Solution to Problem # 5, Problem Set #1

1–5: A problem that might floor you. For any real number x, define $\lfloor x \rfloor$ to be the *floor* of x, namely the greatest integer which is less than or equal to x. Define $q(n) \equiv \left\lfloor \frac{n}{\lfloor \sqrt{n} \rfloor} \right\rfloor$ for $n = 1, 2, 3, \ldots$. Determine all positive integers n for which $q(n) > q(n+1)$.

SOLUTION: Let x^2 be the greatest perfect square less than or equal to n, so $x^2 \leq n < (x+1)^2$. We can write $n = x^2 + k$. Notice that

$$x^2 \leq n = x^2 + k < (x+1)^2 = x^2 + 2x + 1$$

implies that $0 \leq k \leq 2x$.

Now,

$$\begin{aligned} q(n) &= \left\lfloor \frac{n}{\lfloor \sqrt{n} \rfloor} \right\rfloor \\ &= \left\lfloor \frac{x^2 + k}{\lfloor \sqrt{x^2 + k} \rfloor} \right\rfloor \\ &= \left\lfloor \frac{x^2 + k}{x} \right\rfloor \\ &= x + \left\lfloor \frac{k}{x} \right\rfloor \\ &= \begin{cases} x & \text{if } 0 \leq k < x \\ x+1 & \text{if } x \leq k < 2x \\ x+2 & \text{if } k = 2x \end{cases} \end{aligned}$$

We see that $q(n)$ cannot exceed $q(n+1)$ unless $n = x^2 + 2x$. In this case,

$$q(x^2 + 2x) = x + 2 > q(x^2 + 2x + 1) = q((x+1)^2) = x + 1.$$

Thus, $q(n) > q(n+1)$ if and only if $k = 2x$, namely when n is a positive integer one less than a perfect square.

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