Optical Flow in the Presence of Spatially-Varying Motion Blur: Algorithm Details

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1 Blurred image derivatives

Recall from the main text that the observed images, I_1 and I_2, are blurred by each other's spatially-varying kernels:

$$B_1(\mathbf{x}) = (I_1 * k_2)(\mathbf{x}), \tag{1}$$
$$B_2(\mathbf{x}) = (I_2 * k_1)(\mathbf{x}), \tag{2}$$

where k_1 and k_2 are the motion blur kernels at the corresponding points in I_1 and I_2, respectively. In effect, we have

$$B_1(\mathbf{x}) = \left[I_1 * k_2\left(\mathbf{x} + \mathbf{w}(\mathbf{x})\right)\right](\mathbf{x}), \tag{3}$$
$$B_2\left(\mathbf{x} + \mathbf{w}(\mathbf{x})\right) = \left[I_2 * k_1(\mathbf{x})\right]\left(\mathbf{x} + \mathbf{w}(\mathbf{x})\right). \tag{4}$$

Note that we use the function notation with the kernels to mean the particular kernel at that location in the image, not the value of the element at that position in the kernel.

We are interested in obtaining the derivatives of the blurred images with respect to the current flow estimate $\mathbf{w} = (u, v)$. To distinguish this flow from the precomputed baseline flows to neighboring frames I_0 and I_3, we will replace the preceding notation with $\mathbf{w}_{12} = (u_{12}, v_{12})$ while similarly defining $\mathbf{w}_{10}$ and $\mathbf{w}_{23}$ for the baseline flows.

Recall that we use the following piecewise linear blur kernel approximations:

$$k_1(\mathbf{x}) = \tfrac{1}{2}\left(k_{\mathbf{w}_{10}(\mathbf{x})\tau_1/2} + k_{\mathbf{w}_{12}(\mathbf{x})\tau_1/2}\right), \tag{5}$$
$$k_2(\mathbf{x} + \mathbf{w}_{12}(\mathbf{x})) = \tfrac{1}{2}\left(k_{-\mathbf{w}_{12}(\mathbf{x})\tau_2/2} + k_{\mathbf{w}_{23}(\mathbf{x}+\mathbf{w}_{12}(\mathbf{x}))\tau_2/2}\right), \tag{6}$$

where τ_i is the duty cycle of frame i and $k_{\mathbf{w}}$ denotes a blur kernel consisting of a line segment from the origin to $\mathbf{w}$. Since each estimated kernel is composed of two parts, we write

$$B_1(\mathbf{x}) = \tfrac{1}{2}\left[B_{11}(\mathbf{x}) + B_{12}(\mathbf{x})\right], \tag{7}$$
$$B_2\left(\mathbf{x} + \mathbf{w}(\mathbf{x})\right) = \tfrac{1}{2}\left[B_{21}\left(\mathbf{x} + \mathbf{w}(\mathbf{x})\right) + B_{22}\left(\mathbf{x} + \mathbf{w}(\mathbf{x})\right)\right], \tag{8}$$

where

$$B_{11}(\mathbf{x}) = \left(I_1 * k_{-\mathbf{w}_{12}(\mathbf{x})\tau_2/2}\right)(\mathbf{x}), \tag{9}$$
$$B_{12}(\mathbf{x}) = \left(I_1 * k_{\mathbf{w}_{23}(\mathbf{x}+\mathbf{w}_{12}(\mathbf{x}))\tau_2/2}\right)(\mathbf{x}), \tag{10}$$
$$B_{21}\left(\mathbf{x} + \mathbf{w}(\mathbf{x})\right) = \left(I_2 * k_{\mathbf{w}_{10}(\mathbf{x})\tau_1/2}\right)\left(\mathbf{x} + \mathbf{w}(\mathbf{x})\right), \tag{11}$$
$$B_{22}\left(\mathbf{x} + \mathbf{w}(\mathbf{x})\right) = \left(I_2 * k_{\mathbf{w}_{12}(\mathbf{x})\tau_1/2}\right)\left(\mathbf{x} + \mathbf{w}(\mathbf{x})\right). \tag{12}$$

For the first part of B_1, we have

$$\left.\frac{\partial B_{11}}{\partial u_{12}}\right|_{\mathbf{x}} = \left(I_1 * \frac{\partial}{\partial u_{12}} k_{-\mathbf{w}_{12}(\mathbf{x})\tau_2/2}\right)(\mathbf{x})$$
$$= -\frac{\tau_2}{2}\left[\left.\frac{\partial(I_1 * k_{\mathbf{w}})}{\partial u}\right|_{\mathbf{w}=-\mathbf{w}_{12}(\mathbf{x})\tau_2/2}\right]_{\mathbf{x}}. \tag{13}$$

The derivative with respect to v_{12} can be obtained by replacing ∂u with ∂v. The second part of B_2 is more complicated since we are warping the baseline flow $\mathbf{w}_{23}$ based on the current flow estimate:

$$\left.\frac{\partial B_{12}}{\partial u_{12}}\right|_{\mathbf{x}} = \left(I_1 * \frac{\partial}{\partial u_{12}} k_{\mathbf{w}_{23}(\mathbf{x}+\mathbf{w}_{12}(\mathbf{x}))\tau_2/2}\right)(\mathbf{x})$$
$$= \left(I_1 * \left[\frac{\partial}{\partial u_{23}} k_{\mathbf{w}_{23}(\mathbf{x}+\mathbf{w}_{12}(\mathbf{x}))\tau_2/2}\left[\frac{\partial u_{23}}{\partial x}\right]_{\mathbf{x}+\mathbf{w}_{12}(\mathbf{x})}\right.\right.$$
$$\left.\left.+\frac{\partial}{\partial v_{23}} k_{\mathbf{w}_{23}(\mathbf{x}+\mathbf{w}_{12}(\mathbf{x}))\tau_2/2}\left[\frac{\partial v_{23}}{\partial x}\right]_{\mathbf{x}+\mathbf{w}_{12}(\mathbf{x})}\right]\frac{\partial x}{\partial u_{12}}\right)(\mathbf{x})$$
$$= \frac{\tau_2}{2}\left\langle\left[\nabla_{\mathbf{w}}\left(I_1 * k_{\mathbf{w}}\right)\Big|_{\mathbf{w}=\mathbf{w}_{23}(\mathbf{x}+\mathbf{w}_{12}(\mathbf{x}))\tau_2/2}\right]_{\mathbf{x}}, \left[\frac{\partial \mathbf{w}_{23}}{\partial x}\right]_{\mathbf{x}+\mathbf{w}_{12}(\mathbf{x})}\right\rangle, \tag{14}$$

where $\langle\cdot,\cdot\rangle$ denotes the inner-product and $\nabla_{\mathbf{w}}$ denotes the gradient with respect to u and v. The derivative with respect to v_{12} can be obtained by replacing ∂x with ∂y. The derivatives of B_{21} and B_{22} have forms similar to Eq. (13):

$$\left.\frac{\partial B_{21}}{\partial u_{12}}\right|_{\mathbf{x}+\mathbf{w}(\mathbf{x})} = \frac{\tau_1}{2}\left[\left.\frac{\partial(I_2 * k_{\mathbf{w}})}{\partial u}\right|_{\mathbf{w}=\mathbf{w}_{10}(\mathbf{x})\tau_1/2}\right]_{\mathbf{x}+\mathbf{w}(\mathbf{x})}, \tag{15}$$

and

$$\left.\frac{\partial B_{22}}{\partial u_{12}}\right|_{\mathbf{x}+\mathbf{w}(\mathbf{x})} = \frac{\tau_1}{2}\left[\left.\frac{\partial(I_2 * k_{\mathbf{w}})}{\partial u}\right|_{\mathbf{w}=\mathbf{w}_{12}(\mathbf{x})\tau_1/2}\right]_{\mathbf{x}+\mathbf{w}(\mathbf{x})}. \tag{16}$$

Again, the derivatives with respect to v_{12} are obtained by replacing ∂u with ∂v. Terms of the form $\partial(I * k_{\mathbf{w}})/\partial u$ and $\nabla_{\mathbf{w}}(I * k_{\mathbf{w}})$ are evaluated from the blurred image grids using finite differences.

2 Grid construction

Suppose valid optical flows are in the range $u \in [u_{\min}, u_{\max}]$ and $v \in [v_{\min}, v_{\max}]$. We compute these ranges from the baseline flow fields. For simpler notation, assume the duty cycles of the frames in question are one so that we do not need to scale the flows to get the blur kernels. Our goal is to construct a two-dimensional regular grid covering the given range. We place a few constraints on the grid:

1. The total number of grid points must not exceed one.
2. Grid spacing must be at least one pixel in both directions.

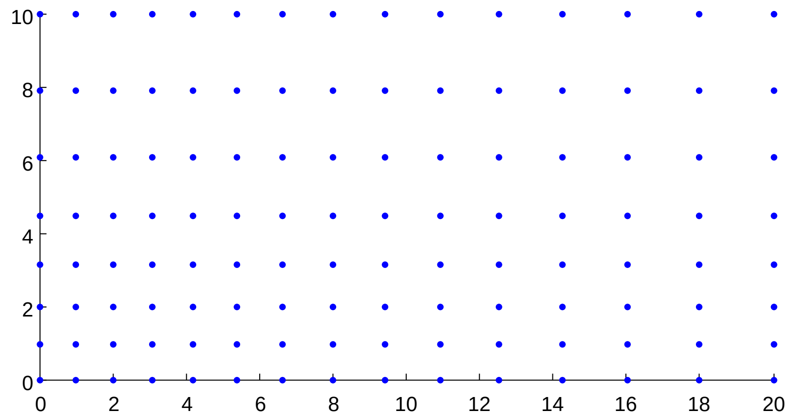

Figure 1: An example of the spacing used for our blurred image grids.

3. If one of the dimensions has both positive and negative values, then 0 must be one of the grid values for that dimension. This guarantees that we never interpolate between two images blurred in opposite directions.

Let n be the number of points along the u-axis and m be the number of points along the v-axis. The first constraint is $n \cdot m \leq M$. We set

$$n = \left\lfloor \min(M, \max(1, \sqrt{M \cdot w/h})) \right\rfloor, \tag{17}$$

$$m = \left\lfloor \frac{M}{n} \right\rfloor, \tag{18}$$

where $w = u_{\max} - u_{\min}$ and $h = v_{\max} - v_{\min}$.

As mentioned in the main text, we would like the grid spacing to be smaller closer to the origin. We will show how the grid spacing is determined for the u-axis (the spacing for the v-axis is similarly determined). For simplicity, assume that $u_{\min} > 0$ and $0 < n - 1 < u_{\max} - u_{\min}$.[1] We found that the following recurrence relation provides good spacing near the origin without sacrificing too much accuracy away from the origin:

$$u_{i+1} = u_i + \beta(u_i - u_{i-1}), \quad \text{for} \quad i > 1, \tag{19}$$

where $u_1 = u_{\min}$, $u_2 = u_1 + 1$, $u_n = u_{\max}$, and $\beta > 1$. To determine β, we solve the following equation:

$$u_n = u_1 + \sum_{i=0}^{n-2} \beta^i = u_1 + \frac{1 - \beta^{n-1}}{1 - \beta}. \tag{20}$$

This gives

$$\beta^{n-1} + (u_1 - u_n)\beta + u_n - u_1 - 1 = 0, \tag{21}$$

where the largest real root is the desired value of β. An example of the spacing produced by this method is shown in Figure 1.

[1] If $u_{\min}$ is negative, we can compute the spacing for negative values seperately in an equivalent manner. If $n - 1 \geq u_{\max} - u_{\min}$, then we just use a one pixel spacing between all the points.