Math 340 — Problem Solving Seminar, Fall 2001, Problem Set #1

Please give each of these problems a try. We do not expect you to solve them all.

Problem 1: The banana boat problem. A person is 100 miles from her destination along a river. She has 300 bananas and a boat. The boat can hold the person and up to 100 bananas. The person would like to deliver as many bananas to the destination as possible. She may safely leave bananas at the edge of the river and load and unload bananas at no cost. However, when rowing the boat up or down the river, the person must eat one banana (from those in the boat) per mile rowed (either upstream or downstream). What is the maximum number of bananas that can be delivered to the destination?

Problem 2: A mathematics contest. Twenty-one boys and twenty-one girls took part in a mathematics contest. Each contestant solved at most six problems. For each boy-girl pairing, there exists a problem that both solved. Prove that there is at least one contest problem that at least three boys and at least three girls solved.

Problem 3: A funny function. A function f is defined for all positive integers and satisfies $f(1) = 2001$ and $\sum_{i=1}^{n} f(i) = n^2 f(n)$ for all $n > 1$. Calculate the exact value of $f(2001)$.

Problem 4: A triangle and two circles. Let ABC be an acute-angled triangle and let O be its circumcenter. (The circumcenter of a triangle is the center of the circle that passes through all its vertices.) The points B, C, and O lie on another circle S. The line AB intersects the circle S at B and at a different point P. The line AC intersects the circle S at C and at a different point Q. Prove that lines AO and PQ are perpendicular.

Problem 5: A problem that might floor you. For any real number x, define $\lfloor x \rfloor$ to be the *floor* of x, namely the greatest integer which is less than or equal to x. Define $q(n) \equiv \left\lfloor \frac{n}{\lfloor \sqrt{n} \rfloor} \right\rfloor$ for $n = 1, 2, 3, \ldots$. Determine all positive integers n for which $q(n) > q(n+1)$.

Problem 6: A three color theorem? If every point of the plane is colored one of three colors, do there necessarily exist two points of the same color exactly one unit apart?