CS559: Computer Graphics

Lecture 20: Curves

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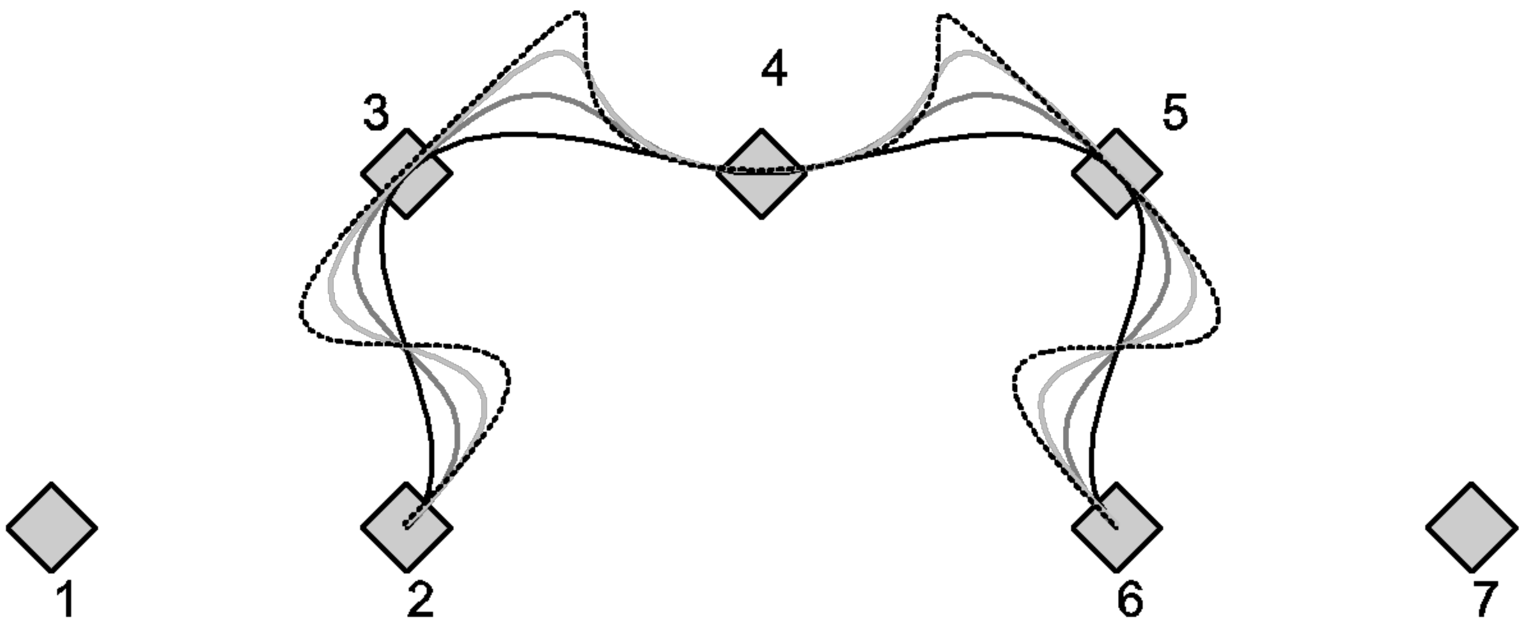
Today

- Continue on curve modeling

- Reading
  - Shirley: Ch 15.6.1 – 15.6.2
  - Moller and Haines: *Real-Time Rendering, 2e*, Ch 12.1.1, 12.12, 12.13, except (Rational Bezier Curves)
    - Linux: /p/course/cs559-lizhang/public/readings/rtr-12-curves-surfaces.pdf
    - Windows: P:\course\cs559-lizhang\public\readings\rtr-12-curves-surfaces.pdf

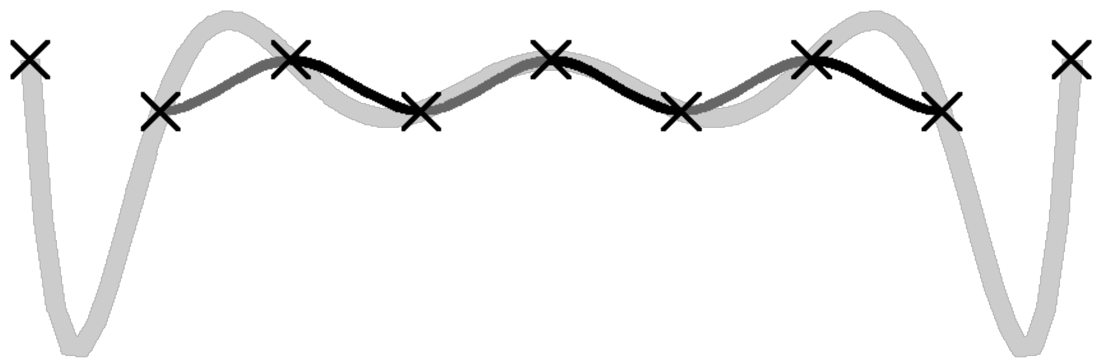
Cardinal Cubic Curves

- 4 different s values



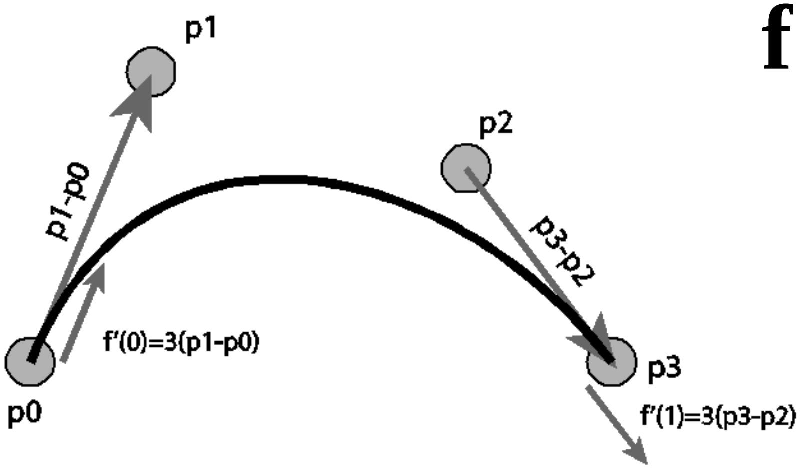
	Interpolate control points	Has local control	C2 continuity
Natural cubics	Yes	No	Yes
Hermite cubics	Yes	Yes	No
Cardinal Cubics	Yes	Yes	No

Cardinal Cubics vs Lagrange Polynomials



- Sacrifice higher order smoothness for
  - Local control
  - Avoid overshooting
- When higher order smoothness may be important?

Bezier Cubics



$$\mathbf{f}(t) = \mathbf{a}_0 + \mathbf{a}_1 t + \mathbf{a}_2 t^2 + \mathbf{a}_3 t^3$$

$$\mathbf{C}_{Hermite} \begin{bmatrix} \mathbf{a}_0 \\ \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \end{bmatrix} = \begin{bmatrix} \mathbf{f}(0) \\ \mathbf{f}'(0) \\ \mathbf{f}(1) \\ \mathbf{f}'(1) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -3 & 3 & 0 & 0 \\ 0 & 0 & -3 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{p}_0 \\ \mathbf{p}_1 \\ \mathbf{p}_2 \\ \mathbf{p}_3 \end{bmatrix}$$

$$\mathbf{f}(0) = \mathbf{p}_0$$

$$\mathbf{f}'(0) = 3 \cdot (\mathbf{p}_1 - \mathbf{p}_0)$$

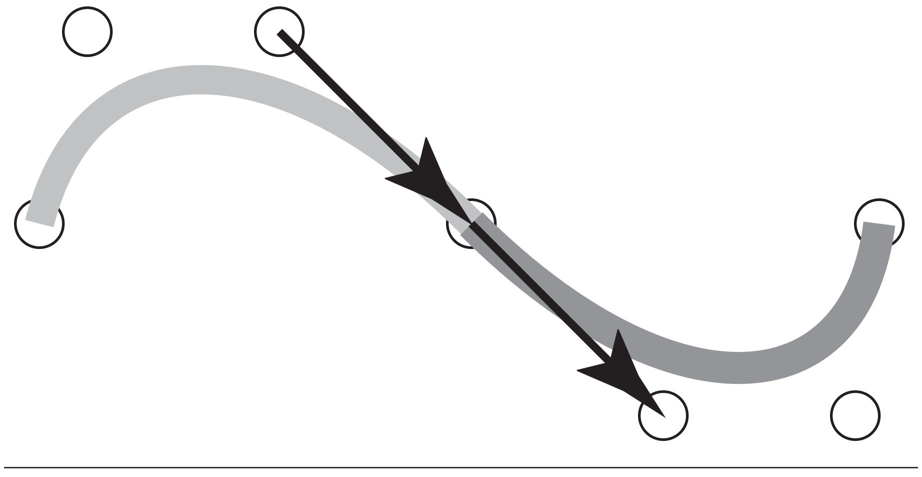
$$\mathbf{f}(1) = \mathbf{p}_3$$

$$\mathbf{f}'(1) = 3 \cdot (\mathbf{p}_3 - \mathbf{p}_2)$$

$$\begin{bmatrix} \mathbf{a}_0 \\ \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \end{bmatrix} = \mathbf{B}_{Bezier} \begin{bmatrix} \mathbf{p}_0 \\ \mathbf{p}_1 \\ \mathbf{p}_2 \\ \mathbf{p}_3 \end{bmatrix}$$

$$\mathbf{B}_{Bezier} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ -3 & 3 & 0 & 0 \\ 3 & -6 & 3 & 0 \\ -1 & 3 & -3 & 1 \end{bmatrix}$$

Bezier Cubic Curves



- Demo
  - http://www.cse.unsw.edu.au/~lambert/splines/Bezier.html

Bezier Cubic Curves in Illustrator

- Demo

- Changing one control point will change 2 neighboring segments

Generalization of Bezier

$$\mathbf{f}(t) = \begin{bmatrix} 1 & t & t^2 & t^3 \end{bmatrix} \mathbf{B}_{Bezier} \begin{bmatrix} \mathbf{p}_0 \\ \mathbf{p}_1 \\ \mathbf{p}_2 \\ \mathbf{p}_3 \end{bmatrix} \qquad \mathbf{B}_{Bezier} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -3 & 3 & 0 & 0 \\ 3 & -6 & 3 & 0 \\ -1 & 3 & -3 & 1 \end{bmatrix}$$

$$\mathbf{f}(t) = b_0(t)\mathbf{p}_0 + b_1(t)\mathbf{p}_1 + b_2(t)\mathbf{p}_2 + b_3(t)\mathbf{p}_3$$

$$b_0(t) = (1-t)^3$$

$$b_1(t) = 3t(1-t)^2$$

$$b_2(t) = 3t^2(1-t)$$

$$b_3(t) = t^3$$

$$\sum_{i=0}^{n} b_i(t) = ? \quad \equiv 1$$

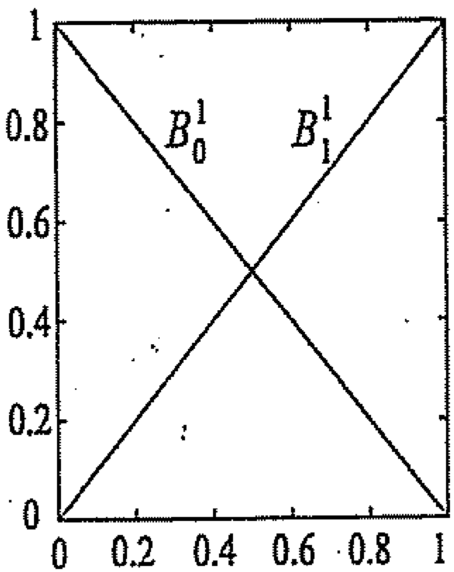
$$1 \equiv \left(\left(1-t\right)+t\right)^3$$

Generalization of Bezier

- If we have $\mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_2, \cdots \mathbf{p}_n$ $\quad \mathbf{f}(t) = \sum_{i=0}^{n} b_i(t)\mathbf{p}_i$

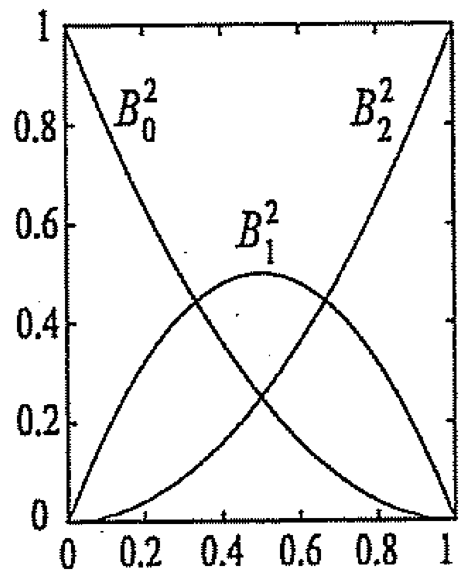
$$1 = \left(\left(1-t\right)+t\right)^n = \sum_{i=0}^{n} \binom{n}{i} t^i (1-t)^{n-i} \qquad \binom{n}{i} = \frac{n!}{i!(n-i)!}$$

$(1-t)\mathbf{p}_0 + t\mathbf{p}_1$

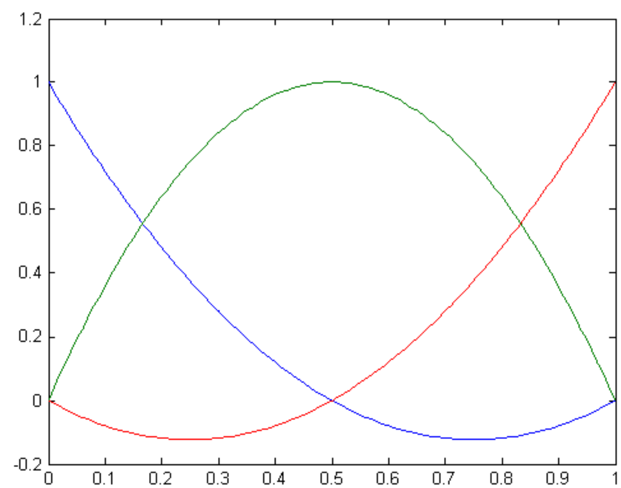


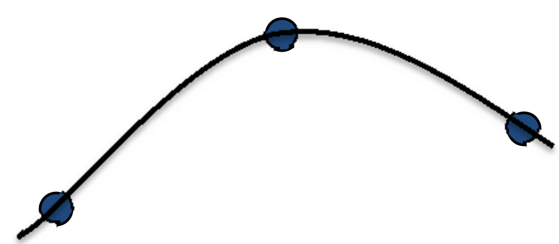
n=1

$(1-t)^2\mathbf{p}_0 + 2t(1-t)\mathbf{p}_1 + t^2\mathbf{p}_2$



n=2



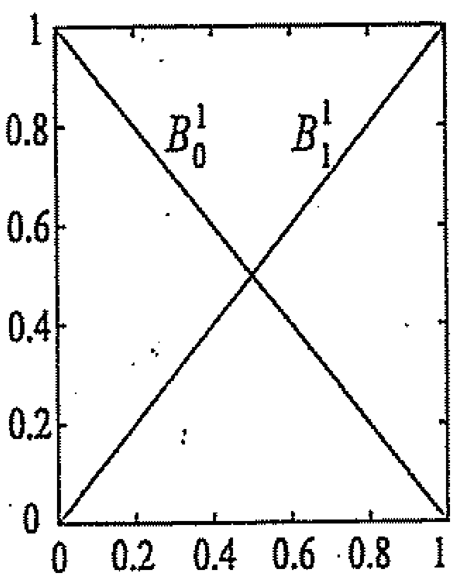


Generalization of Bezier

- If we have $\mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_2, \cdots \mathbf{p}_n$ $\quad \mathbf{f}(t) = \sum_{i=0}^{n} b_i(t) \mathbf{p}_i$

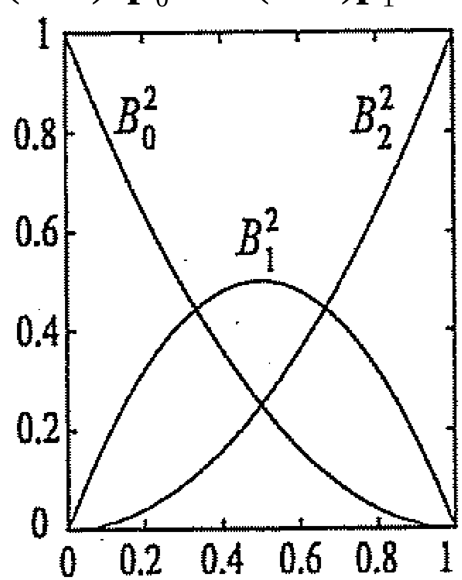
$$1 = \left((1-t)+t\right)^n = \sum_{i=0}^{n} \binom{n}{i} t^i (1-t)^{n-i} \qquad \binom{n}{i} = \frac{n!}{i!(n-i)!}$$

$(1-t)\mathbf{p}_0 + t\mathbf{p}_1$

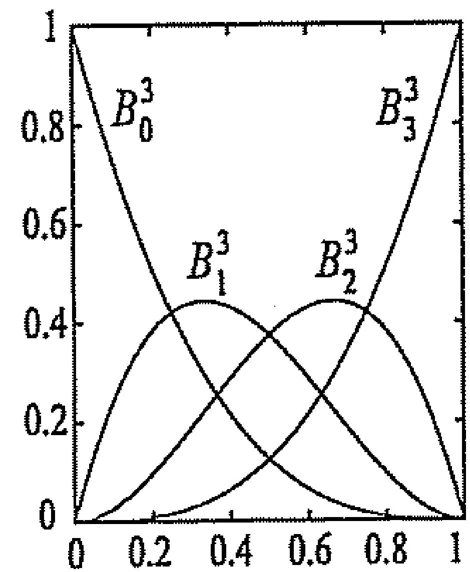


n=1

$(1-t)^2\mathbf{p}_0 + 2t(1-t)\mathbf{p}_1 + t^2\mathbf{p}_2$



n=2

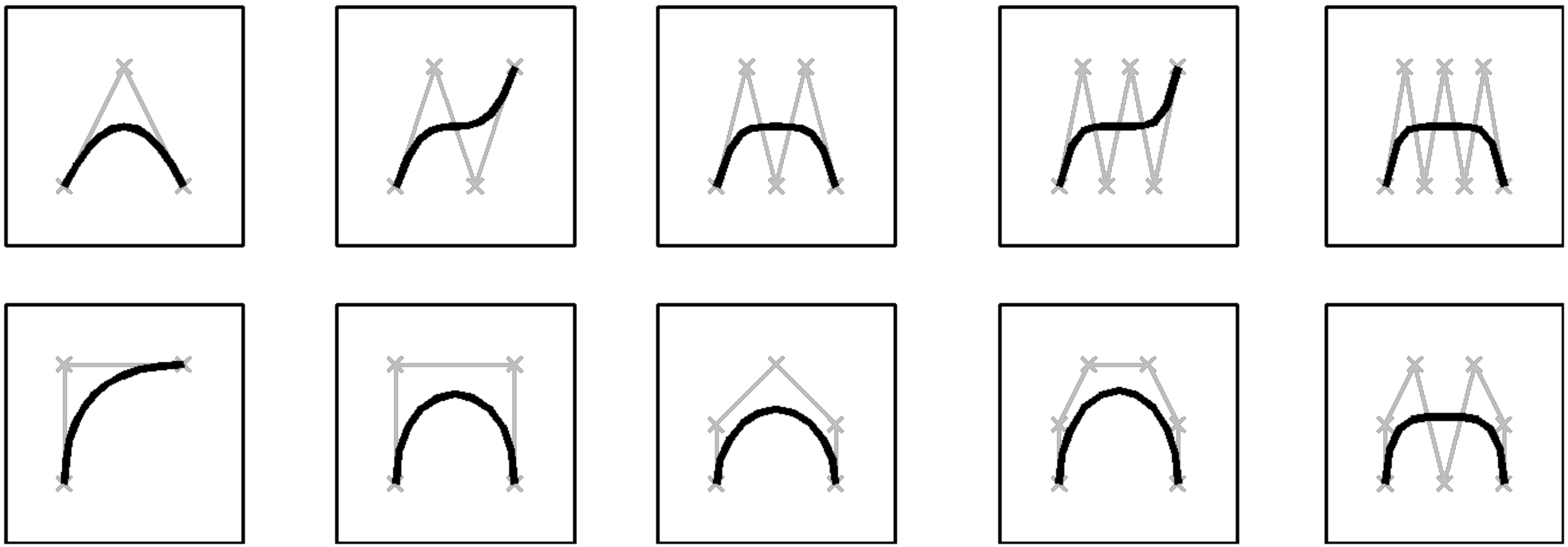


n=3

Generalization of Bezier

- If we have $\mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_2, \cdots \mathbf{p}_n$ $\quad \mathbf{f}(t) = \sum_{i=0}^{n} b_i(t)\mathbf{p}_i$

$$1 = \left(\left(1-u\right)+u\right)^n = \sum_{i=0}^{n} \binom{n}{i} u^i (1-u)^{n-i} \qquad \binom{n}{i} = \frac{n!}{i!(n-i)!}$$



n=2 n=3 n=4 n=5 n=6

Bezier Curve Properties

- The first and last control points are interpolated
- The tangent to the curve at the first control point is along the line joining the first and second control points

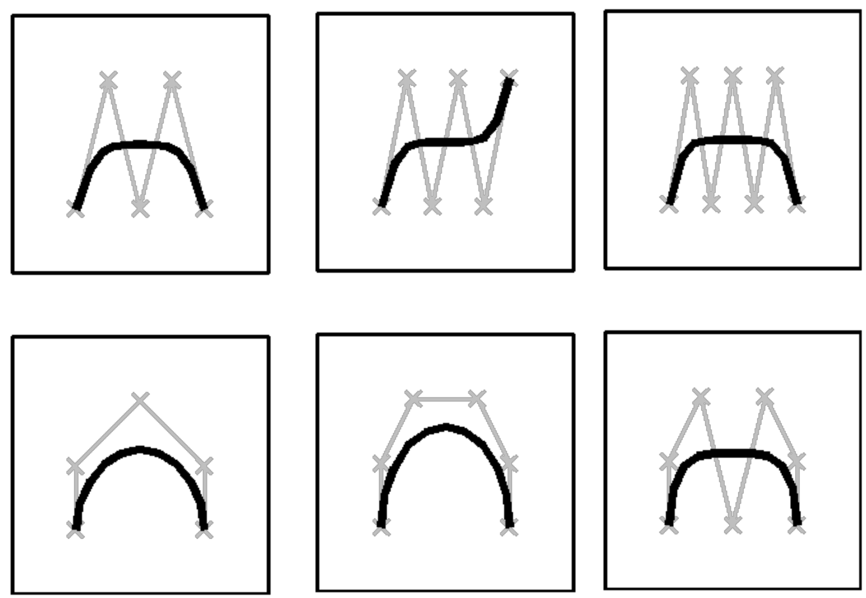
$$\mathbf{f}(t) = \sum_{i=0}^{n} \binom{n}{i} t^i (1-t)^{n-i} \mathbf{p}_i$$

$$\mathbf{f}'(0) = ? \ = n \cdot (\mathbf{p}_1 - \mathbf{p}_0)$$

$$\mathbf{f}'(1) = n \cdot (\mathbf{p}_n - \mathbf{p}_{n-1})$$

- The tangent at the last control point is along the line joining the second last and last control points
- K’th order derivative at 0 depends on p0 … pK
  - Why important?
    - Design curves with higher order continuity

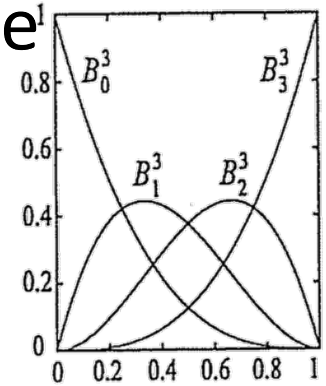
- The curve lies entirely within the convex hull of its control points
  - The basis functions sum to 1 and are everywhere positive → convex combination
  - Terminologies:
    - Convex shape
    - convex hull
    - convex combination
  - Why important?
    - Collision detection



Bezier Curve Properties

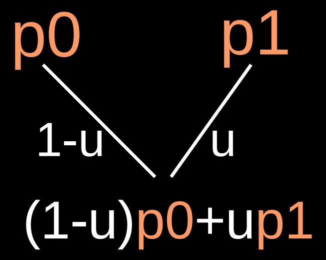
- Reversing the control points yields the same curve $\mathbf{f}_{bezier}(t;\mathbf{p}_0,\mathbf{p}_1,\cdots\mathbf{p}_n)=\mathbf{f}_{bezier}(1-t;\mathbf{p}_n,\mathbf{p}_{n-1},\cdots\mathbf{p}_0)$
  - why?

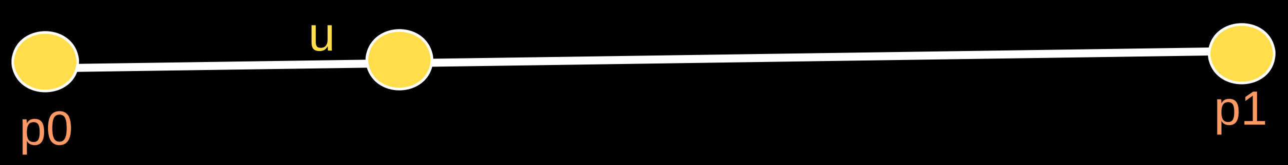
$$\binom{n}{i}=\binom{n}{n-i}$$



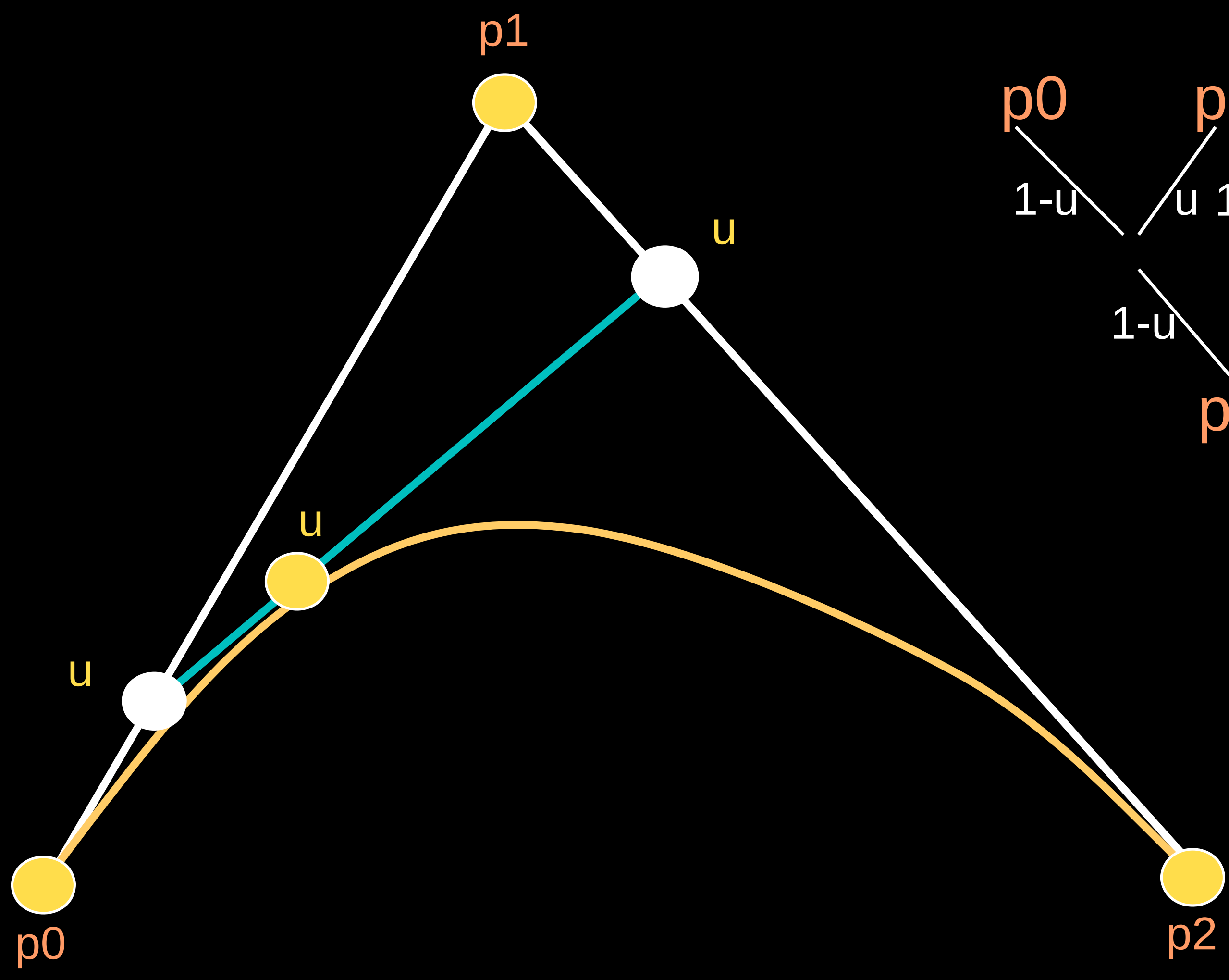
- The curves are affine invariant – Translating, scaling, rotating, or skewing the control points is the same as performing those operations on the curve itself.
  - Why important?
    - Efficient rendering
- Subdivision for efficient rendering, editing, approximating.

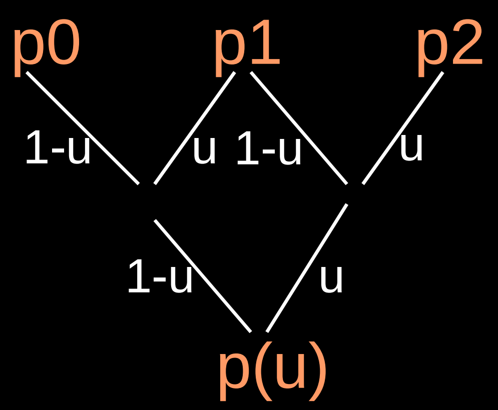
Geometric Interpretation: Line





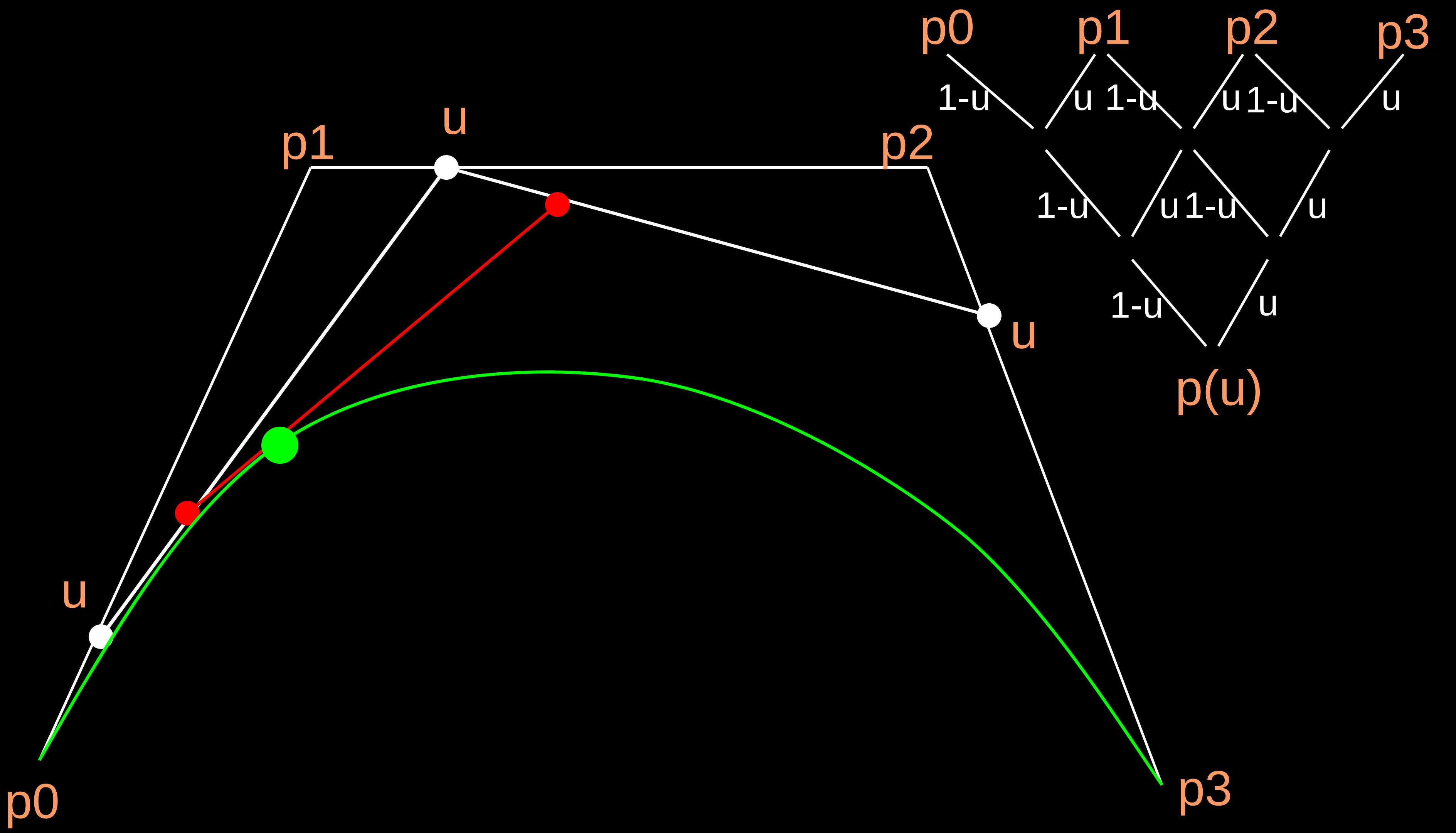
Geometric Interpretation: Quadratic





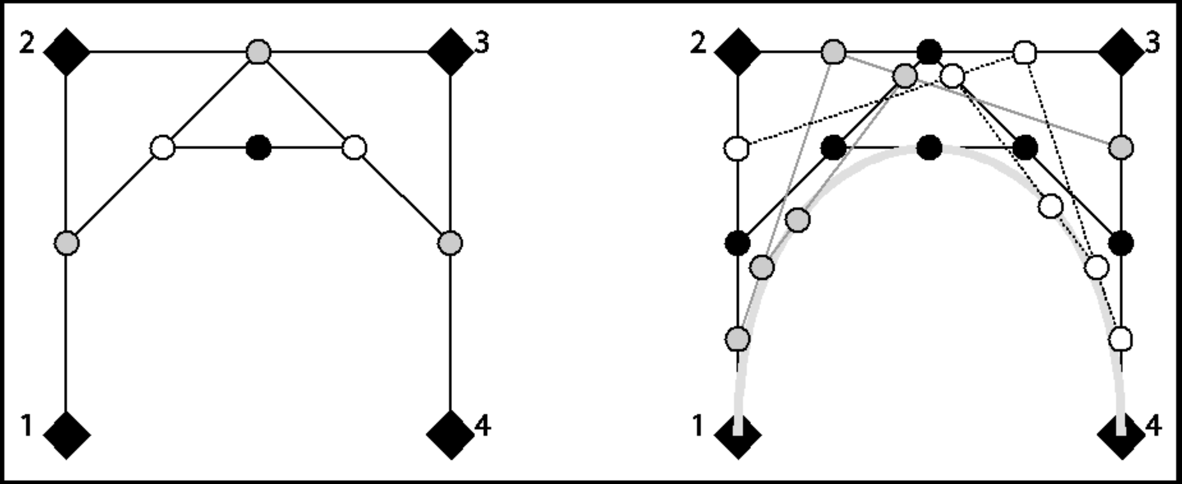
http://www1.cs.columbia.edu/~cs4160/html06f/

Geometric Interpretation: Cubic



http://www1.cs.columbia.edu/~cs4160/html06f/

Changing u



u=0.5

u=0.25, u=0.5, u=0.75

De Casteljau algorithm, Recursive Rendering