Math 340–Problem Solving Seminar, Fall 2001, Problem Set 4

(1) Find all integers a, b, and c for which

$$(x-a)(x-10)+1=(x+b)(x+c) \text{ for all } x.$$

(2) Let A be a positive real number. What are the possible values of $\Sigma_{j=0}^{\infty} x_j^2$, given that x_0,x_1,... are positive numbers for which $\Sigma_{j=0}^{\infty} x_j = A$.

(3) Prove that there exist infinitely many integers n such that n,$n+1$,$n+2$ are each the sum of the squares of two integers. (For example, $0 = 0^2 + 0^2$, $1 = 0^2 + 1^2$, $2 = 1^2 + 1^2$.)

(4) For any two integers m and n with $0 \leq m \leq n$, numbers $d(m,n)$ are defined by

$$d(n,0) = d(n,n) = 1 \text{ for all } n \geq 0$$

and

$$m \cdot d(n,m) = m \cdot d(n-1,m) + (2n-m) \cdot d(n-1,m-1) \text{ for } 0 < m < n.$$

Prove that all of the $d(n,m)$ are integers.