Vector spaces

A vector space is an abstract set of objects that can be added together and scaled according to a specific set of axioms. The notion of "scaling" is addressed by the mathematical object called a field. Most commonly, the field we use are the real numbers $\mathbb{R}$. That is, vectors in some vector space $\mathcal{V}$ defined on a field $\mathcal{F}$ can interact with both other vectors in $\mathcal{V}$ as well as with elements in the field F. In order for $(\mathcal{V}, \mathcal{F})$ to be a vector space it must follow the axioms set in Definition 1.

Definition 1 *Given a set of objects $\mathcal{V}$ called vectors and a field $\mathcal{F} := (C, +, \cdot, -, ^{-1}, 0, 1)$ where C is the set of elements in the field, called scalars, the tuple $(\mathcal{V}, \mathcal{F})$ is a **vector space** if for all $\mathbf{v}, \mathbf{u}, \mathbf{w} \in \mathcal{V}$ and $c, d \in C$, it obeys the following ten axioms:*

1. $\mathbf{u} + \mathbf{v} \in \mathcal{V}$
2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
3. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
4. *There exists a zero vector $\mathbf{0} \in \mathcal{V}$ such that $\mathbf{u} + \mathbf{0} = \mathbf{u}$*
5. *For each $\mathbf{u} \in \mathcal{V}$ there exists a $\mathbf{u}' \in \mathcal{V}$ such that $\mathbf{u} + \mathbf{u}' = \mathbf{0}$. We call $\mathbf{u}'$ the negative of $\mathbf{u}$ and denote it as $-\mathbf{u}$*
6. *The scalar multiple of $\mathbf{u}$ by c, denoted by $c\mathbf{u}$ is in $\mathcal{V}$*
7. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$
8. $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$
9. $c(d\mathbf{u}) = (cd)\mathbf{u}$
10. $1\mathbf{u} = \mathbf{u}$

Properties

The zero vector is unique

There is only one zero-vector in a vector space. This does not need to be an axiom because it can be proven from the current 10 axioms as is done in the following theorem:

Theorem 1 *Given vector space $(\mathcal{V}, \mathcal{F})$, the zero vector is unique.*

Proof: Assume for the sake of contradiction that there exists a vector $\mathbf{a}$ such that $\mathbf{a} \neq \mathbf{0}$ and that $\forall \mathbf{v} \in \mathcal{V}$

$$\mathbf{a} + \mathbf{v} = \mathbf{v}$$

Then, this implies that :

$$\mathbf{a} + \mathbf{0} = \mathbf{0}$$

However, axiom 4 states that for the zero-vector

$$\mathbf{a} + \mathbf{0} = \mathbf{a}$$

Since $\mathbf{a} \neq \mathbf{0}$, we reach a contradiction. Therefore, there does not exist a vector $\mathbf{a} \neq \mathbf{0}$ for which $\forall \mathbf{v} \in \mathcal{V} \quad \mathbf{a} + \mathbf{v} = \mathbf{v}$. Thus, the zero-vector is unique.

□

Any vector multiplied by the zero scalar is the zero vector

The zero scalar multiplied by any vector produces the zero vector.

Theorem 2 *Given a vector space $(\mathcal{V}, \mathcal{F})$*

$$\forall \mathbf{v} \in \mathcal{V}, 0\mathbf{v} = \mathbf{0}$$

Proof: Assume for the sake of contradiction that there exists a vector $\mathbf{a} \neq \mathbf{0}$ such that

$$0\mathbf{v} = \mathbf{a}$$

Now, for any scalar $c \neq 0$, we have

$$\begin{aligned} c\mathbf{v} &= (c + 0)\mathbf{v} \\ &= c\mathbf{v} + 0\mathbf{v} && \text{by axiom 8} \\ &= c\mathbf{v} + \mathbf{a} \end{aligned}$$

Our assumption assumed that $\mathbf{a} \neq \mathbf{0}$ must be false because by Theorem 1 the only vector $\mathbf{a}$ for which $c\mathbf{v} + \mathbf{a} = c\mathbf{v}$ would be true is the zero-vector.

□

The negative vector of a given vector is unique

Each vector in a vector space has only one corresponding negative vector.

Theorem 3 *Given a vector space* $(\mathcal{V}, \mathcal{F})$ *and vector* $\mathbf{v} \in \mathcal{V}$*, its negative,* $-\mathbf{v}$*, is unique. That is,*

$$\mathbf{v} + \mathbf{a} = \mathbf{0} \iff \mathbf{a} = -\mathbf{v}$$

Proof:

We need only prove $\mathbf{v} + \mathbf{a} = \mathbf{0} \implies \mathbf{a} = -\mathbf{v}$. The other direction is stated in the axioms.

$$\begin{aligned} & \mathbf{v} + \mathbf{a} = \mathbf{0} & \\ \implies & -\mathbf{v} + \mathbf{v} + \mathbf{a} = -\mathbf{v} + \mathbf{0} & \\ \implies & [-\mathbf{v} + \mathbf{v}] + \mathbf{a} = -\mathbf{v} & \\ \implies & \mathbf{0} + \mathbf{a} = -\mathbf{v} & \text{by axiom 5} \\ \implies & \mathbf{a} = -\mathbf{v} & \text{by axiom 4} \end{aligned}$$

□

The negative of a vector is the vector multiplied by -1

The negative of a vector is obtained by multiplying it by the scalar -1.

Theorem 4 *Given a vector* $\mathbf{v} \in \mathcal{V}$*, it's negative is* $(-1)\mathbf{v}$*. That is,*

$$-\mathbf{v} = (-1)\mathbf{v}$$

Proof:

$$\begin{aligned} \mathbf{v} + (-1)\mathbf{v} &= (1)\mathbf{v} + (-1)\mathbf{v} && \text{by axiom 10} \\ &= (1-1)\mathbf{v} && \text{by axiom 8} \\ &= 0\mathbf{v} && \\ &= \mathbf{0} && \text{by Theorem 2} \end{aligned}$$

Then, by axiom 5, it must be that $(-1)\mathbf{v} = -\mathbf{v}$.

□

The zero vector multiplied by a scalar is the zero vector

The zero vector multiplied by any scalar yields the zero vector. This result says that the zero vector does not grow or shrink when multiplied by a scalar.

Theorem 5 *Given a vector space* $(\mathcal{V}, \mathcal{F})$

$$c\mathbf{0} = \mathbf{0} \iff \mathbf{a} = \mathbf{0}$$

Proof:

$$\begin{aligned} \mathbf{0} + \mathbf{0} &= \mathbf{0} && \text{by axiom 4} \\ c(\mathbf{0} + \mathbf{0}) &= c\mathbf{0} \\ c\mathbf{0} + c\mathbf{0} &= c\mathbf{0} && \text{by axiom 8} \end{aligned}$$

By Theorem 1, the only vector $\mathbf{a}$ in $\mathcal{V}$ for which $\mathbf{a} + \mathbf{v} = \mathbf{v}$ for all vectors $\mathbf{v} \in \mathcal{V}$ is the zero vector $\mathbf{0}$. Thus, $c\mathbf{0} = \mathbf{0}$.

□

The only vector whose negative is not distinct from itself is the zero vector

The only vector whose negative is not distinct from itself is the zero vector. This result says that all vectors besides the zero vector must be added to a unique and different *different* vector to yield the zero vector.

Theorem 6 *Given a vector space* $(\mathcal{V}, \mathcal{F})$

$$-\mathbf{0} = \mathbf{0}$$

Proof:

$$\begin{aligned} \mathbf{a} + -\mathbf{a} &= \mathbf{0} && \text{by axiom 5} \\ \mathbf{a} + \mathbf{a} &= \mathbf{0} && \text{assume } \mathbf{a} = -\mathbf{a} \\ \implies 2\mathbf{a} &= \mathbf{0} \\ \implies \mathbf{a} &= \mathbf{0} && \text{by Theorem 5} \end{aligned}$$

Thus, if we assume $\mathbf{a} = -\mathbf{a}$, then $\mathbf{a}$ must be the zero vector.

□

Vector Subspaces

A vector space can be induced by an appropriate subset of vectors from some larger vector space. We call such a subspace a **vector subspace**. By merits of the original vector space, seven out of 10 axioms will always hold; however, there are three axioms that may not hold that must be verified whenever a subset of vectors from a vector space are to considered as a vector space in their own right:

Definition 2 *A subset of vectors $H \subseteq \mathcal{V}$ from a vector space $(\mathcal{V}, \mathcal{F})$ forms a* ***vector subspace*** *if the following three properties hold:*

1. *The zero vector of $\mathcal{V}$ is in H.*
2. *H is closed under vector addition. That is, for each $\mathbf{u}, \mathbf{v} \in H$, the vector $\mathbf{u} + \mathbf{v}$ is also in H.*
3. *H is closed under scalar multiplication. That is, for each $\mathbf{u} \in H$ and each scalar $c \in C$, the vector $c\mathbf{u}$ is in H.*