Theorems

1.1 De Morgan's law relates all three basic operations: $(A \cup B)^c = A^c \cap B^c$.

1.2 For an event space $B = \{B_1, B_2, \ldots\}$ and any event A in the sample space, let $C_i = A \cap B_i$. For $i \neq j$, the events C_i and C_j are mutually exclusive and $A = C_1 \cup C_2 \cup \ldots$.

1.3 For mutually exclusive events A1 and A2, $P[A_1 \cup A_2] = P[A_1] + P[A_2]$.

1.4 If $A = A_1 \cup A_2 \cup \ldots \cup A_m$ and $A_i \cap A_j = \phi$ for $i \neq j$, then $P[A_1] = \sum_{i=1}^{m} P[A_i]$.

1.5 The probability of an event $B = \{s_1, s_2, \ldots, s_m\}$ is the sum of the probabilities of the outcomes contained in the event: $P[B] = \sum_{i=1}^{m} P[\{s_i\}]$.

1.6 For an experiment with sample space $S = \{s_1, \ldots, s_n\}$ in which each outcome s_i is equally likely, $P[s_i] = 1/n 1 \leq i \leq n$.

1.7 The probability measure P[.] satisfies (a) $P[\phi] = 0$. (b) $P[A^c] = 1 - P[A]$. (c) For any A and B (not necessarily disjoint), $P[A \cup B] = P[A] + P[B] - P[A \cap B]$. (d) If $A \subset B$, then $P[A] \leq P[B]$.

1.8 For any event A, and event space $\{B_1, B_2, \ldots, B_m\}$, $P[A] = \sum_{i=1}^{m} P[A \cap B_i]$.

1.9 A conditional probability measure $P[A|B]$ has the following properties that correspond to the axioms of probability. Axiom 1: $P[A|B] \geq 0$. Axiom 2: $P[B|B] = 1$. Axiom 3: If $A = A_1 \cup A_2 \cup \ldots$ with $A_i \cap A_j = \phi$ for $i \neq j$, then $P[A|B] = P[A_1|B] + P[A_2|B] + \ldots$

1.10 Law of Total Probability: For an event space $\{B_1, B_2, \ldots, B_m\}$ with $P[B_i] > 0$ for all i, $P[A] = \sum_{i=1}^{m} P[A|B_i]P[B_i]$.

1.11 Bayes' Theorem: $P[B|A] = \frac{P[A|B]P[B]}{P[A]}$.

1.12 The number of k-permutations of n distinguishable objects is $(n)_k = n(n-1)(n-2)\ldots(n-k+1) = \frac{n!}{(n-k)!}$.

1.13 The number of ways to choose k objects out of n distinguishable objects is $\binom{n}{k} = (n)_k/k! = \frac{n!}{k!(n-k)!}$.

1.14 Given m distinguishable objects, there are m^n ways to choose with replacement an ordered sample of n objects.

1.15 For n repetitions of a subexperiment with sample space $S = \{s_0, \ldots, s_{m-1}\}$, there are m^n possible observation sequences.

1.16 The number of observation sequences for n subexperiments with sample space $S = \{0, 1\}$ with 0 appearing n_0 times and 1 appearing $n_1 = n - n_0$ times is $\binom{n}{n_1}$.

1.17 For n repetitions of a subexperiment with sample space $S = \{s_0, \ldots, s_{m-1}\}$, the number of length $n = n_0 + \ldots + n_{m-1}$ observation sequences with s_i appearing n_i times is $\binom{n}{n_0, \ldots, n_{m-1}} = \frac{n!}{n_0! n_1! \ldots n_{m-1}!}$.

1.18 The probability of n_0 failures and n_1 successes in $n = n_0 + n_1$ independent trails is $P[S_{n_0,n_1}] = \binom{n}{n_1}(1-p)^{n-n_1}p^{n_1} = \binom{n}{n_0}(1-p)^{n_0}p^{n-n_0}$.

1.19 A subexperiment has sample space $S = \{s_0, \ldots, s_{m-1}\}$ with $P[s_i] = p_i$. For $n = n_0 + \ldots + n_{m-1}$ independent trails, the probability if n_i occurrences of s_i, $i = 0, 1, \ldots, m-1$, is $P[S_{n_0,\ldots,n_{m-1}}] = \binom{n}{n_0, \ldots, n_{m-1}} p_0^{n_0} \ldots p_0^{n_{m-1}}$.

2.1 For a discrete random variable X with PMF $P_X(x)$ and range S_X: (a) For any x, $P_X \geq 0$. (b) $\sum_{x \epsilon S_X} P_X(x) = 1$. (c) For any event $B \subset S_X$, the probability that X is in the set B is $P[B] = \sum_{x \epsilon B} P_X(x)$.

2.2 For any discrete random variable X with range $S_X = \{x_1, x_2, \ldots\}$ satisfying $x_1 \leq x_2 \leq \ldots$, (a) Going from left to right on the x-axis, $F_X(x)$ starts at zero and ends at one. (b) The CDF never decreases as it goes from left to right. (c) For a discrete random Variable X, there is a jump (discontinuity) at each value of $x_i \epsilon S_X$. The height of the jump at x_i is $P_X(x_i)$. (d) Between jumps, the graph of the CDF of the discrete random variable X is a horizontal line.

2.3 For all $b \geq a$, $F_X(b) - F_X(a) = P[a < X \leq b]$.

2.4 The Bernoulli (p) random variable X has expected value $E[X] = p$.

2.5 The geometric (p) random variable X has expected value $E[X] = 1/p$.

2.6 The Poisson (α) random variable in Definition 2.10 has expected value $E[X] = \alpha$.

2.7 (a) For the binomial (n, p) random variable X of Definition 2.7, $E[X] = np$. (b) For the Pascal (k, p) random variable X of Definition 2.8, $E[X] = k/p$. (c) For the discrete uniform (k, l) random variable X of Definition 2.9, $E[X] = (k + l)/2$.

2.8 Perform n Bernoulli trails. In each trail, let the probability of success be α/n, where $\alpha > 0$ is a constant and $n > \alpha$. Let the random variable K_n be the number of successes in the n trails. As $n \to \infty$, $P_{K_n}(k)$ converges to the PMF of a Poisson (α) random variable.

2.9 For a discrete random variable X, the PMF of $Y = g(X)$ is $P_Y(y) = \sum_{x:g(x)=y} P_X(x)$.

2.10 Given a random variable X with PMF $P_X(x)$ and the derived random variable $Y = g(X)$, the expected value of Y is $E[Y] = \mu_Y = \sum_{x \epsilon S_X} g(x) P_X(x)$.

2.11 For any random variable X, $E[X - \mu_X] = 0$.

2.12 For any random variable X, $E[aX + b] = aE[X] + b$.

2.13 $Var[X] = E[X^2] - \mu_X^2 = E[X^2] - (E[X])^2$.

2.14 $Var[aX + b] = a^2 Var[X]$.

2.15 (a) If X is Bernoulli (p), then $Var[X] = p(1-p)$. (b) if X is geometric, then $Var[X] = (1-p)/p^2$. (c) If X is binomial (n, p), then $Var[X] = np(1-p)$. (d) If X is Pascal (k, p), then $Var[X] = k(1=p)/p^2$. (e) If X is Poisson (α), the $Var[X] = \alpha$. (f) If X is discrete uniform (k, l), then $Var[X] = (l-k)(l-k+2)/12$.

2.16 A random variable X resulting from an experiment with event space $B_1, \ldots, B_m$ has PMF $P_X(x) = \sum_{i=1}^{m} P_{X|Bi}(x)P[B_i]$.

2.17 $P_{X|B}(x) = \begin{cases} \frac{P_X(x)}{P[L]} & x = 5, 6, 7, 8, \\ 0 & otherwise. \end{cases}$

2.18 (a) For any $x \epsilon B$, $P_{X|B}(x) \geq 0$. (b) $\sum_{x \epsilon B} P_{X|B}(x) = 1$. (c) For any event $C \subset B$, $P[C|B]$, the conditional probability that X is in the set C, is $P[C|B] = \sum_{x \epsilon C} P_{X|B}(x)$.

2.19 For a random variable X resulting from an experiment with event space $B_1, \ldots, B_m$, $E[X] = \sum_{i=1}^{m} E[X|B_i]P[B_i]$.

2.20 The conditional expected value of $Y = g(X)$ given condition B is $E[Y|B] = E[g(X)|B] = \sum_{x \epsilon B} g(x)|P_{X|B}(x)$.

3.1 For any random variable X, (a) $F_X(-\infty) = 0$, (b) $F_X(\infty) = 1$, (c) $P[x_1 < X \leq x_2] = F_X(x_2) - F_X(x_1)$.

3.2 For a continuous random variable X with PDF $f_X(x)$, (a) $f_X(x) \geq 0$ for all x, (b) $F_X(x) = \int_{-\infty}^{x} f_X(u)du$, (c) $\int_{-\infty}^{\infty} f_X(x)dx = 1$.

3.3 $P[x_1 < X \leq x_2] = \int_{x_1}^{x_2} f_X(x)dx$.

3.4 The expected value of a function, g(X), of random variable X is $E[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x)dx$.

3.5 For any random variable X, (a) $E[X - \mu_X] = 0$, (b) $E[aX + b] = aE[X] + b$, (c) $Var[X] = E[X^2] - \mu_X^2$, (d) $Var[aX + b] = a^2 Var[X]$.

3.6 If X is a uniform (a, b) random variable, (a) The CDF of X is $F_X(x) = \begin{cases} 0 & x \leq a \\ \frac{x-a}{b-a} & a < x \leq b \\ 1 & x > b. \end{cases}$ (b) The expected value of X is $E[X] = (b + a)/2$. (c) The variance of X is $Var[X] = (b - a)^2/12$.

3.7 Let X be a uniform (a, b) random variable, where a and b are both integers. Let $K = \lceil X \rceil$. Then K is a discrete uniform (a + 1, b) random variable.

3.8 If X is an exponential (λ) random variable, (a) $F_X(x) = \begin{cases} 1 - e^{-\lambda x} & x \geq 0, \\ 0 & otherwise, \end{cases}$ (b) $E[X] = 1/\lambda$, (c) $Var[X] = 1/\lambda^2$.

3.9 If X is an exponential (λ) random variable, then $K = \lceil X \rceil$ is a geometric (p) random variable with $p = 1 - e^{-\lambda}$.

3.10 If X is an Erlang (n, λ) random variable, then $E[X] = n/\lambda$, $Var[X] = n/\lambda^2$.

3.11 Let K_α denote a Poisson (α) random variable. For any $x > 0$, the CDF of an Erlang (n, λ) random variable X satisfies $F_X(x) = 1 - F_{K_{\lambda x}}(n-1) = 1 - \sum_{k=0}^{n-1} \frac{(\lambda x)^k e^{-\lambda x}}{k!}$.

Definitions

1.1 Outcome: An outcome of an experiment is any possible observation of that experiment.

1.2 Sample space: The sample space of an experiment is the finest-grain, mutually exclusive, collectively exhaustive set of all possible outcomes.

1.3 Event: An event is a set of outcomes of an experiment.

1.4 Event Space: An event space is a collectively exhaustive, mutually exclusive set of events.

1.5 Axioms of Probability: A probability measure P[.] is a function that maps events in the sample space to real numbers such that Axiom 1: For any event A, $P[A] \geq 0$. Axiom 2: $P[S] = 1$. Axiom 3: For any countable collection $A_1, A_2, \ldots$ of mutually exclusive events, $P[A_1 \cup A_2 \cup \ldots] = P[A_1] + P[A_2] + \ldots$

1.6 Conditional Probabilty: The conditional probability of the even A given the occurrence of the event B is $P[A|B] = \frac{P[AB]}{P[B]}$.

1.7 Two independent Events: Events A and B are independent if and only if $P[AB] = P[A]P[B]$. When events A and B have nonzero probabilities, the following formulas are equivalent to the definition of independent events: $P[A|B] = P[A]$, $P[B|A] = P[B]$.

1.8 3 Independent Events: A_1, A_2, and A_3 are independent if and only if (a) A_1 and A_2 are independent, (b) A_2 and A_3 are independent, (c) A_1 and A_3 are independent, (d) $P[A_1 \cap A_2 \cap A_3] = P[A_1]P[A_2]P[A_3]$.

1.9 More than Two Independent Events: If $n \geq 3$, the sets $A_1, A_2, \ldots, A_n$ are independent if and only if (a) every set of n-1 sets from $A_1, A_2, \ldots, A_n$ are independent, (b) $P[A_1 \cap A_2 \cap \ldots \cap A_n] = P[A_1]P[A_2]\ldots P[A_n]$.

1.10 Fundamental Principle of Counting: If subexperiment A has n possible outcomes, and subexperiment B has k possible outcomes, then there are nk possible outcomes when you perform both experiments.

1.11 n choose k: For an integer $n \geq 0$, we define $\binom{n}{k} = \begin{cases} \frac{n!}{k!(n-k)!} & k = 0, 1, \ldots, n, \\ 0 & otherwise. \end{cases}$

1.10 Multinomial Coefficient: For an integer $n \geq 0$, we define $\binom{n}{n_0, \ldots, n_{m-1}} = \begin{cases} \frac{n!}{n_0! n_1! \ldots n_{m-1}!} & n_0 + \cdots + n_{m-1} = n; n_i \epsilon\, 0, 1, \ldots, n\}, i = 0, 1, \ldots, m-1, \\ 0 & otherwise. \end{cases}$

2.1 Random Variable: A random variable consists of an experiment with a probability measure P[.] defined on a sample space S and a function that assigns a real number to each outcome in the sample space of the experiment.

2.2 Discrete Random Variable: X is a discrete random variable if the range of X is a countable set $S_X = \{x_1, x_2, \ldots\}$.

2.3 Finite Random Variable: X is a finite random variable iff the range is a finite set $S_X = \{x_1, x_2, \ldots, x_n\}$.

2.4 Probability Mass Function (PMF): The probability mass function (PMF) of the discrete random variable X is $P_X(x) = P[X=x]$.

2.5 Bernoulli (p) Random Variable: X is a Bernoulli (p) random variable if the PMF of X has the form $P_X(x) = \begin{cases} 1-p & x = 0 \\ p & x = 1 \\ 0 & otherwise, \end{cases}$ where the parameter p is in the range $0 < p < 1$.

2.6 Geometric (p) Random Variable: X is a geometric (p) random variable if the PMF of X has the form $P_X(x) = \begin{cases} p(1-p)^{x-1} & x = 1, 2, \ldots \\ 0 & otherwise, \end{cases}$ where the parameter p is in the range $0 < p < 1$.

2.7 Binomial (n, p) Random Variable: X is a binomial (n, p) random variable if the PMF of X has the form $P_X(x) = \binom{n}{x}p^x(1-p)^{n-x}$ where $0 < p < 1$ and n is an integer such that $n \geq 1$.

2.8 Pascal (k, p) Random Variable: X is a Pascal (k, p) random variable if the PMF of X has the form $P_X(x) = \binom{x-1}{k-1}p^k(1-p)^{x-k}$ where $0 < p < 1$ and k is an integer such that $k \geq 1$.

2.9 Discrete Uniform (k, l) Random Variable: X is a discrete uniform (k, l) random variable if the PMF of X has the form $P_X(x) = \begin{cases} \frac{1}{l-k+1} & x = k, k+1, k+2, \ldots, l \\ 0 & otherwise, \end{cases}$ where the parameters k and *l* are integers such that $k < l$.

2.10 Poisson (α) Random Variable: X is a Poisson (α) random variable if the PMF of X has the form $P_X(x) = \begin{cases} \frac{\alpha^x e^{-\alpha}}{x!} & x = 0, 1, 2, \ldots \\ 0 & otherwise, \end{cases}$ where the parameter α is in the range $\alpha > 0$. A Poisson model often specifies an average rate, λ arrivals per second and a time interval, T seconds. In this time interval, the number of arrivals X has a Poisson PMF with $\alpha = \lambda T$.

2.11 Cumulative Distribution Function (CDF): The cumulative distribution function (CDF) of random variable X is $F_X(x) = P[X \leq x]$.

2.12 Mode: A mode of random variable X is a number x_{mod} satisfying $P_X(x_{mod}) \geq P_X(x)$ for all x.

2.13 Median: A median, x_{med}, of random variable X is a number that satisfies $P[X < x_{med}] = P[X > x_{med}]$.

2.14 Expected Value: The expected value of X is $E[X] = \mu_X = \sum_{x \epsilon S_X} x P_X(x)$.

2.15 Derived Random Variable: Each sample value y of a derived random variable Y is a mathematical function g(x) of a sample value x of another random variable X. We adopt the notation $Y = g(X)$ to describe the relationship of the two random variables.

2.16 Variance: The variance of random variable X is $Var[X] = E[(X - \mu_X)^2]$.

2.17 Standard Deviation: The standard deviation of random variable X is $\sigma_X = \sqrt{Var[X]}$.

2.18 Moments: For random variable X: (a) The nth moment is $E[X^n]$. (b) The nth central moment is $E[(X - \mu_X)^n]$.

2.19 Conditional PMF: Given the event B, with $P[B] > 0$, the conditional probability mass function of X is $P_{X|B}(x) = P[X = x|B]$.

2.20 Conditional Expected Value: The conditional expected value of random variable X given conditional B is $E[X|B] = \mu_{X|B} = \sum_{x \epsilon B} x P_{X|B}(x)$.

3.1 Cumulative Distribution Function (CDF): The cumulative distribution function (CDF) of random variable X is $F_X(x) = P[X \leq x]$.

3.2 Continuous Random Variable: X is a continuous random variable if the CDF $F_X(x)$ is a continuous function.

3.3 Probability Density Function (PDF): The probability density function (PDF) of a continuous random variable X is $f_X(x) = \frac{dF_X(x)}{dx}$.

3.4 Expected Value: The expected value of a continuous random variable X is $E[X] = \int_{-\infty}^{\infty} x f_X(x) dx$.

3.5 Uniform Random Variable: X is a uniform (a, b) random variable if the PDF of X is $f_X(x) = \begin{cases} \frac{1}{b-a} & a \leq x < b \\ 0 & otherwise, \end{cases}$ where the two parameters are $b > a$.

3.6 Exponential Random Variable: X is an exponential (λ) random variable if the PDF of X is $f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0, \\ 0 & otherwise, \end{cases}$ where the parameter $\lambda > 0$.

3.7 Erlang Random Variable: X is an Erlang (n, λ) random variable if the PDF of X is $f_X(x) = \begin{cases} \lambda^n x^{n-1} e^{-\lambda x} & x \geq 0 \\ 0 & otherwise, \end{cases}$ where the parameter $\lambda > 0$, and the parameter $n \geq 1$ is an integer.

<u>Math Facts</u>

B.1 Half Angle Formulas: (a) $\cos(A + B) = \cos A \cos B - \sin A \sin B$, (b) $\sin(A + B) = \sin A \cos B + \cos A \sin B$, (c) $\cos 2A = \cos^2 A - \sin^2 A$, (d) $\sin 2A = 2 \sin A \cos A$.

B.2 Products of Sinusoids: (a) $\sin A \sin B = .5[\cos(A - B) - \cos(A + B)]$, (b) $\cos A \cos B = .5[\cos(A - B) + \cos(A + B)]$ (c) $\sin A \cos B = .5[\sin(A + B) + \sin(A - B)]$.

B.4 Finite Geometric Series: The finite geometric series is $\sum_{i=0}^{n} q^i = 1 + q + q^2 + \ldots + q^n = \frac{1-q^{n+1}}{1-q}$.

B.5 Infinite Geometric Series: When $|q| < 1$, $\sum_{i=0}^{\infty} q^i = \lim_{n\to\infty} \sum_{i=0}^{n} q^i = \frac{1}{1-q}$.

B.6 $\sum_{i=1}^{n} i q^i = \frac{q(1-q^n[1+n(1-q)])}{(1-q)^2}$.

B.7 If $|q| < 1$, $\sum_{i=1}^{\infty} i q^i = \frac{q}{(1-q)^2}$.

B.8 $\sum_{j=1}^{n} j = \frac{n(n+1)}{2}$.

B.9 $\sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}$.

B.10 Integration by Parts: The integration by parts formula is $\int_a^b u\,dv = uv\big|_a^b - \int_a^b v\,du$.