CS559: Computer Graphics

Lecture 19: Curves

Li Zhang

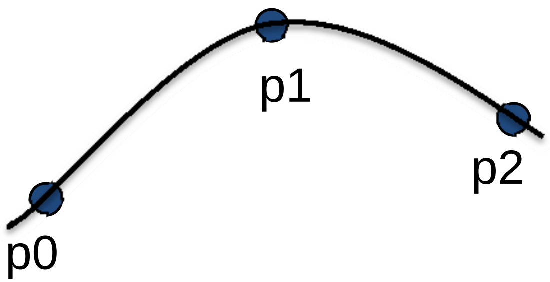
Spring 2008

Today

- Continue on curve modeling

- Reading
  - Shirley: Ch 15.1 – 15.5

More control points



$$\mathbf{f}(t) = \mathbf{a}_0 + \mathbf{a}_1 t + \mathbf{a}_2 t^2$$

$$\mathbf{p}_0 = \mathbf{f}(0) = \mathbf{a}_0 + 0\mathbf{a}_1 + 0^2\mathbf{a}_2$$

$$\mathbf{p}_1 = \mathbf{f}(0.5) = \mathbf{a}_0 + 0.5\mathbf{a}_1 + 0.5^2\mathbf{a}_2$$

$$\mathbf{p}_2 = \mathbf{f}(1.0) = \mathbf{a}_0 + 1\mathbf{a}_1 + 1^2\mathbf{a}_2$$

$$\begin{bmatrix} \mathbf{p}_0 \\ \mathbf{p}_1 \\ \mathbf{p}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0.5 & 0.25 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{a}_0 \\ \mathbf{a}_1 \\ \mathbf{a}_2 \end{bmatrix} = \mathbf{C} \begin{bmatrix} \mathbf{a}_0 \\ \mathbf{a}_1 \\ \mathbf{a}_2 \end{bmatrix}$$

$$\begin{bmatrix} \mathbf{a}_0 \\ \mathbf{a}_1 \\ \mathbf{a}_2 \end{bmatrix} = \mathbf{C}^{-1} \begin{bmatrix} \mathbf{p}_0 \\ \mathbf{p}_1 \\ \mathbf{p}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 4 & -1 \\ 2 & -4 & 2 \end{bmatrix} \begin{bmatrix} \mathbf{p}_0 \\ \mathbf{p}_1 \\ \mathbf{p}_2 \end{bmatrix}$$

$$\mathbf{f}(t) = \begin{bmatrix} 1 & t & t^2 \end{bmatrix} \begin{bmatrix} \mathbf{a}_0 \\ \mathbf{a}_1 \\ \mathbf{a}_2 \end{bmatrix}$$

$$= \mathbf{t}\mathbf{C}^{-1} \begin{bmatrix} \mathbf{p}_0 \\ \mathbf{p}_1 \\ \mathbf{p}_2 \end{bmatrix} = \mathbf{t}\mathbf{B} \begin{bmatrix} \mathbf{p}_0 \\ \mathbf{p}_1 \\ \mathbf{p}_2 \end{bmatrix}$$

By solving a matrix equation that satisfies the constraints,
we can get polynomial coefficients

Two views on polynomial curves

$$\mathbf{f}(t) = \begin{bmatrix} 1 & t & t^2 \end{bmatrix} \mathbf{B} \begin{bmatrix} \mathbf{p}_0 \\ \mathbf{p}_1 \\ \mathbf{p}_2 \end{bmatrix}$$

$$\mathbf{f}(t) = \begin{bmatrix} 1 & t & t^2 \end{bmatrix} \left(\mathbf{B} \begin{bmatrix} \mathbf{p}_0 \\ \mathbf{p}_1 \\ \mathbf{p}_2 \end{bmatrix} \right)$$

$$\mathbf{f}(t) = \mathbf{a}_0 + \mathbf{a}_1 t + \mathbf{a}_2 t^2$$

From control points **p**,
we can compute coefficients **a**

$$\mathbf{f}(t) = \left(\begin{bmatrix} 1 & t & t^2 \end{bmatrix} \mathbf{B} \right) \begin{bmatrix} \mathbf{p}_0 \\ \mathbf{p}_1 \\ \mathbf{p}_2 \end{bmatrix}$$

$$= \begin{bmatrix} b_0(t) & b_1(t) & b_2(t) \end{bmatrix} \begin{bmatrix} \mathbf{p}_0 \\ \mathbf{p}_1 \\ \mathbf{p}_2 \end{bmatrix}$$

$$\mathbf{f}(t) = b_0(t)\mathbf{p}_0 + b_1(t)\mathbf{p}_1 + b_2(t)\mathbf{p}_2$$

Each point on the curve is a linear
blending of the control points

What are b0(t), b1(t), …?

Two control point case:

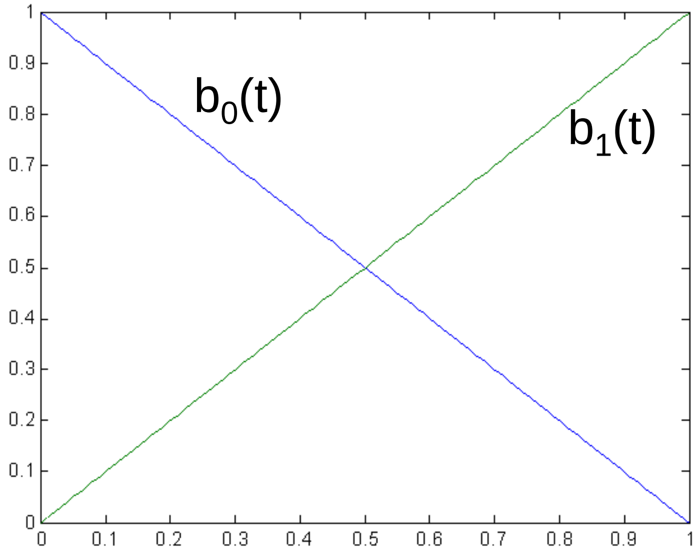
$$\mathbf{f}(t) = \left(\begin{bmatrix} 1 & t \end{bmatrix} \mathbf{B}\right) \begin{bmatrix} \mathbf{p}_0 \\ \mathbf{p}_1 \end{bmatrix}$$

$$\mathbf{f}(t) = b_0(t)\mathbf{p}_0 + b_1(t)\mathbf{p}_1$$

$$\begin{bmatrix} \mathbf{a}_0 \\ \mathbf{a}_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{p}_0 \\ \mathbf{p}_1 \end{bmatrix}$$

$$\begin{bmatrix} b_0(t) & b_1(t) \end{bmatrix} = \begin{bmatrix} 1 & t \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1-t & t \end{bmatrix}$$



What are b0, b1, …?

Three control point case:

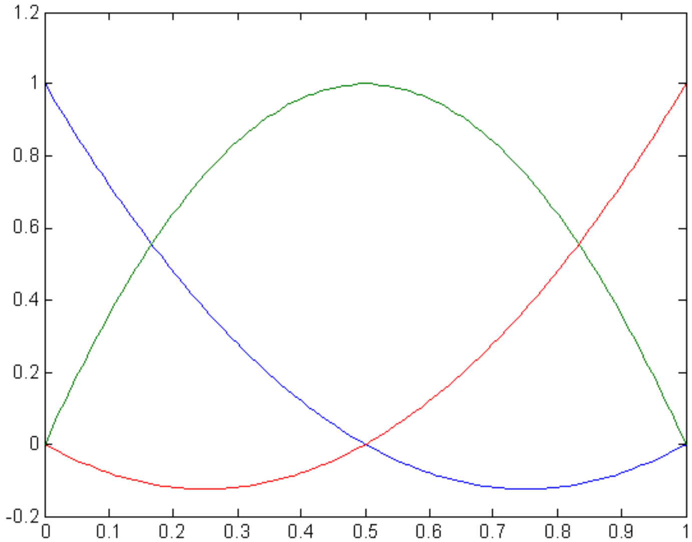
$$\mathbf{f}(t) = \left(\begin{bmatrix}1 & t & t^2\end{bmatrix}\mathbf{B}\right)\begin{bmatrix}\mathbf{p}_0 \\ \mathbf{p}_1 \\ \mathbf{p}_2\end{bmatrix}$$

$$\mathbf{f}(t) = b_0(t)\mathbf{p}_0 + b_1(t)\mathbf{p}_1 + b_2(t)\mathbf{p}_2$$

$$\begin{bmatrix}\mathbf{a}_0 \\ \mathbf{a}_1 \\ \mathbf{a}_2\end{bmatrix} = \begin{bmatrix}1 & 0 & 0 \\ -3 & 4 & -1 \\ 2 & -4 & 2\end{bmatrix}\begin{bmatrix}\mathbf{p}_0 \\ \mathbf{p}_1 \\ \mathbf{p}_2\end{bmatrix}$$

$$\begin{bmatrix}b_0(t) & b_1(t) & b_2(t)\end{bmatrix} = \begin{bmatrix}1 & t & t^2\end{bmatrix}\begin{bmatrix}1 & 0 & 0 \\ -3 & 4 & -1 \\ 2 & -4 & 2\end{bmatrix}$$

$$= \begin{bmatrix}1 - 3t + 2t^2 & 4t - 4t^2 & -t + 2t^2\end{bmatrix}$$



Which is $b_0(t)$?

$$b_0(t) + b_1(t) + b_2(t) = ? \quad \equiv 1$$

What are b0, b1, …?

Three control point case:

$$\mathbf{f}(t) = \left(\begin{bmatrix} 1 & t & t^2 \end{bmatrix} \mathbf{B}\right) \begin{bmatrix} \mathbf{p}_0 \\ \mathbf{p}_1 \\ \mathbf{p}_2 \end{bmatrix}$$

$$\mathbf{f}(t) = b_0(t)\mathbf{p}_0 + b_1(t)\mathbf{p}_1 + b_2(t)\mathbf{p}_2$$

- Why $b_0(t) + b_1(t) + b_2(t) \equiv 1$ is important?
  - Translation-invariant interpolation

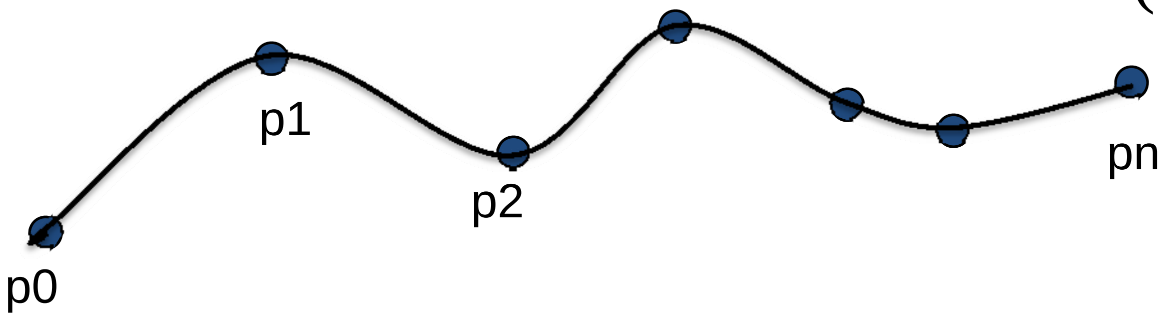
$$\begin{bmatrix} \mathbf{p}_0 \\ \mathbf{p}_1 \\ \mathbf{p}_2 \end{bmatrix} \Rightarrow \begin{bmatrix} \mathbf{p}_0 + \mathbf{d} \\ \mathbf{p}_1 + \mathbf{d} \\ \mathbf{p}_2 + \mathbf{d} \end{bmatrix}$$

$$\begin{aligned} \mathbf{f}_{new}(t) &= b_0(t)(\mathbf{p}_0 + \mathbf{d}) + b_1(t)(\mathbf{p}_1 + \mathbf{d}) + b_2(t)(\mathbf{p}_2 + \mathbf{d}) \\ &= b_0(t)\mathbf{p}_0 + b_1(t)\mathbf{p}_1 + b_2(t)\mathbf{p}_2 \\ &\quad + \left(b_0(t) + b_1(t) + b_2(t)\right)\mathbf{d} \\ &= \mathbf{f}(t) + \mathbf{d} \end{aligned}$$

for any t

Many control points

$$\mathbf{f}(t) = \sum_{i=0}^{n} \mathbf{a}_i t^i \qquad \mathbf{a}_i \in R^3$$



$$\mathbf{f}(t_0) = \mathbf{a}_0 + \mathbf{a}_1 t_0 + \mathbf{a}_2 t_0^{\ 2} + \cdots + \mathbf{a}_n t_0^{\ n} = \mathbf{p}_0$$

$$\mathbf{f}(t_1) = \mathbf{a}_0 + \mathbf{a}_1 t_1 + \mathbf{a}_2 t_1^{\ 2} + \cdots + \mathbf{a}_n t_1^{\ n} = \mathbf{p}_1$$

$$\cdots$$

$$\mathbf{f}(t_n) = \mathbf{a}_0 + \mathbf{a}_1 t_n + \mathbf{a}_2 t_n^{\ 2} + \cdots + \mathbf{a}_n t_n^{\ n} = \mathbf{p}_n$$

$$\begin{bmatrix} 1 & t_0 & \cdots & t_0^n \\ 1 & t_1 & \cdots & t_1^n \\ & \cdots & & \\ 1 & t_n & \cdots & t_n^n \end{bmatrix} \begin{bmatrix} \mathbf{a}_0 \\ \mathbf{a}_1 \\ \vdots \\ \mathbf{a}_n \end{bmatrix} = \begin{bmatrix} \mathbf{p}_0 \\ \mathbf{p}_1 \\ \vdots \\ \mathbf{p}_n \end{bmatrix}$$

Straightforward, but not intuitive

Many control points

- A shortcut $$\mathbf{f}(t)=\sum_{i=0}^{n} b_i(t)\mathbf{p}_i$$

Getting $b_i(t)$ is easier!

- Goal: $$\mathbf{f}(t_i)=\mathbf{p}_i$$

$$\sum_{i=0}^{n} b_i(t) = ? \equiv 1$$ Why?

- Idea: $$b_i(t_i)=1$$ $$b_i(t_j)=0 \quad j \neq i$$

$$\sum_{i=0}^{n} b_i(t_j) = ? = 1, \forall j = 0,1,\cdots n$$

$$B(t)=\sum_{i=0}^{n} b_i(t)$$ is a polynomial of degree n

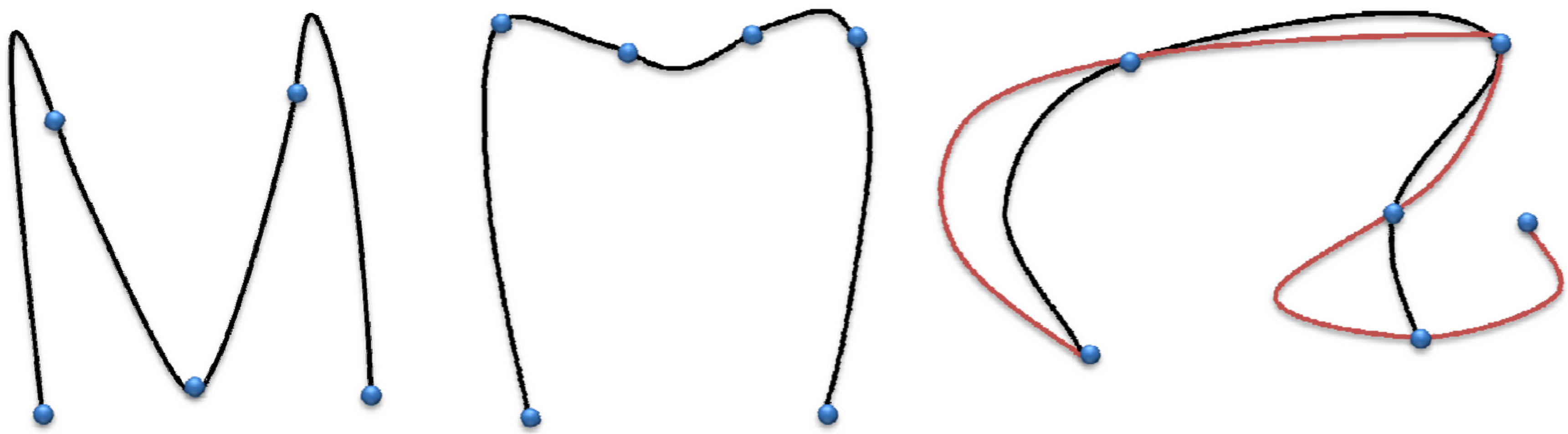
- Magic: $$b_i(t)=\prod_{j=0,\, j\neq i}^{n} \frac{t-t_j}{t_i-t_j}$$

If an n-degree polynomial has the same value at n+1 locations, it must be a constant polynomial

$$=\frac{t-t_0}{t_i-t_0}\frac{t-t_1}{t_i-t_1}\cdots\frac{t-t_{i-1}}{t_i-t_{i-1}}\cancel{\frac{t-t_i}{t_i-t_i}}\frac{t-t_{i+1}}{t_i-t_{i+1}}\cdots\frac{t-t_n}{t_i-t_n}$$

Lagrange Interpolation

Lagrange Polynomial Interpolation

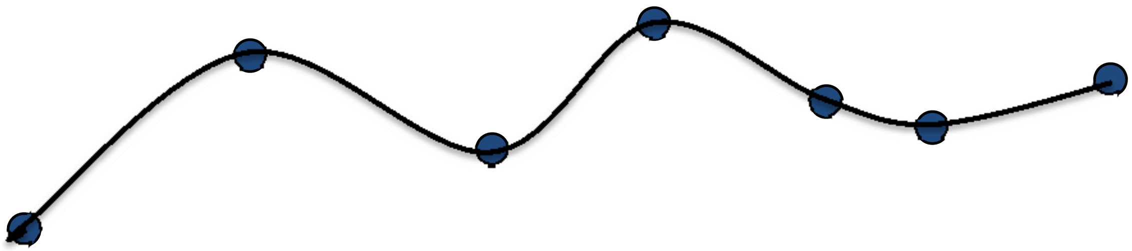


Lagrange Interpolation Demo

- http://www.math.ucla.edu/~baker/java/hoefer/Lagrange.htm

- Properties:
  - The curve passes through all the control points
  - Very smooth: C^n for n control points
  - Do not have local control
  - Overshooting

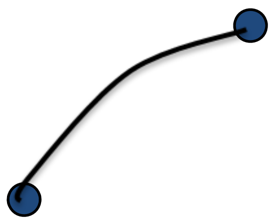
Piecewise Cubic Polynomials



$$\mathbf{f}(t) = \mathbf{a}_0 + \mathbf{a}_1 t + \mathbf{a}_2 t^2 + \mathbf{a}_3 t^3$$

- Desired Features:
  - Interpolation
  - Local control
  - C1 or C2

Natural Cubics



$$\mathbf{f}(t) = \mathbf{a}_0 + \mathbf{a}_1 t + \mathbf{a}_2 t^2 + \mathbf{a}_3 t^3$$

$$\mathbf{p}_0 = \mathbf{f}(0) = \mathbf{a}_0 \quad + 0\mathbf{a}_1 \quad + 0^2\mathbf{a}_2 \quad + 0^3\mathbf{a}_3$$

$$\mathbf{p}_1 = \mathbf{f}'(0) = \quad \mathbf{a}_1 \quad + 2 \cdot 0\mathbf{a}_2 \quad + 3 \cdot 0^2\mathbf{a}_3$$

$$\mathbf{p}_2 = \mathbf{f}''(0) = \quad + 2\mathbf{a}_2 \quad + 6 \cdot 0\mathbf{a}_3$$

$$\mathbf{p}_3 = \mathbf{f}(1) = \mathbf{a}_0 \quad + \mathbf{a}_1 \quad + \mathbf{a}_2 \quad + \mathbf{a}_3$$

$$\begin{bmatrix} \mathbf{p}_0 \\ \mathbf{p}_1 \\ \mathbf{p}_2 \\ \mathbf{p}_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{a}_0 \\ \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \end{bmatrix}$$

$$\begin{bmatrix} \mathbf{a}_0 \\ \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0.5 & 0 \\ -1 & -1 & -0.5 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{p}_0 \\ \mathbf{p}_1 \\ \mathbf{p}_2 \\ \mathbf{p}_3 \end{bmatrix}$$

Natural Cubics

- If we have n points, how to use natural cubic to interpolate them?
  - Define the first and second derivatives for the starting point of the first segment.
  - Compute the cubic for the first segment
  - Copy the first and second derivatives for the end point of the first segment to the starting point for the second segment
- How many segments do we have for n control points?
  - n-1

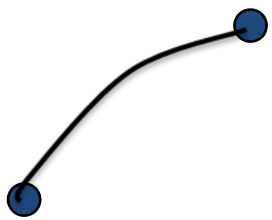
Natutral Cubic Curves

- Demo: http://www.cse.unsw.edu.au/~lambert/splines/

Natural Cubics

	Interpolate control points	Has local control	C2 continuity
Natural cubics			

Hermit Cubics



$$\mathbf{f}(t) = \mathbf{a}_0 + \mathbf{a}_1 t + \mathbf{a}_2 t^2 + \mathbf{a}_3 t^3$$

$$\mathbf{p}_0 = \mathbf{f}(0) = \mathbf{a}_0 \quad + 0\mathbf{a}_1 \quad + 0^2\mathbf{a}_2 \quad + 0^3\mathbf{a}_3$$

$$\mathbf{p}_1 = \mathbf{f}'(0) = \quad \mathbf{a}_1 \quad + 2\cdot 0\mathbf{a}_2 \quad + 3\cdot 0^2\mathbf{a}_3$$

$$\mathbf{p}_2 = \mathbf{f}(1) = \mathbf{a}_0 \quad + \mathbf{a}_1 \quad + \mathbf{a}_2 \quad + \mathbf{a}_3$$

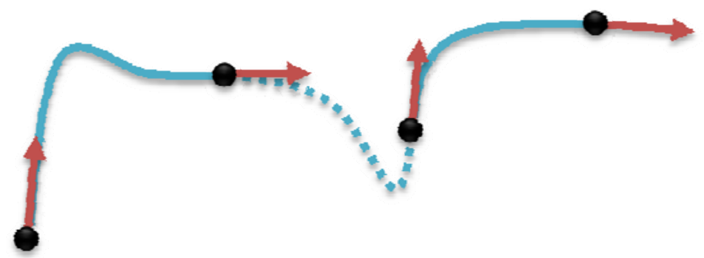
$$\mathbf{p}_3 = \mathbf{f}'(1) = \quad \mathbf{a}_1 \quad + 2\mathbf{a}_2 \quad + 3\mathbf{a}_3$$

$$\begin{bmatrix} \mathbf{p}_0 \\ \mathbf{p}_1 \\ \mathbf{p}_2 \\ \mathbf{p}_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} \mathbf{a}_0 \\ \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \end{bmatrix}$$

$$\begin{bmatrix} \mathbf{a}_0 \\ \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -3 & -2 & 3 & -1 \\ 2 & 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{p}_0 \\ \mathbf{p}_1 \\ \mathbf{p}_2 \\ \mathbf{p}_3 \end{bmatrix}$$

Hermite Cubic Curves

- If we have n points, how to use Hermite cubic to interpolate them?
  - For each pair, using the first derivatives at starting and ending points to define the inbetween

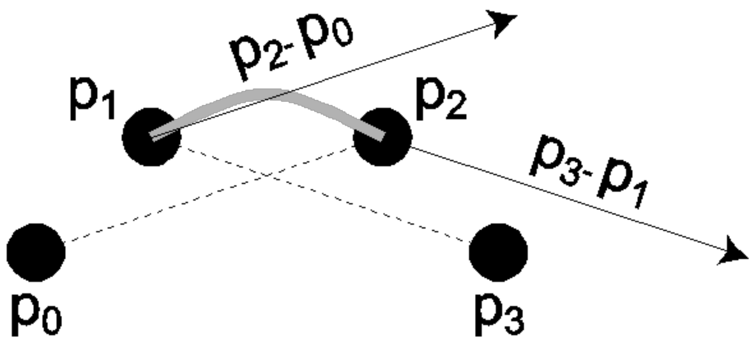


- How many segments do we have for n controls?
  - n/2-1

Hermite Cubies

	Interpolate control points	Has local control	C2 continuity
Natural cubics	Yes	No	Yes
Hermite cubics			

Catmull-Rom Cubics



$$\mathbf{C}_{Hermite}\begin{bmatrix}\mathbf{a}_0\\ \mathbf{a}_1\\ \mathbf{a}_2\\ \mathbf{a}_3\end{bmatrix}=\begin{bmatrix}\mathbf{f}(0)\\ \mathbf{f}'(0)\\ \mathbf{f}(1)\\ \mathbf{f}'(1)\end{bmatrix}=\begin{bmatrix}0 & 1 & 0 & 0\\ -0.5 & 0 & 0.5 & 0\\ 0 & 0 & 1 & 0\\ 0 & -0.5 & 0 & 0.5\end{bmatrix}\begin{bmatrix}\mathbf{p}_0\\ \mathbf{p}_1\\ \mathbf{p}_2\\ \mathbf{p}_3\end{bmatrix}$$

$$\begin{bmatrix}\mathbf{a}_0\\ \mathbf{a}_1\\ \mathbf{a}_2\\ \mathbf{a}_3\end{bmatrix}=\mathbf{B}_{Catmull}\begin{bmatrix}\mathbf{p}_0\\ \mathbf{p}_1\\ \mathbf{p}_2\\ \mathbf{p}_3\end{bmatrix}$$

$$\mathbf{f}(0)=\mathbf{p}_1$$

$$\mathbf{f}'(0)=\frac{1}{2}\left(\mathbf{p}_2-\mathbf{p}_0\right)$$

$$\mathbf{f}(1)=\mathbf{p}_2$$

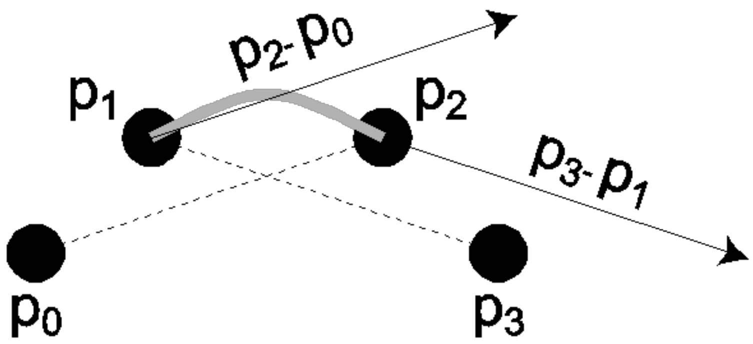
$$\mathbf{f}'(1)=\frac{1}{2}\left(\mathbf{p}_3-\mathbf{p}_1\right)$$

Catmull-Rom Cubic Curves

- Demo
- http://www.cse.unsw.edu.au/~lambert/splines/CatmullRom.html

- n control points define n-2 segments
- Changing 1 point affects neighboring 4 segments

Cardinal Cubics



$$\mathbf{C}_{Hermite}\begin{bmatrix}\mathbf{a}_0\\ \mathbf{a}_1\\ \mathbf{a}_2\\ \mathbf{a}_3\end{bmatrix} = \begin{bmatrix}\mathbf{f}(0)\\ \mathbf{f}'(0)\\ \mathbf{f}(1)\\ \mathbf{f}'(1)\end{bmatrix} = \begin{bmatrix}0 & 1 & 0 & 0\\ -s & 0 & s & 0\\ 0 & 0 & 1 & 0\\ 0 & -s & 0 & s\end{bmatrix}\begin{bmatrix}\mathbf{p}_0\\ \mathbf{p}_1\\ \mathbf{p}_2\\ \mathbf{p}_3\end{bmatrix}$$

$$\mathbf{f}(0) = \mathbf{p}_1$$

$$\mathbf{f}'(0) = s \cdot (\mathbf{p}_2 - \mathbf{p}_0)$$

$$\mathbf{f}(1) = \mathbf{p}_2$$

$$\mathbf{f}'(1) = s \cdot (\mathbf{p}_3 - \mathbf{p}_1)$$

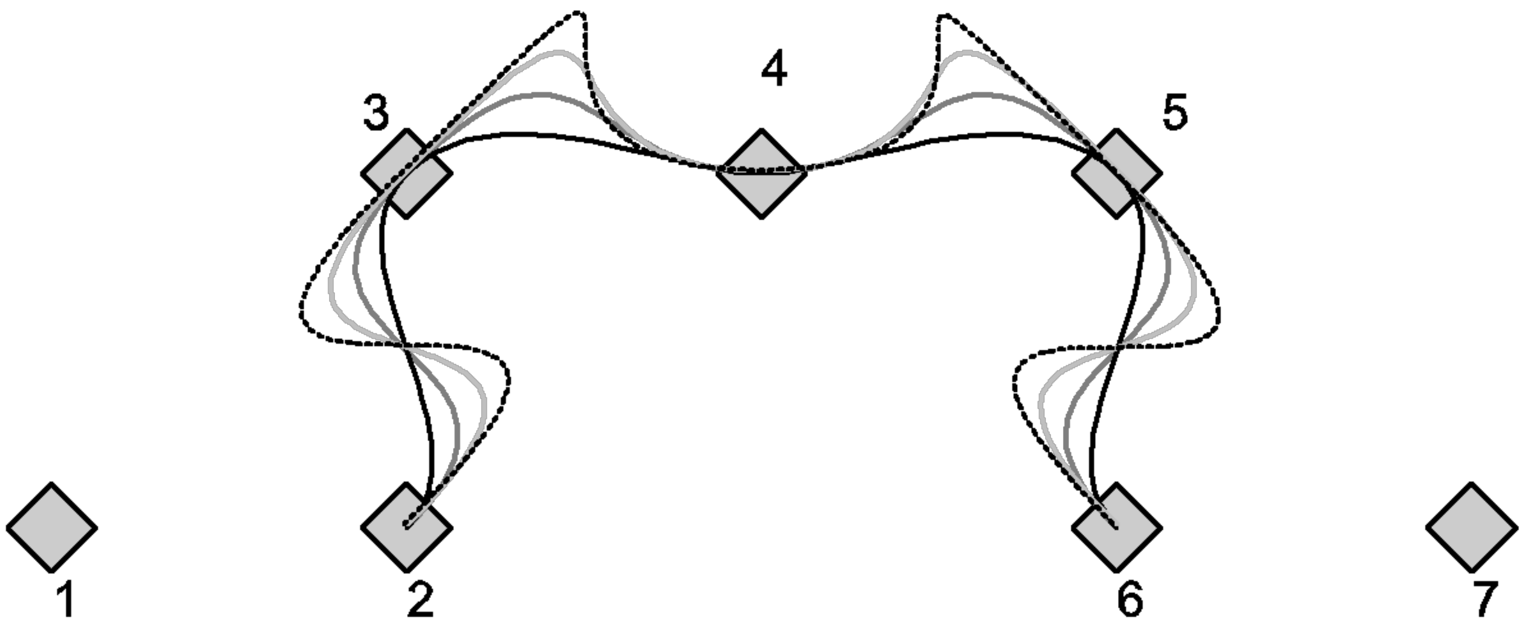
$$\begin{bmatrix}\mathbf{a}_0\\ \mathbf{a}_1\\ \mathbf{a}_2\\ \mathbf{a}_3\end{bmatrix} = \mathbf{B}_{Cardinal}\begin{bmatrix}\mathbf{p}_0\\ \mathbf{p}_1\\ \mathbf{p}_2\\ \mathbf{p}_3\end{bmatrix}$$

$$\begin{bmatrix}\mathbf{a}_0\\ \mathbf{a}_1\\ \mathbf{a}_2\\ \mathbf{a}_3\end{bmatrix} = \begin{bmatrix}0 & 1 & 0 & 0\\ -s & 0 & s & 0\\ 2s & s-3 & 3-2s & -s\\ -s & 2-s & s-2 & s\end{bmatrix}\begin{bmatrix}\mathbf{p}_0\\ \mathbf{p}_1\\ \mathbf{p}_2\\ \mathbf{p}_3\end{bmatrix}$$

s: tension control

Cardinal Cubic Curves

- 4 different s values



	Interpolate control points	Has local control	C2 continuity
Natural cubics	Yes	No	Yes
Hermite cubics	Yes	Yes	No
Cardinal Cubics			