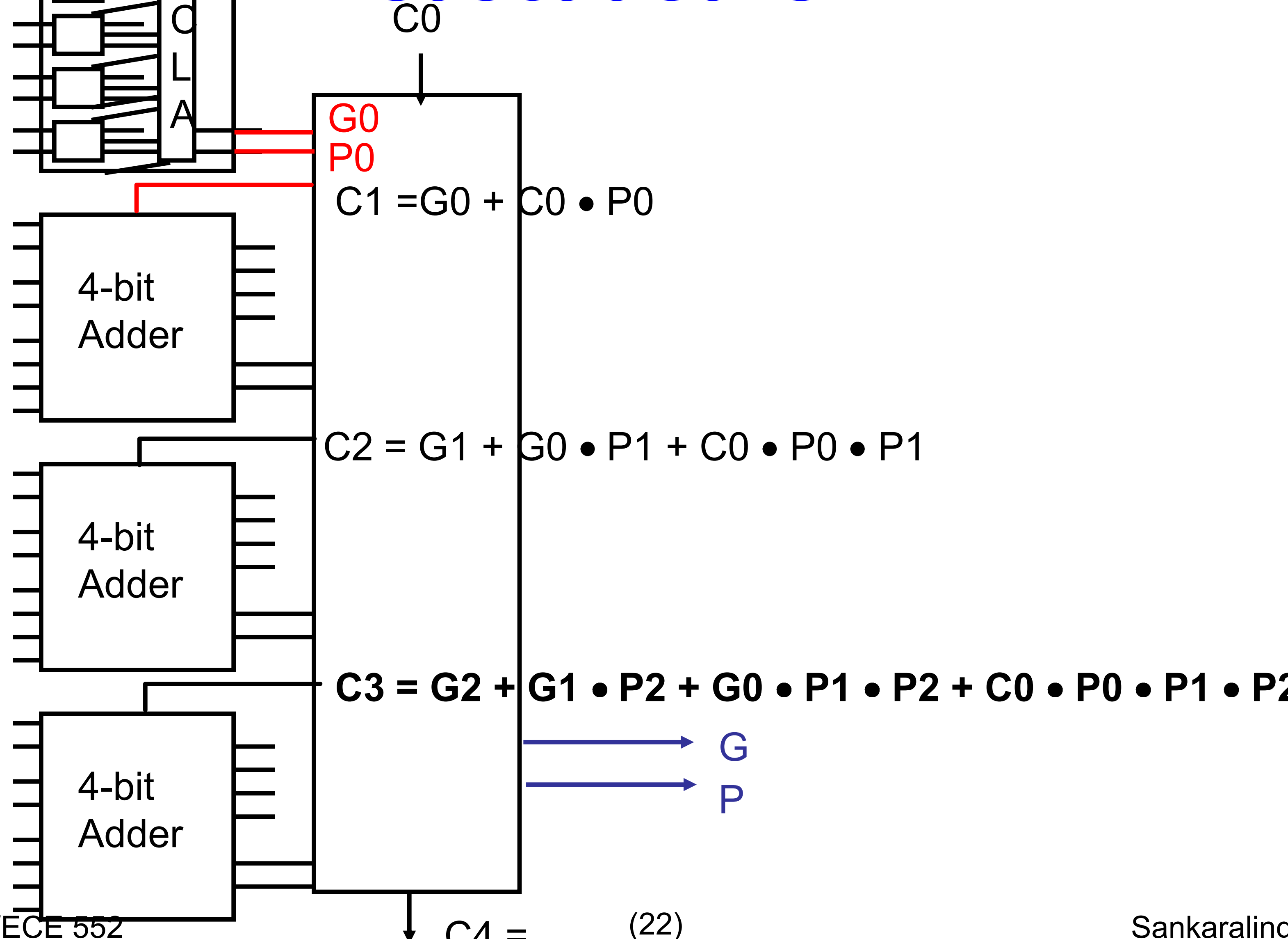



- Height of tree = O(lg(n))
- 32 bit addition : k*lg(32) = k*5

# Block level signals



# Cascaded CLA



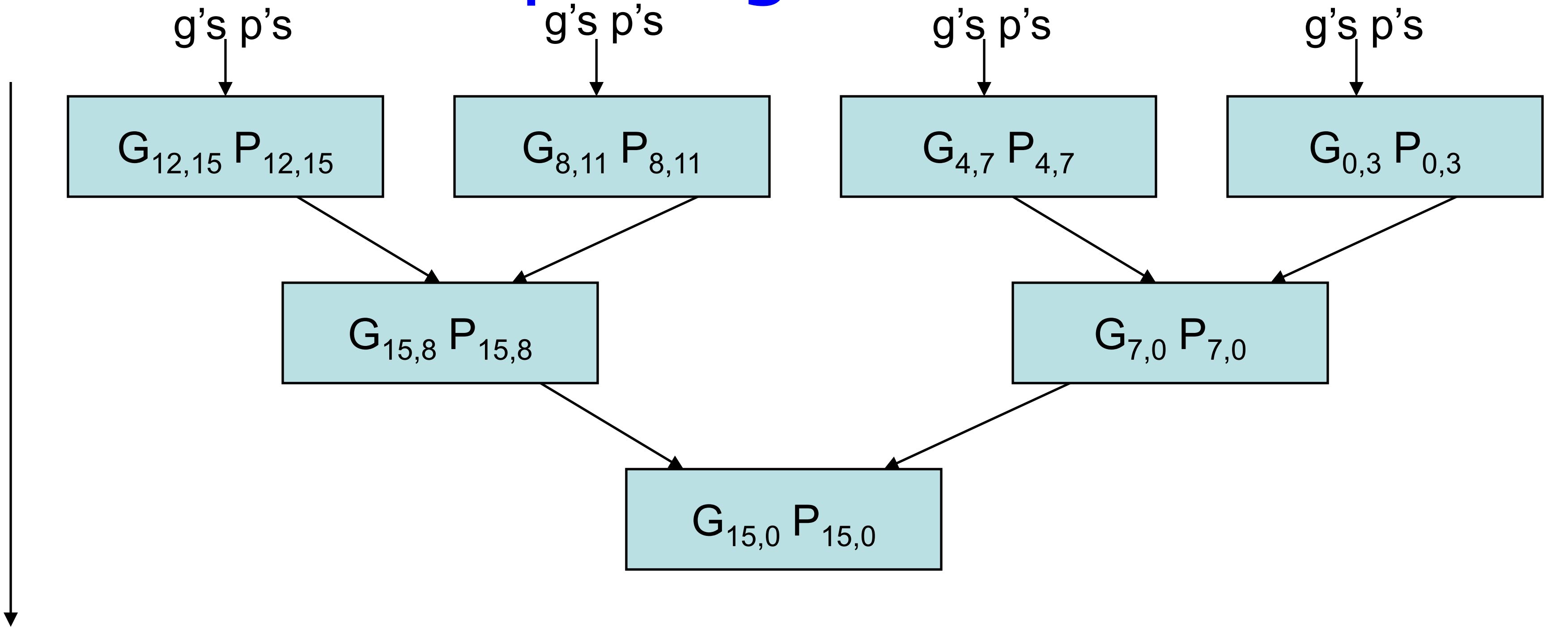
# Carry-Lookahead Adder

- Hierarchy
  - $G_{i,k} = G_{j+1,k} + P_{j+1,k} * G_{i,j}$ (assume $i < j+1 < k$ )
  - $P_{i,k} = P_{i,j} * P_{j+1,k}$
  - $G_{0,7} = G_{4,7} + P_{4,7} * G_{0,3}$
  - $P_{0,7} = P_{0,3} * P_{4,7}$

  - $G_{8,15} = G_{12,15} + P_{12,15} * G_{8,11}$
  - $P_{8,15} = P_{8,11} * P_{12,15}$

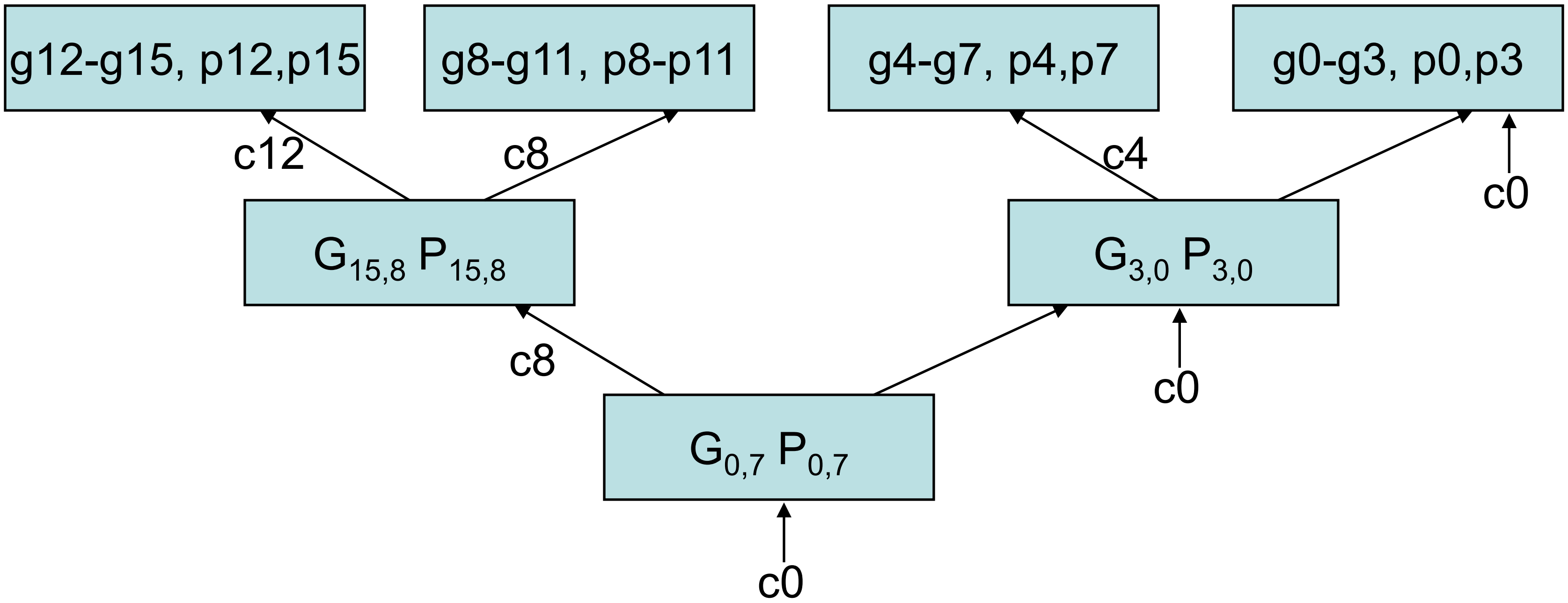
  - $G_{0,15} = G_{8,15} + P_{8,15} * G_{0,7}$
  - $P_{0,15} = P_{0,7} * P_{8,15}$

# Computing G's and Ps




- Parallel algorithm in hardware

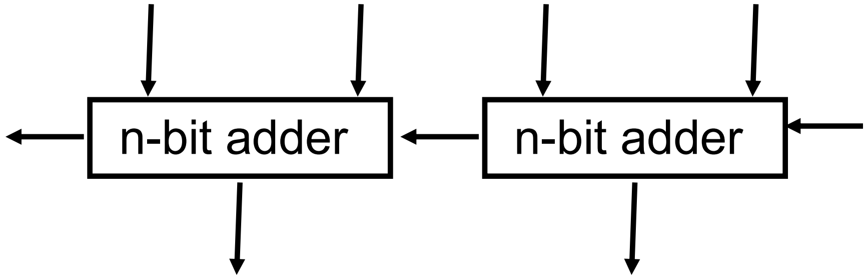
# Computing C's



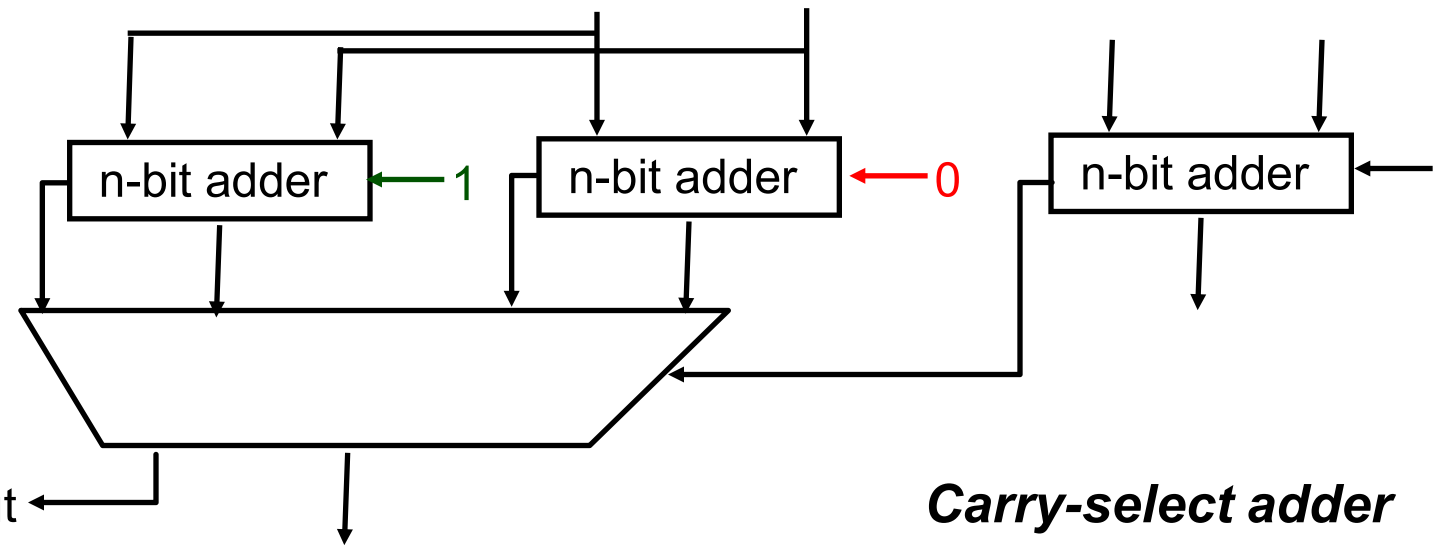

- Different tree.
- Note propagation order
- Worth spending some time to think about the intricacies

# Carry-selection: Guess

CP(2n) = 2*CP(n)




CP(2n) = CP(n) + CP(mux)




***Carry-select adder***

# Overflow

| Decimal | Binary | Decimal | 2's Complement |
|---|---|---|---|
| 0 | 0000 | 0 | 0000 |
| 1 | 0001 | -1 | 1111 |
| 2 | 0010 | -2 | 1110 |
| 3 | 0011 | -3 | 1101 |
| 4 | 0100 | -4 | 1100 |
| 5 | 0101 | -5 | 1011 |
| 6 | 0110 | -6 | 1010 |
| 7 | 0111 | -7 | 1001 |
| | | -8 | 1000 |

- **Examples: 7 + 3 = 10 but ...**
  **- 4 - 5 = - 9 but ...**

| | | | | | |
|---|---|---|---|---|---|
| | 0 | 1 | 1 | 1 | |
| | | 0 | 1 | 1 | 1 | 7 |
| + | | 0 | 0 | 1 | 1 | 3 |
| | | 1 | 0 | 1 | 0 | **– 6** |

| | | | | | |
|---|---|---|---|---|---|
| | 1 | | | | |
| | | 1 | 1 | 0 | 0 | – 4 |
| + | | 1 | 0 | 1 | 1 | – 5 |
| | | 0 | 1 | 1 | 1 | **7** |

# Overflow

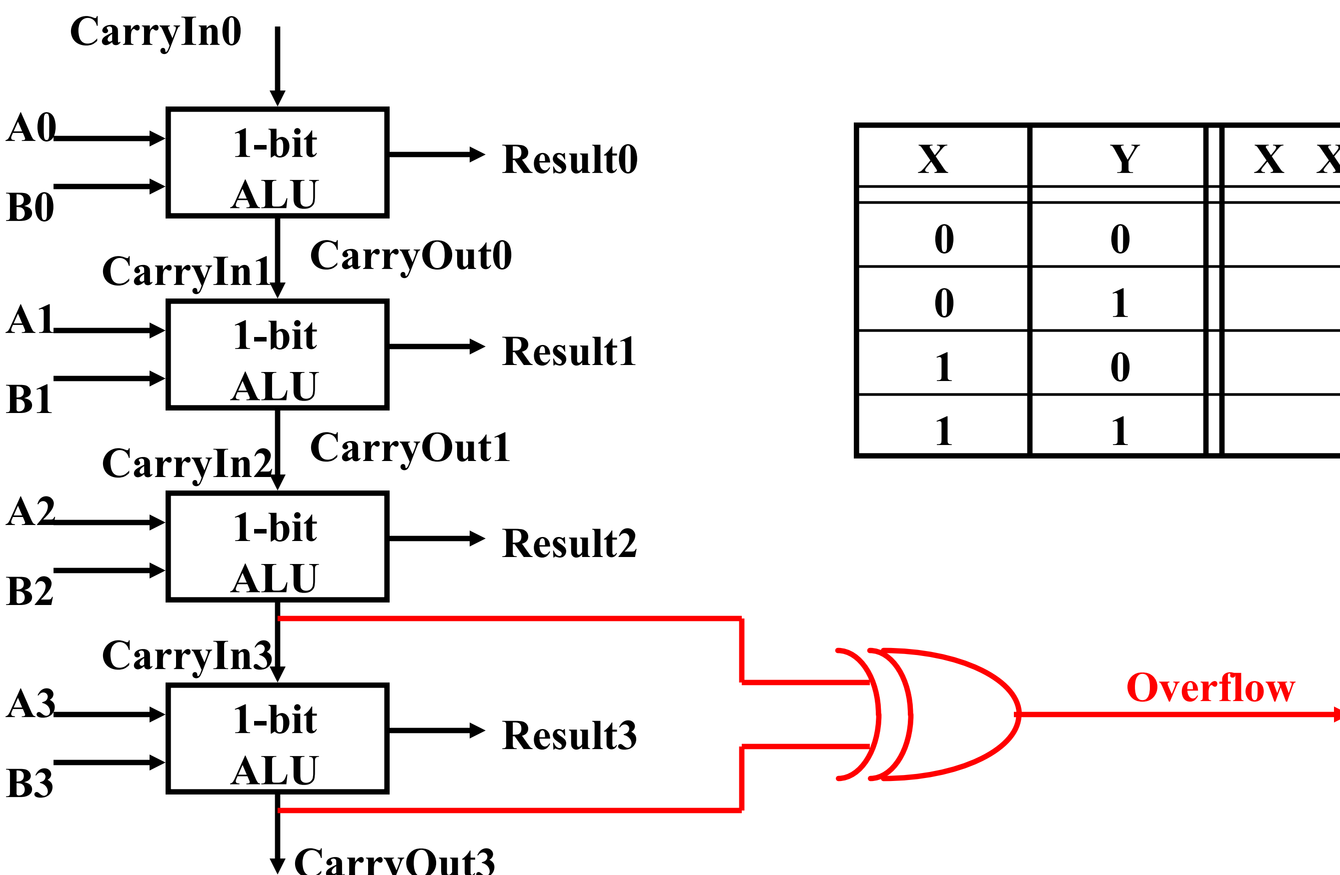
- Overflow: the result is too large (or too small) to represent properly
  - Example: - 8 < = 4-bit binary number <= 7
- When adding operands with different signs, overflow cannot occur!
- Overflow occurs when adding:
  - 2 positive numbers and the sum is negative
  - 2 negative numbers and the sum is positive
- On your own: Prove you can detect overflow by:
  - Carry into MSB ⊕ Carry out of MSB

| | | | | | | |
|---|---|---|---|---|---|---|
| | 0 | 1 | 1 | 1 | | |
| | | 0 | 1 | 1 | 1 | 7 |
| + | | 0 | 0 | 1 | 1 | 3 |
| | | 1 | 0 | 1 | 0 | – 6 |

| | | | | | | |
|---|---|---|---|---|---|---|
| | 1 | 0 | | | | |
| | | 1 | 1 | 0 | 0 | –4 |
| + | | 1 | 0 | 1 | 1 | – 5 |
| | | 0 | 1 | 1 | 1 | 7 |

# Overflow detection

- Carry into MSB $\oplus$ Carry out of MSB
  - For N-bit ALU: Overflow = CarryIn[N - 1] XOR CarryOut[N - 1]




| X | Y | X XOR Y |
|---|---|---|
| 0 | 0 | 0 |
| 0 | 1 | 1 |
| 1 | 0 | 1 |
| 1 | 1 | 0 |

# Negative, Zero

- Required for conditional branches
- Zero
  - How?
  - NOR all 32 bits
  - Avoid $33^{rd}$ bit (carry out)
- Negative may be required on overflow
  - If (a<b) jump : jump taken if a-b is negative
- Tempting to consider MSB
  - E.g. if (-5 < 4) branch
  - Branch should be taken, but (-5-4) computation results in overflow… so MSB is 0
  - E.g. if (7 < -3) branch
  - Branch should not be taken but (7- (-3)) results in overflow… so MSB is 1.

# Shift

- E.g., Shift left logical for d<7:0> and shamt<2:0>
  - Using 2-1 muxes called Mux(select, in0, in1)
  - stage0<7:0> = Mux(shamt<0>,d<7:0>, 0 || d<7:1>)
  - stage1<7:0> = Mux(shamt<1>, stage0<7:0>, 00 || stage0<6:2>)
  - dout<7:0) = Mux(shamt<2>, stage1<7:0>, 0000 || stage1<3:0>)
- Other operations
  - Right shift
  - Arithmetic shifts
  - Rotate

# Barrel Shifter