CS559: Computer Graphics

Lecture 9: 3D Transformation and Projection

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Spring 2010

Most slides borrowed from [Yungyu Chuang]

Last time: Image morphing

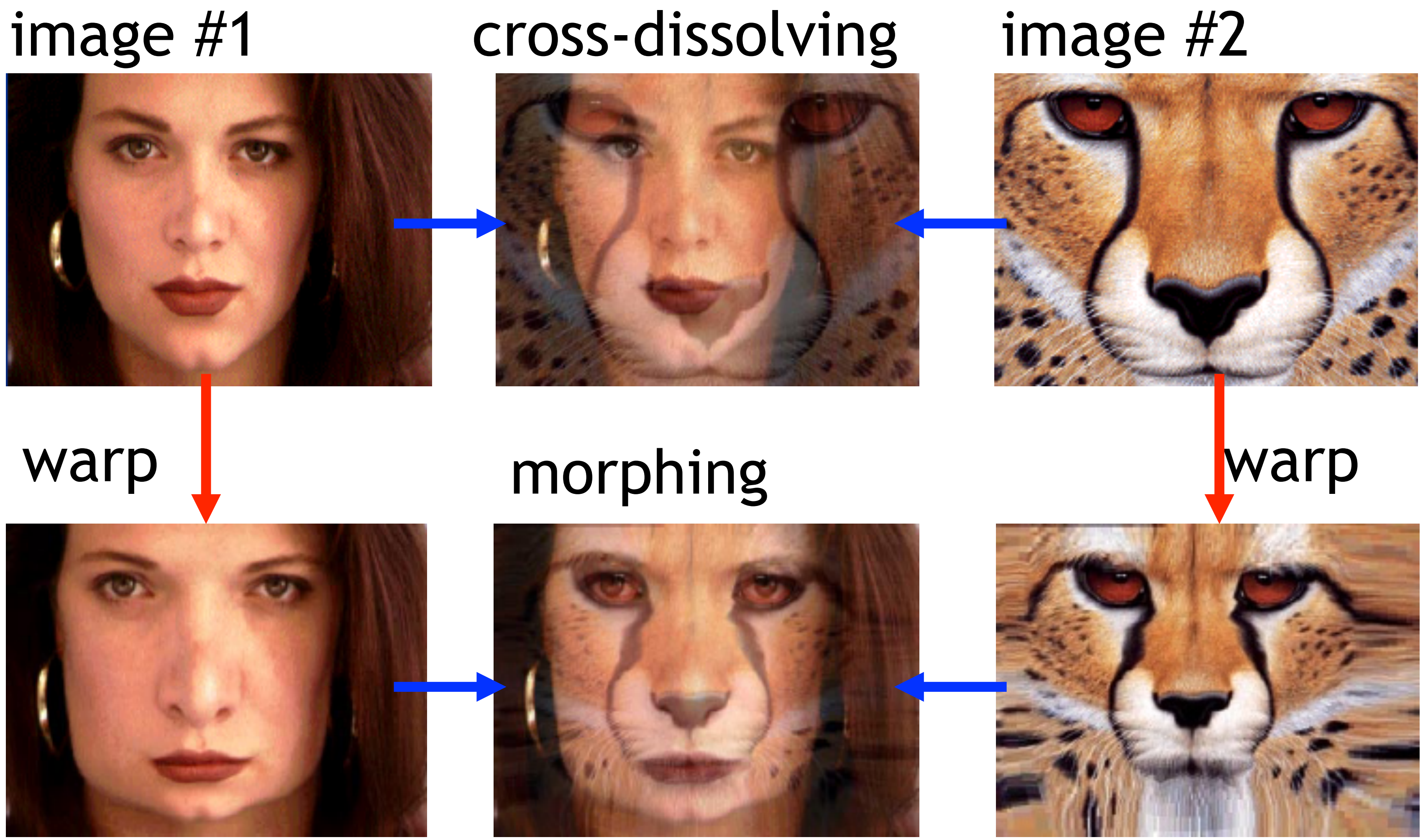
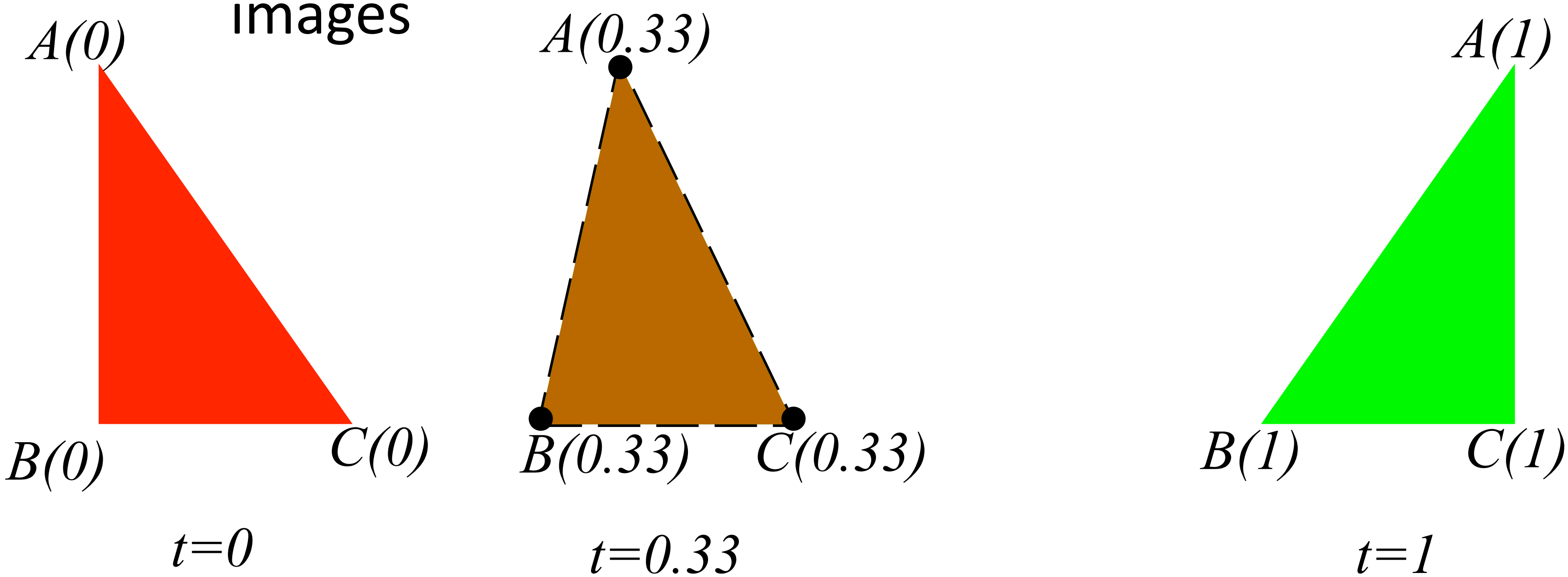


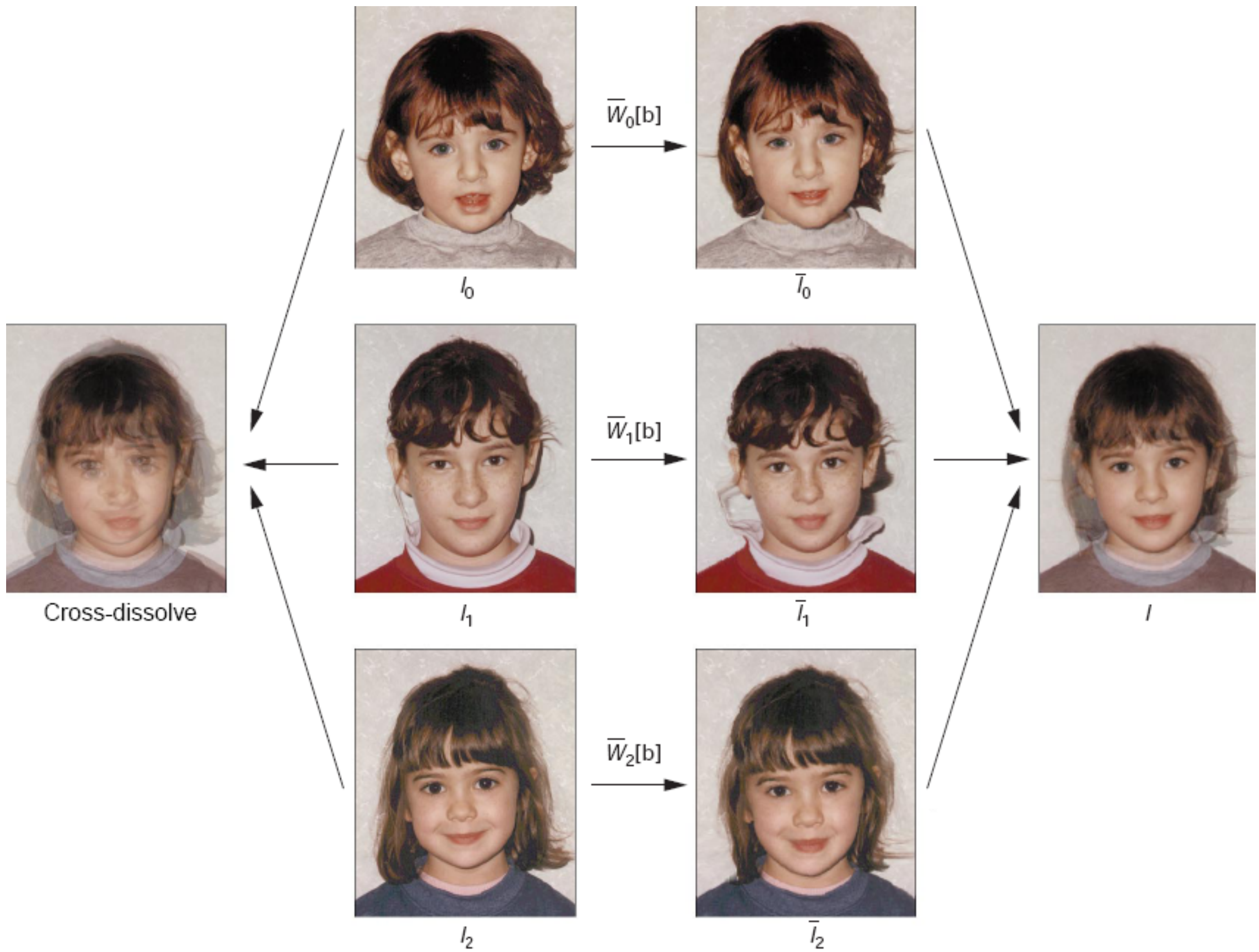
Image morphing

create a morphing sequence: for each time t

1. Create an intermediate warping field (by interpolation)
2. Warp both images towards it
3. Cross-dissolve the colors in the newly warped images



Multi-source morphing



More complex morph

- Triangular Mesh

Homogeneous Directions

- Translation does not affect directions!
- Homogeneous coordinates give us a very clean way of handling this
- The direction *(x,y)* becomes the homogeneous direction *(x,y,0)*

$$\begin{bmatrix} M & \mathbf{t} \\ \mathbf{0}^T & 1 \end{bmatrix} \begin{bmatrix} \mathbf{p} \\ 1 \end{bmatrix} = \begin{bmatrix} M\mathbf{p} + \mathbf{t} \\ 1 \end{bmatrix} \quad \begin{bmatrix} M & \mathbf{t} \\ \mathbf{0}^T & 1 \end{bmatrix} \begin{bmatrix} \mathbf{v} \\ 0 \end{bmatrix} = \begin{bmatrix} M\mathbf{v} \\ 0 \end{bmatrix}$$

- M can be any linear transformation: rotation, scaling, etc
  - Uniform scaling changes the length of the vector, but not the direction
  - How to represent this equivalence

Homogeneous Coordinates

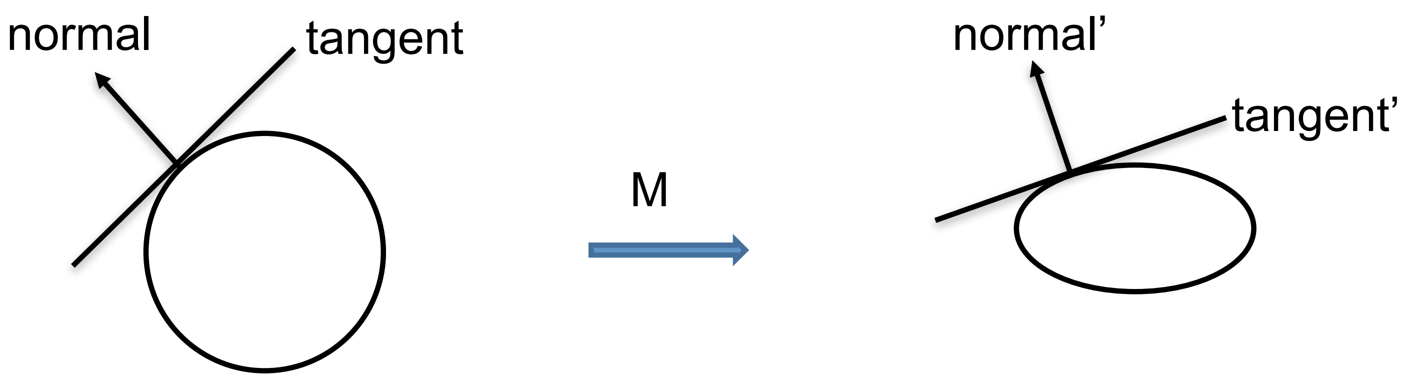
- In general, homogeneous coordinates (x, y, w)

$$\begin{bmatrix} x/w \\ y/w \end{bmatrix} \Leftarrow \begin{bmatrix} x \\ y \\ w \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} \Rightarrow \begin{bmatrix} wx \\ wy \\ w \end{bmatrix} \text{for any w} \neq 0\text{, usually w} = 1$$

- How to interpret the case for w = 0?
  - Point at infinity: directional vector
- (1,2,1) ⇔ (2,4,2)
- (2,3,0) ⇔ (6,9,0)

Transforming normal vectors



$$\mathbf{n}^{T}\mathbf{t} = 0$$

$$\mathbf{t}' = \boxed{\mathbf{M}\mathbf{t}}$$

$$\mathbf{n}'^{T}\,\mathbf{t}' = 0$$

$$(\mathbf{n}^{T}\mathbf{M}^{-1})(\mathbf{M}\mathbf{t}) = 0$$

$$\mathbf{n}' = (\mathbf{n}^{T}\mathbf{M}^{-1})^{T} = \boxed{(\mathbf{M}^{-1})^{T}\mathbf{n}}$$

If M is a rotation,

$$(\mathbf{M}^{-1})^{T} = \mathbf{M}$$

3D Transformations

- Homogeneous coordinates: *(x,y,z)=(wx,wy,wz,w)*

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \Rightarrow \begin{bmatrix} wx \\ wy \\ wz \\ w \end{bmatrix} \text{for any w} \neq 0, \text{usually w} = 1$$

$$\begin{bmatrix} x/w \\ y/w \\ z/w \end{bmatrix} \Leftarrow \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

- Transformations are now represented as 4x4 matrices

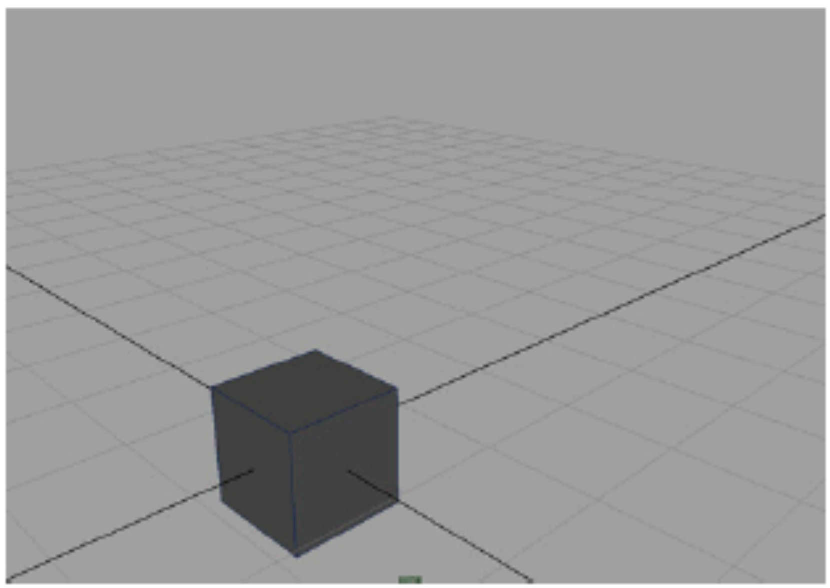
$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

3D Affine Transform

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} a & b & c & t_x \\ d & e & f & t_y \\ g & h & i & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Translation

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$



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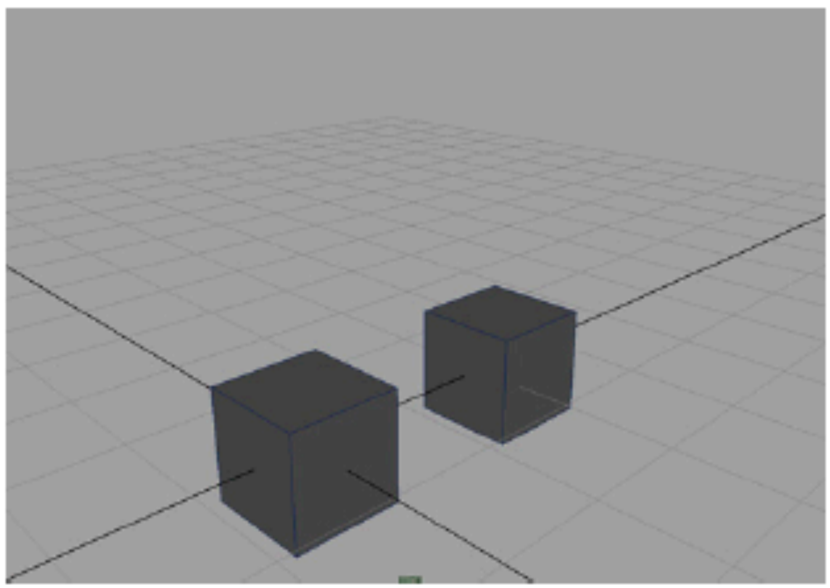

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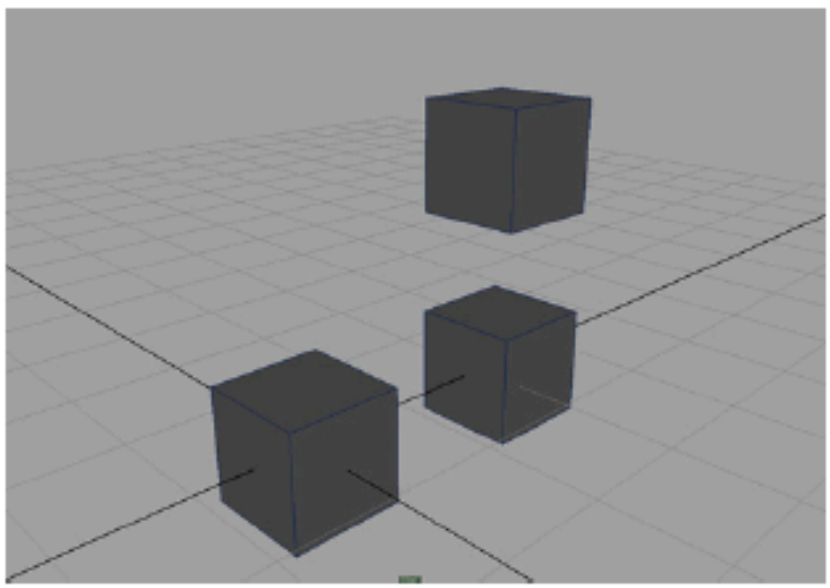

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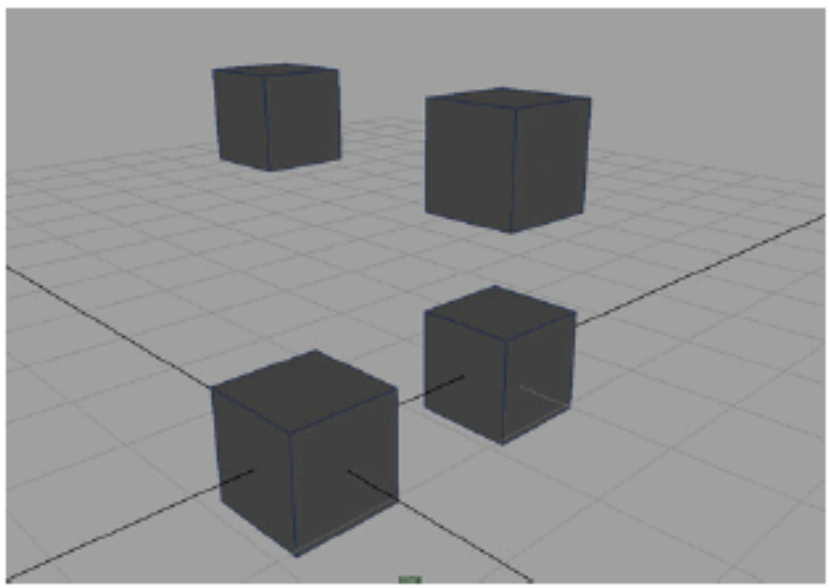
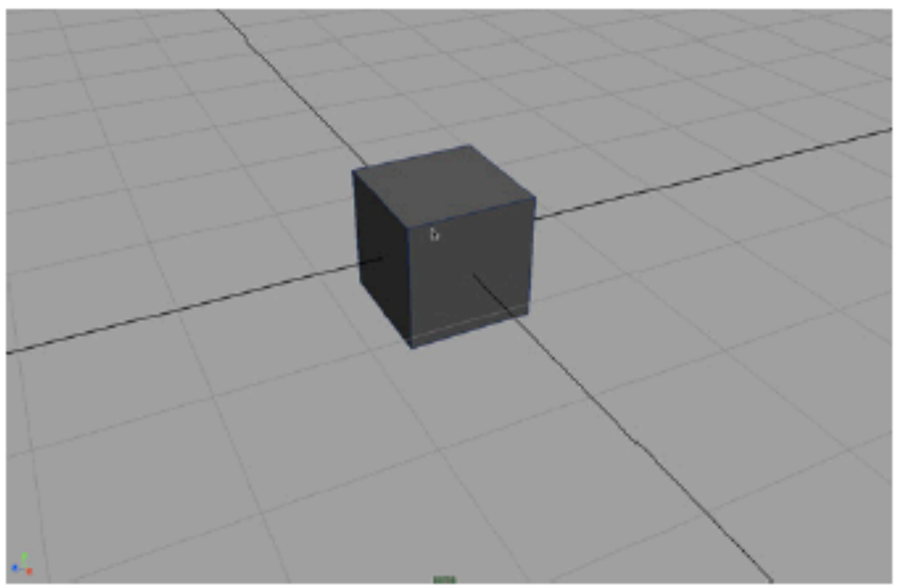


Scaling

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$



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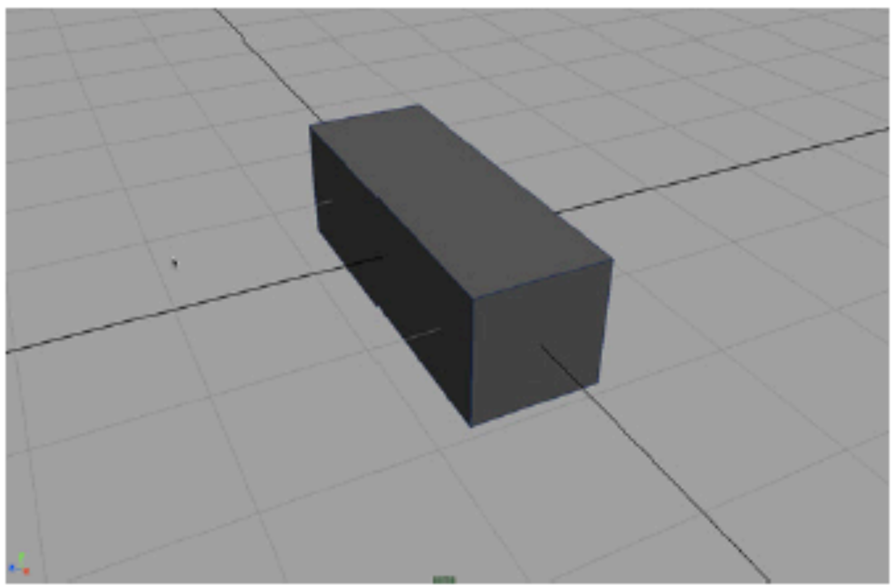

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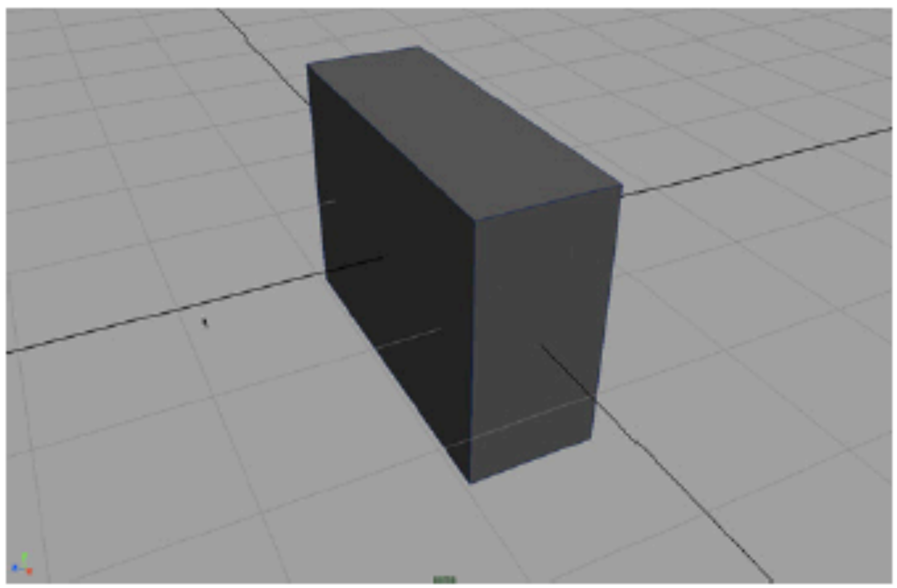

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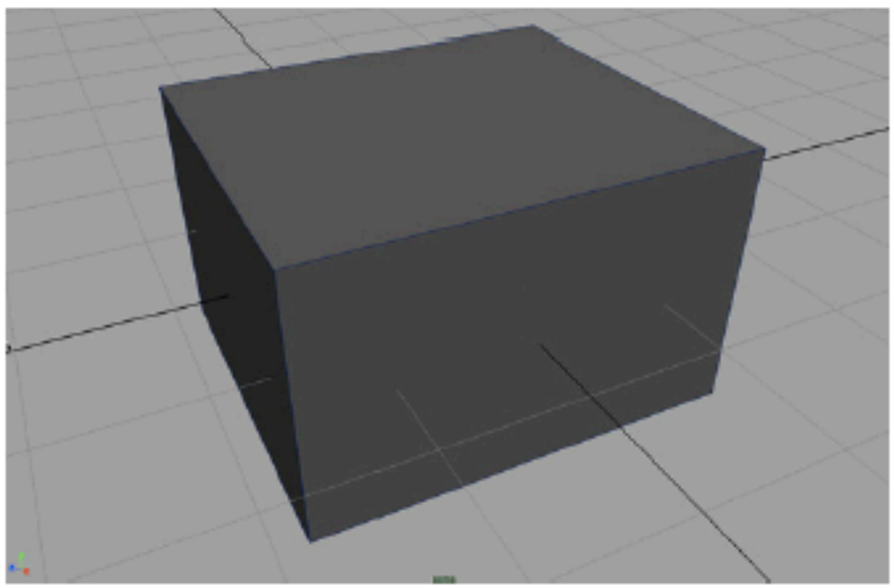
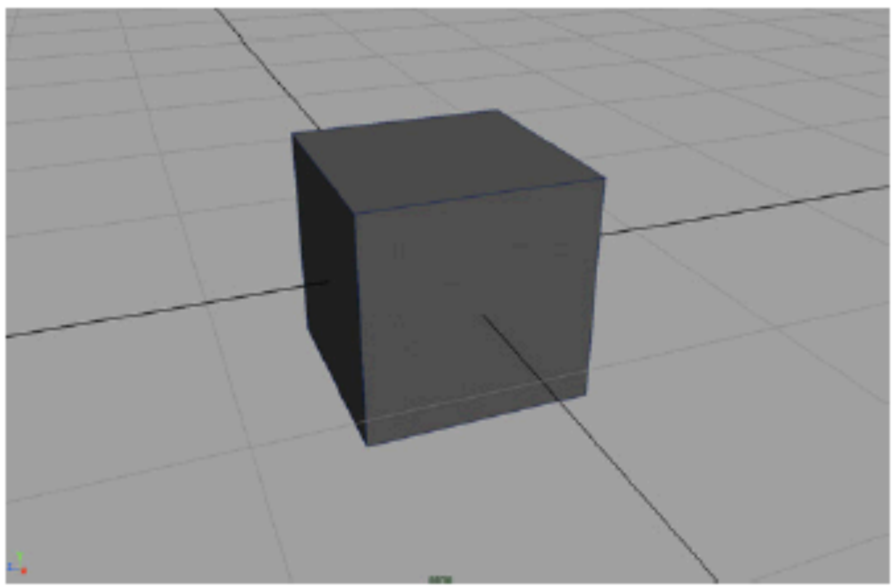


Rotation about z axis

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$



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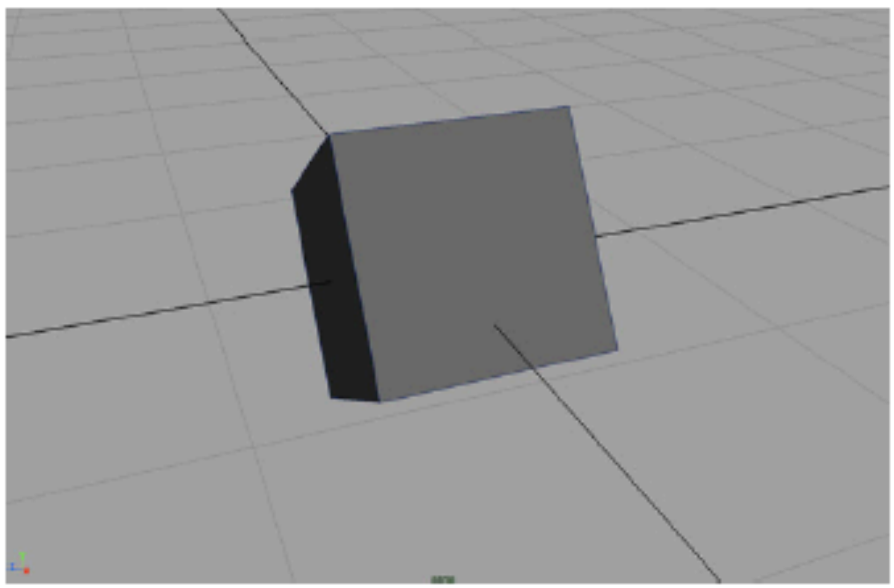
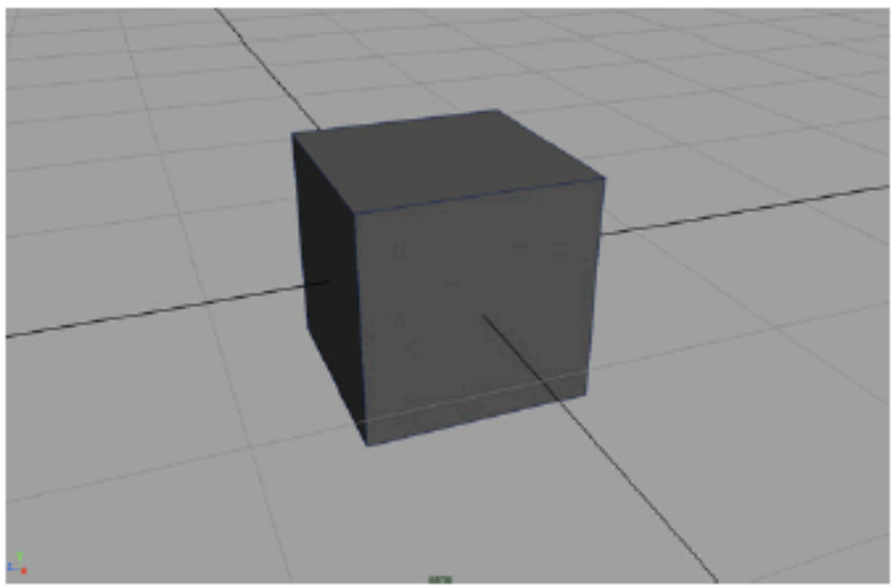


Rotation about x axis

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$



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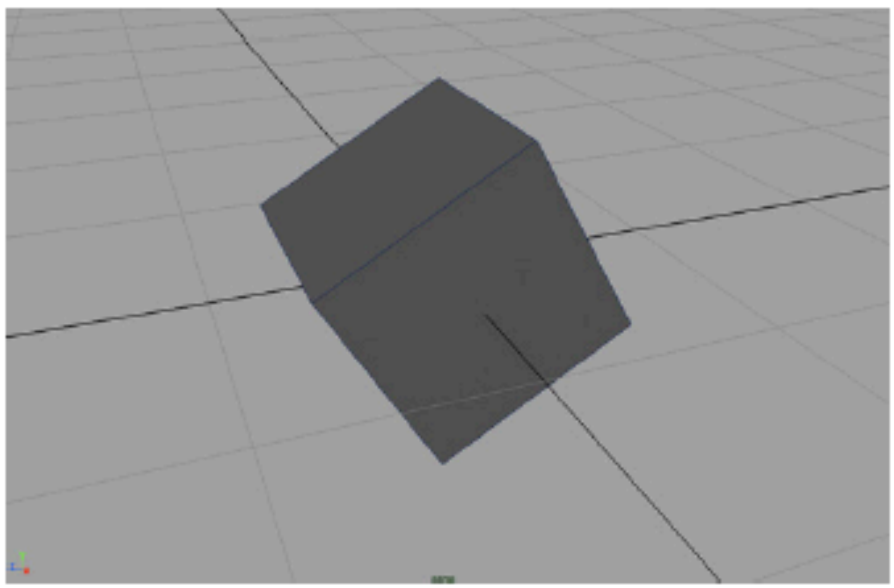
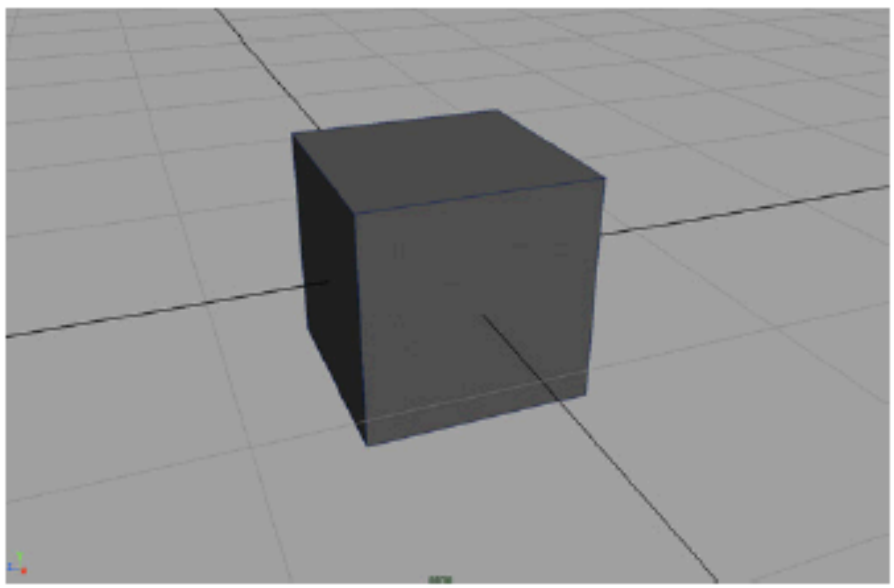


Rotation about y axis

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$



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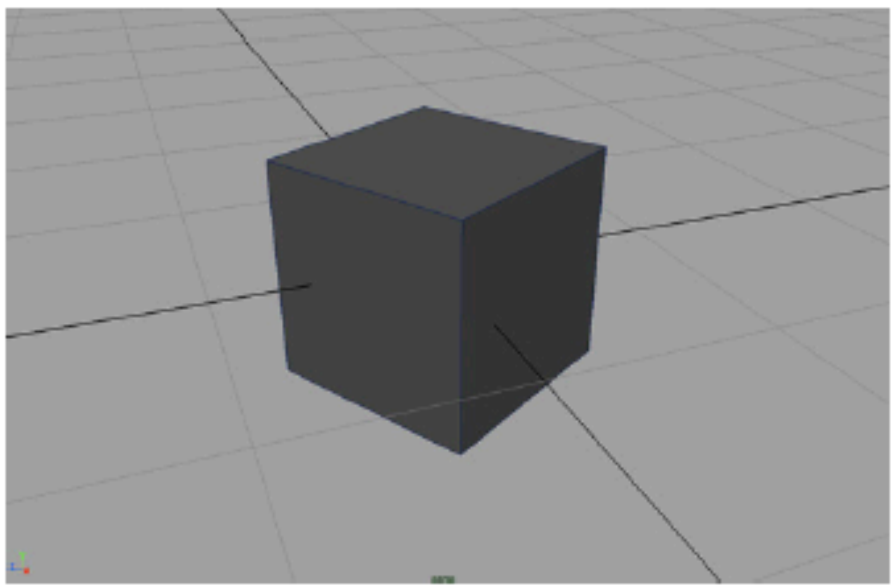
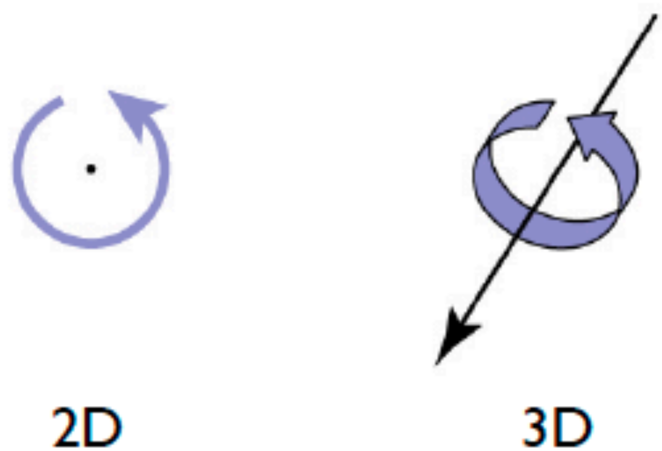


General rotations

- A rotation in 2D is around a point
- A rotation in 3D is around an axis
  - so 3D rotation is w.r.t a line, not just a point
  - there are many more 3D rotations than 2D
    - a 3D space around a given point, not just 1D



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- Using a matrix representation, any matrix with an *orthonormal* top-left 3x3 sub-matrix is a rotation
  - Rows/columns are mutually orthogonal (0 dot product)

$$R = \begin{bmatrix} | & | & | & 0 \\ \mathbf{r}_1 & \mathbf{r}_2 & \mathbf{r}_3 & 0 \\ | & | & | & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{ then } \mathbf{r}_1 \bullet \mathbf{r}_2 = 0, \mathbf{r}_1 \bullet \mathbf{r}_3 = 0, \mathbf{r}_2 \bullet \mathbf{r}_3 = 0, \mathbf{r}_1 \bullet \mathbf{r}_1 = 1, \mathbf{r}_2 \bullet \mathbf{r}_2 = 1, \mathbf{r}_3 \bullet \mathbf{r}_3 = 1.$$

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- Using a matrix representation, any matrix with an *orthonormal* top-left 3x3 sub-matrix is a rotation
  - Rows/columns are mutually orthogonal (0 dot product)
  - Determinant is 1
  - Implies columns are also orthogonal, and that the transpose is equal to the inverse

$$R = \begin{bmatrix} | & | & | & 0 \\ \mathbf{r}_1 & \mathbf{r}_2 & \mathbf{r}_3 & 0 \\ | & | & | & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{ then } \mathbf{r}_1 \bullet \mathbf{r}_2 = 0, \mathbf{r}_1 \bullet \mathbf{r}_3 = 0, \mathbf{r}_2 \bullet \mathbf{r}_3 = 0, \mathbf{r}_1 \bullet \mathbf{r}_1 = 1, \mathbf{r}_2 \bullet \mathbf{r}_2 = 1, \mathbf{r}_3 \bullet \mathbf{r}_3 = 1.$$

Specifying a rotation matrix

- *Euler angles:* Specify how much to rotate about X, then how much about Y, then how much about Z
  - Hard to think about, and hard to compose

$$[\mathbf{R}] = \begin{bmatrix} \cos\gamma & \sin\gamma & 0 \\ -\sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\beta & \sin\beta \\ 0 & -\sin\beta & \cos\beta \end{bmatrix} \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Alternative Representations

- Specify the axis and the angle (OpenGL method)
  - Hard to compose multiple rotations
  - Axis ω, angle θ

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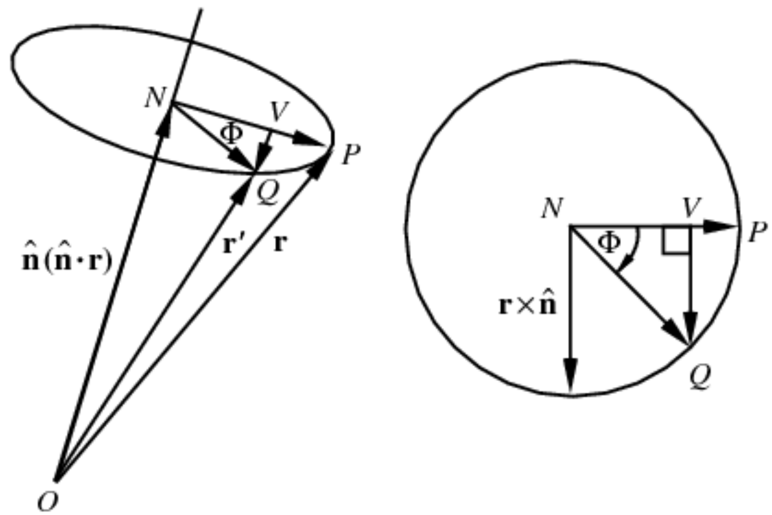

A rotation by an angle $\theta \in \mathbb{R}$ around axis specified by the unit vector $\hat{\omega} = (\omega_x, \omega_y, \omega_z) \in \mathbb{R}^3$ is given by $I + \tilde{\omega}\sin\theta + \tilde{\omega}^2 (1-\cos\theta)$

$$\tilde{\omega} = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix}.$$

http://mathworld.wolfram.com/RodriguesRotationFormula.html

Non-Commutativity

- Not Commutative (unlike in 2D)!!
- Rotate by x, then y is not same as y then x
- Order of applying rotations does matter
- Follows from matrix multiplication not commutative
  - R1 * R2 is not the same as R2 * R1

Other Rotation Issues

- Rotation is about an axis at the origin
  - For rotation about an arbitrary axis, use the same trick as in 2D: Translate the axis to the origin, rotate, and translate back again

Transformation Leftovers

- Shear etc extend naturally from 2D to 3D
- Rotation and Translation are the *rigid-body transformations:*
  - Do not change lengths or angles, so a body does not deform when transformed

Building transforms from points

- Recall2D affine transformation has 6 degrees of freedom (DOFs)
  - this is the number of "knobs" we have to set to define one
- Therefore 6 constraints suffice to define the transformation
  - handy kind of constraint: point **p** maps to point **q** (2 constraints at once)
  - three point constraints add up to constrain all 6 DOFs (i.e. can map any triangle to any other triangle)

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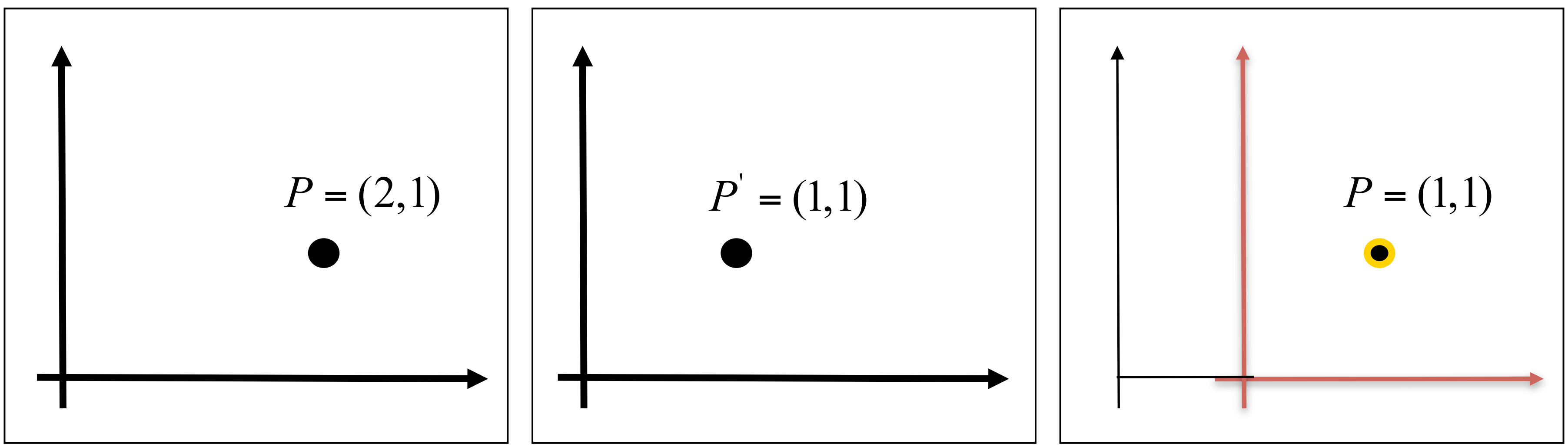
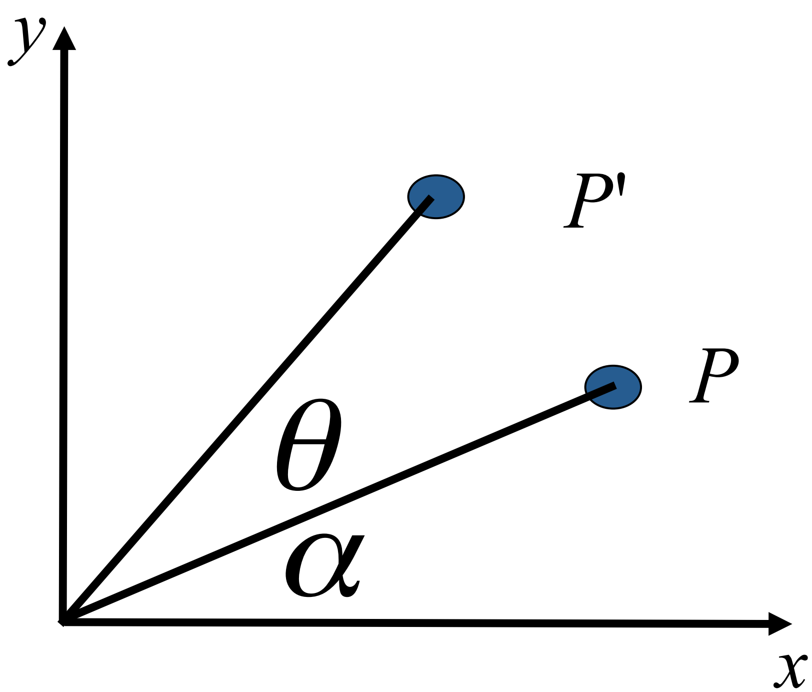
  - handy kind of constraint: point **p** maps to point **q** (2 constraints at once)
  - three point constraints add up to constrain all 6 DOFs
    (i.e. can map any triangle to any other triangle)
- 3D affine transformation has 12 degrees of freedom
  - count them by looking at the matrix entries we're allowed to change
- Therefore 12 constraints suffice to define the transformation
  - in 3D, this is 4 point constraints
    (i.e. can map any tetrahedron to any other tetrahedron)

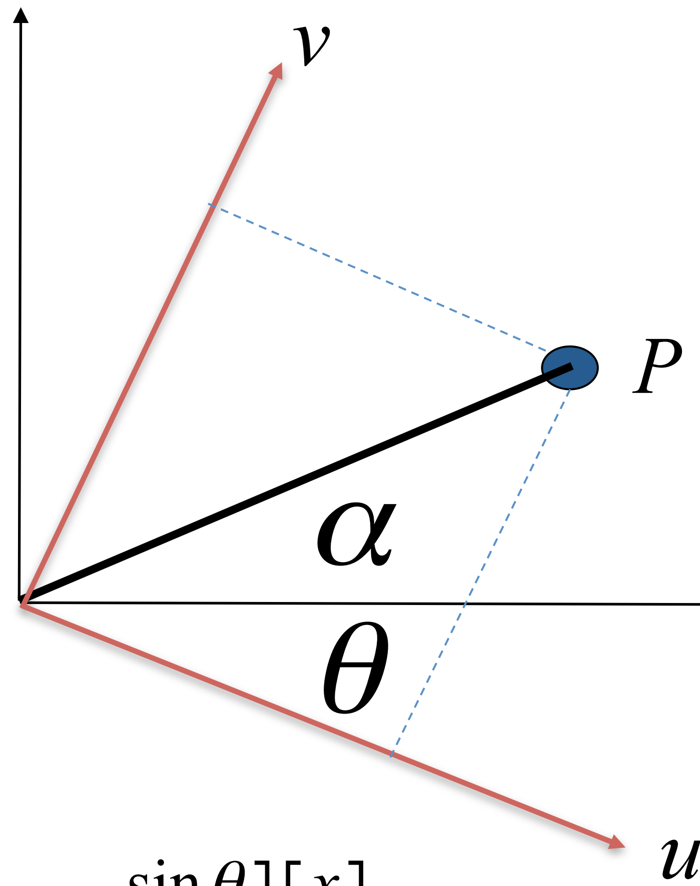
Coordinate Frames

- All of discussion in terms of operating on points
- But can also change coordinate system
- Example, motion means either point moves backward, or coordinate system moves forward



Coordinate Frames: Rotations





$$R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

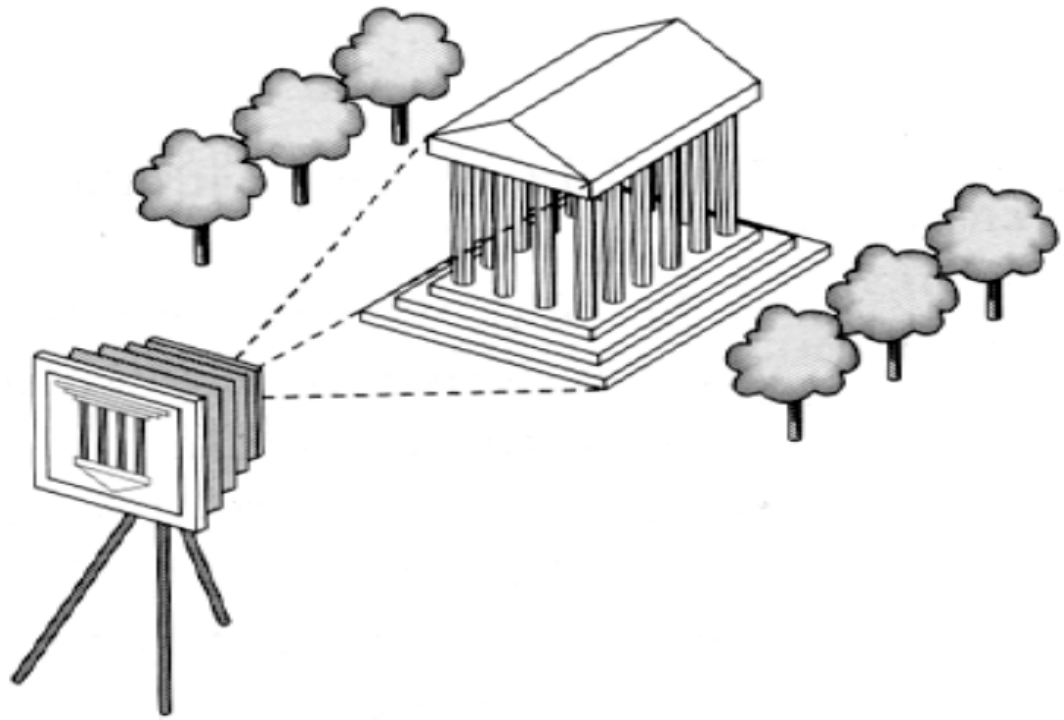
Geometric Interpretation 3D Rotations

- Rows of matrix are 3 unit vectors of new coord frame
- Can construct rotation matrix from 3 orthonormal vectors
- Effectively, projections of point into new coord frame

$$R_{uvw} = \begin{pmatrix} x_u & y_u & z_u \\ x_v & y_v & z_v \\ x_w & y_w & z_w \end{pmatrix}$$

$$Rp = \begin{pmatrix} x_u & y_u & z_u \\ x_v & y_v & z_v \\ x_w & y_w & z_w \end{pmatrix} \begin{pmatrix} x_p \\ y_p \\ z_p \end{pmatrix} = ? \quad \begin{pmatrix} u \bullet p \\ v \bullet p \\ w \bullet p \end{pmatrix}$$

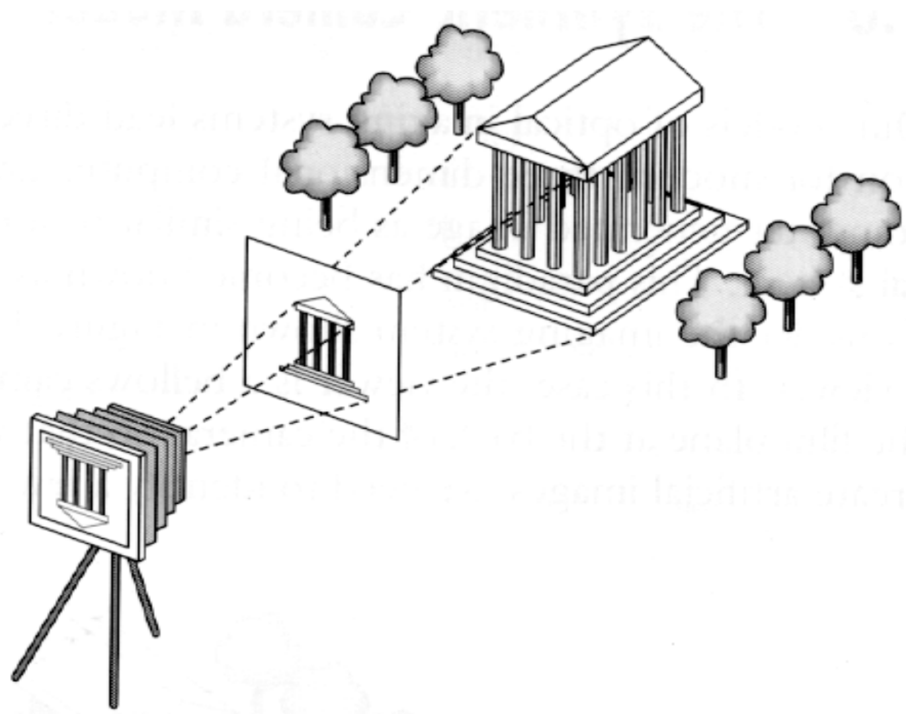
The 3D synthetic camera model



The **synthetic camera model involves two components,** specified *independently:*

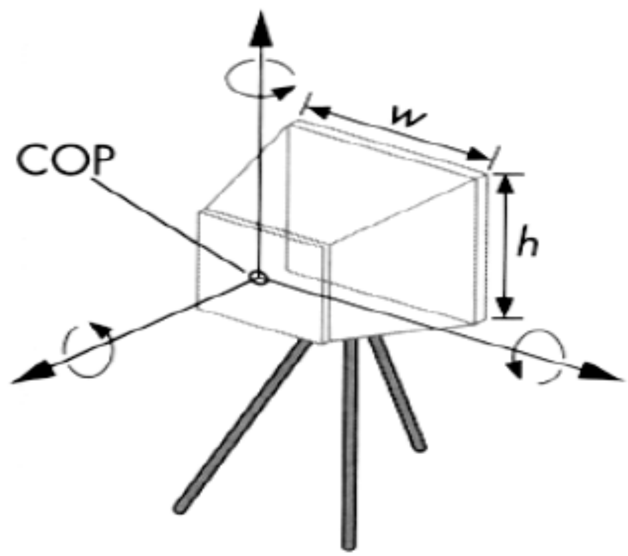
- objects (a.k.a. **geometry)**
- viewer (a.k.a. **camera)**

Imaging with the synthetic camera



The image is rendered onto an **image plane or projection plane (usually** in front of the camera).
Rays emanate from the center of projection (COP) at the center of the lens (or pinhole).
The image of an object point *P is at the intersection of the ray* through *P and the image plane.*

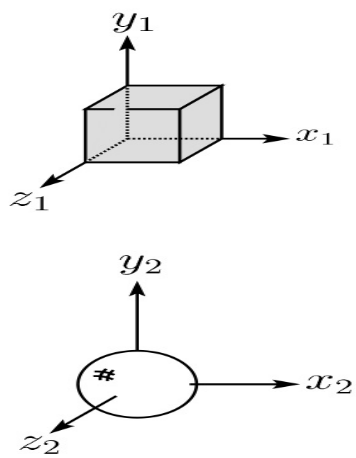
Specifying a viewer



Camera specification requires four kinds of parameters:

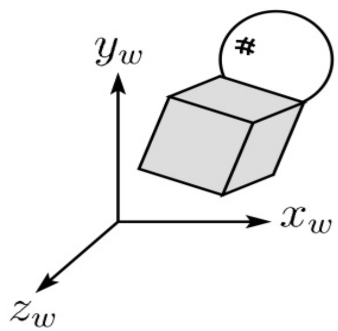
- *Position: the COP.*
- *Orientation: rotations about axes with origin at the COP.*
- *Focal length: determines the size of the image on the film plane, or* the **field of view.**
- *Film plane: its width and height.*

3D Geometry Pipeline



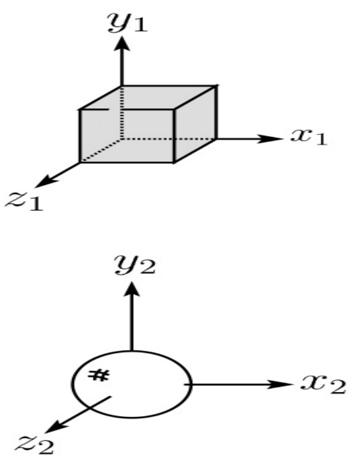
Model Space
(Object Space)

Rotation
Translation
Resizing



World Space

3D Geometry Pipeline



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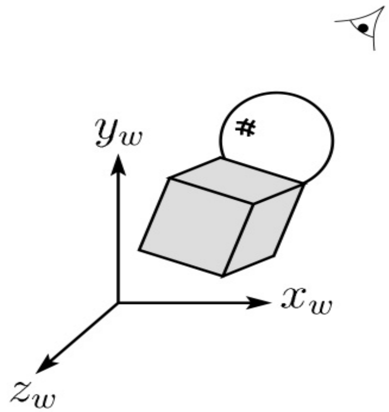
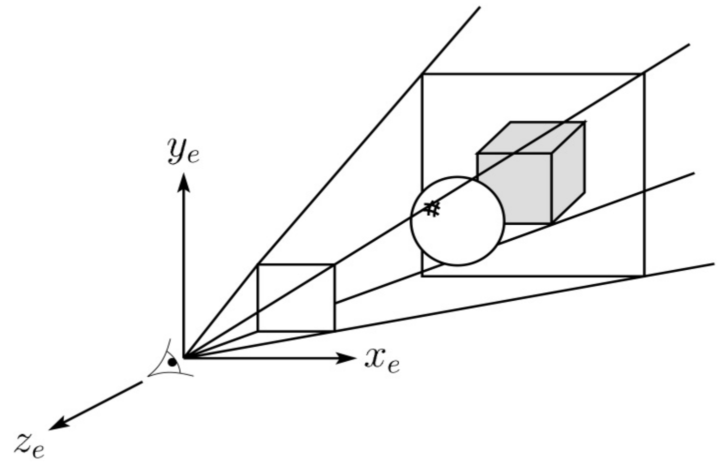


World Space

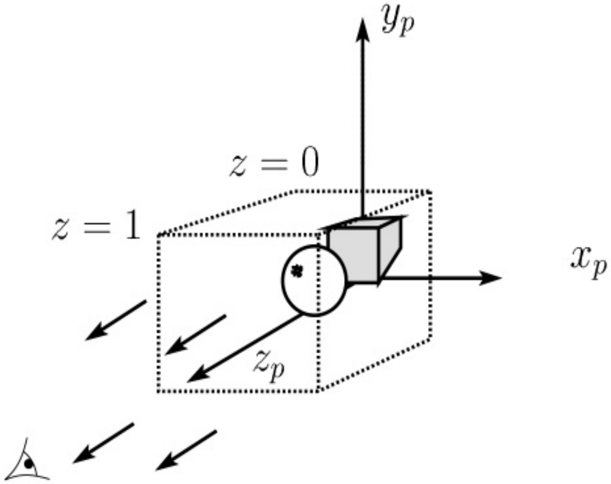
Rotation
Translation

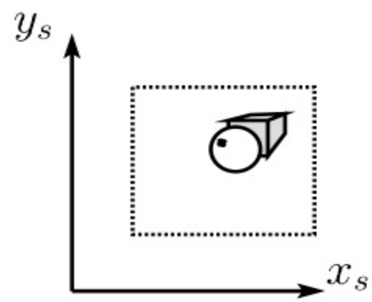


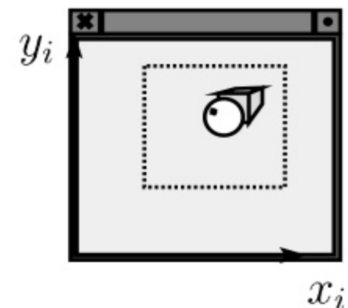
Eye Space
(View Space)

Projective,
Scale
Translation

3D Geometry Pipeline (cont'd)







Normalized Projection Space

Project

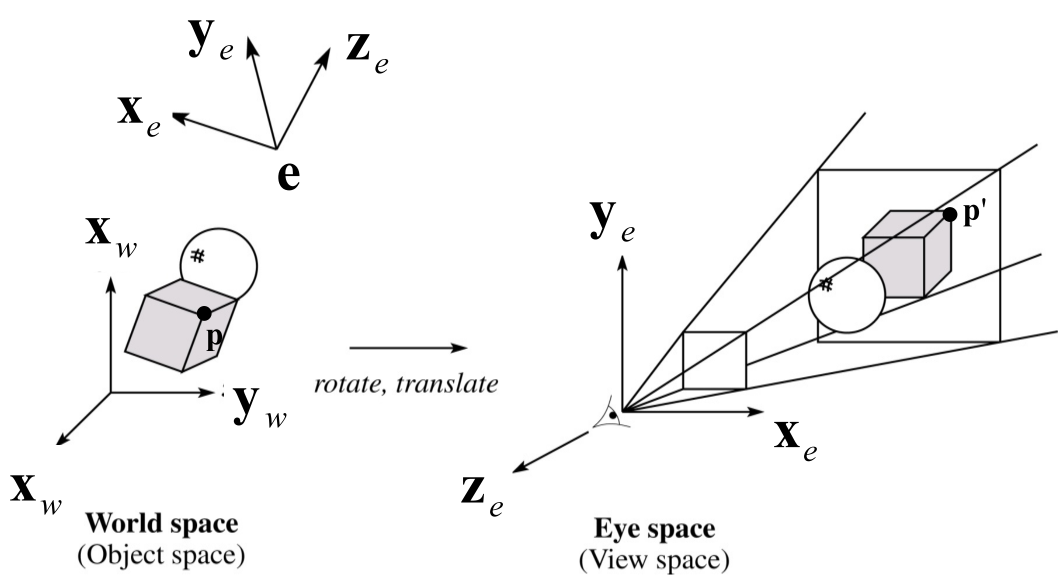
Screen Space (2D)

Scale
Translation

Image Space (pixels)
Raster Space

World -> eye transformation

- Let's look at how we would compute the world->eye transformation.



$$\mathbf{p}' = ?$$

$$x_e = \mathbf{x}_e \bullet \mathbf{p}\,?$$

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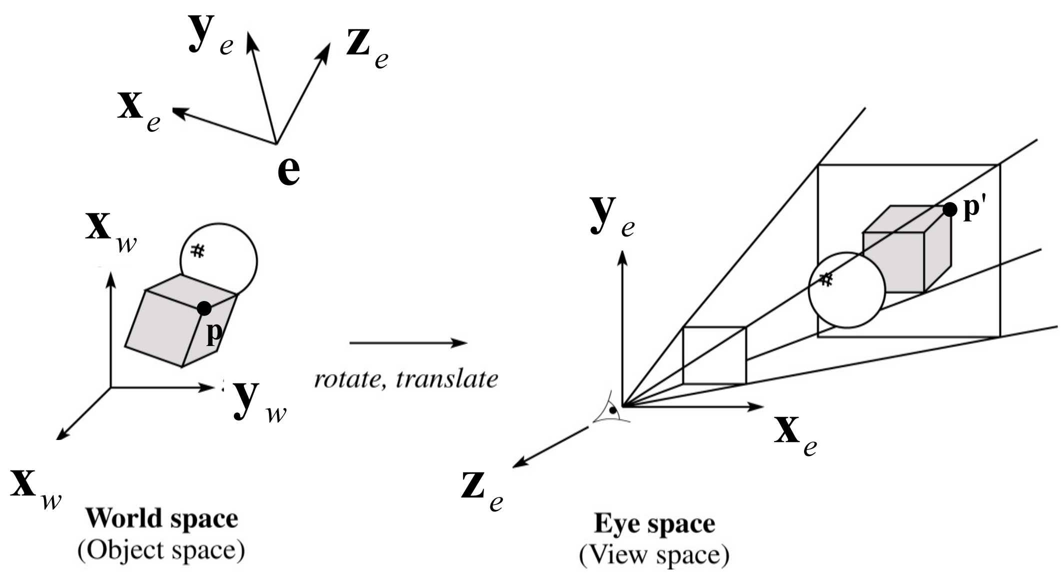


$$\mathbf{p}' = ?$$

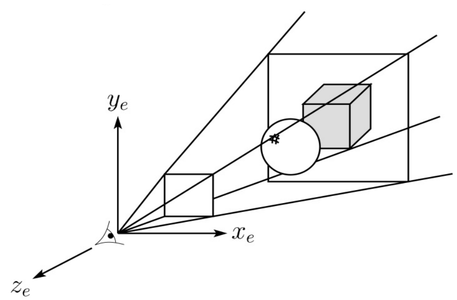
$$x_e = \mathbf{x}_e \bullet (\mathbf{p} - \mathbf{e})$$

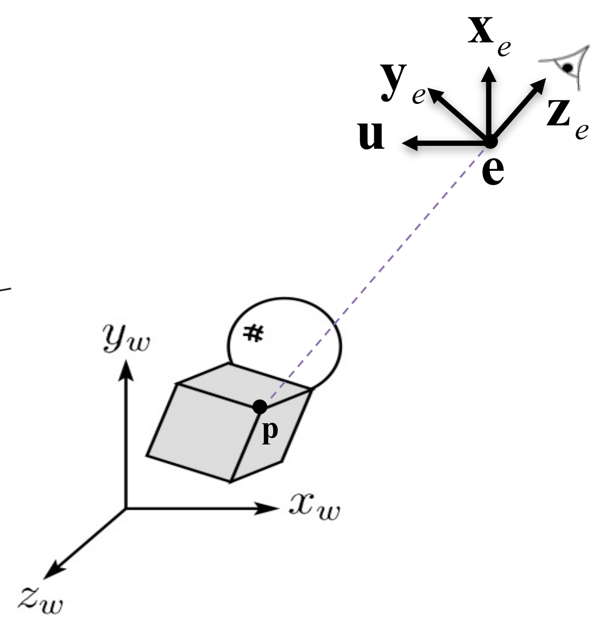
$$y_e = \mathbf{y}_e \bullet (\mathbf{p} - \mathbf{e})$$

$$z_e = \mathbf{z}_e \bullet (\mathbf{p} - \mathbf{e})$$

$$\mathbf{M}_v = \begin{bmatrix} - & \mathbf{x}_e^{\mathrm{T}} & - & 0 \\ - & \mathbf{y}_e^{\mathrm{T}} & - & 0 \\ - & \mathbf{z}_e^{\mathrm{T}} & - & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & | \\ 0 & 1 & 0 & -\mathbf{e} \\ 0 & 0 & 1 & | \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

How to specify eye coordinate?



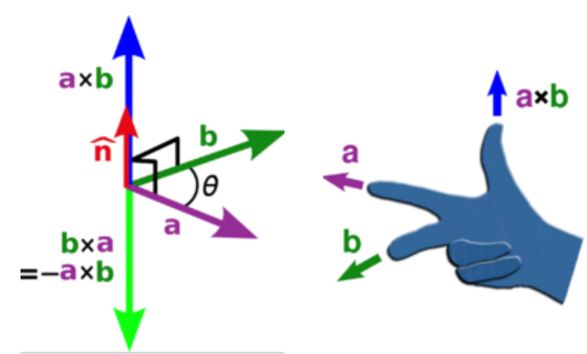


$$\mathbf{z}_e = -\frac{\mathbf{p}-\mathbf{e}}{|\mathbf{p}-\mathbf{e}|}$$

$$\mathbf{x}_e = \frac{\mathbf{u}\times\mathbf{z}_e}{|\mathbf{u}\times\mathbf{z}_e|}$$

$$\mathbf{y}_e = \frac{\mathbf{z}_e\times\mathbf{x}_e}{|\mathbf{z}_e\times\mathbf{x}_e|}$$

1. Give eye location e
2. Give target position p
3. Give upward direction u

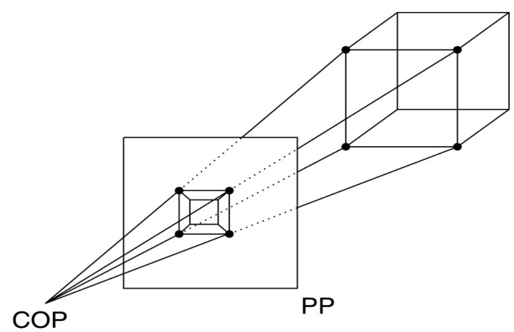


OpenGL has a helper command:

gluLookAt (*eyex*, *eyey*, *eyez*, *px*, *py*, *pz*, *upx*, *upy*, *upz*)

Projections

- **Projections** transform points in *n*-space to *m*-space, where *m*<*n*.
- In 3-D, we map points from 3-space to the **projection plane** (PP) (a.k.a., image plane) along **rays** emanating from the center of projection (COP):



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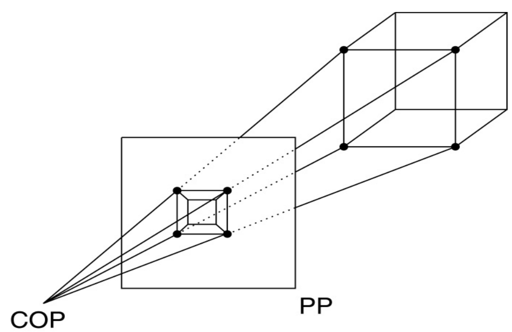

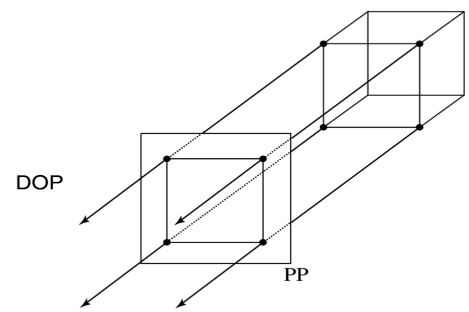

- There are two basic types of projections:
  - Perspective – distance from COP to PP finite
  - Parallel – distance from COP to PP infinite

Parallel projections

- For parallel projections, we specify a **direction of projection** (DOP) instead of a COP.
- We can write orthographic projection onto the *z=0* plane with a simple matrix, such that x'=x, y'=y.

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \qquad\qquad \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

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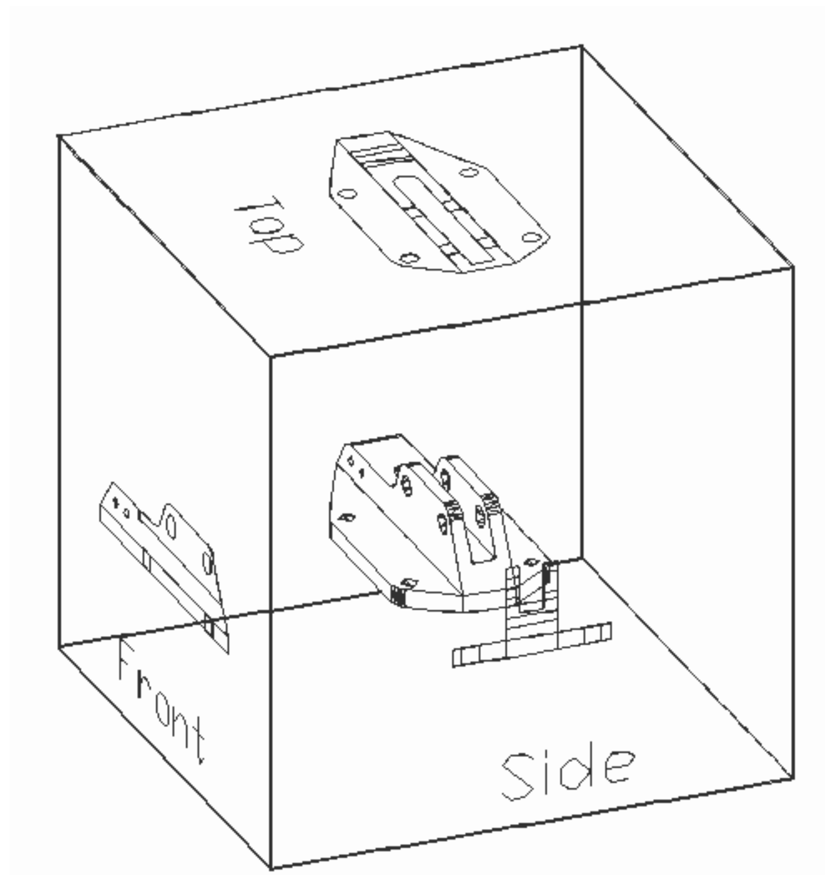
- Normally, we do not drop the z value right away. Why not?

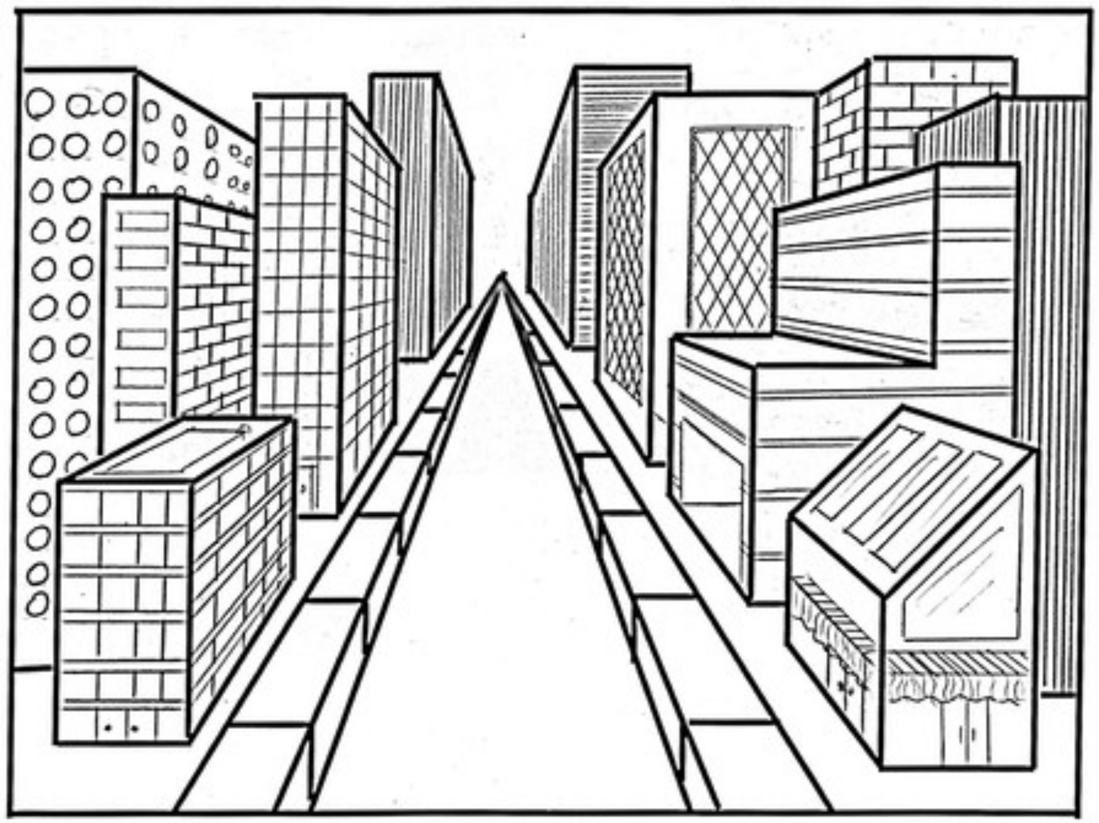
Properties of parallel projection

- Properties of parallel projection:
  - Are actually a kind of affine transformation
    - Parallel lines remain parallel
    - Ratios are preserved
    - Angles not (in general) preserved
  - Not realistic looking
  - Good for exact measurements,
    Most often used in
    - CAD,
    - architectural drawings,
    - etc.,

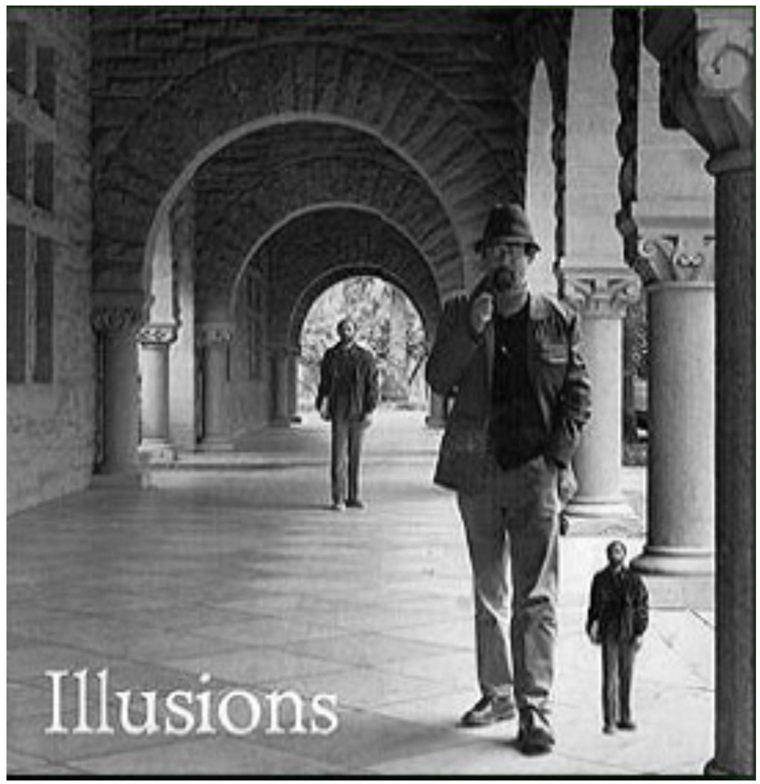
    where taking exact measurement
    is important



Perspective effect



Perspective Illusion



Perspective Illusion