The Radial Basis Function Kernel

The **Radial basis function kernel**, also called the **RBF kernel**, or **Gaussian kernel**, is a kernel that is in the form of a radial basis function (more specifically, a Gaussian function). The RBF kernel is defined as

$$K_{\text{RBF}}(\mathbf{x}, \mathbf{x}') = \exp\left[-\gamma \|\mathbf{x} - \mathbf{x}'\|^2\right]$$

where γ is a parameter that sets the "spread" of the kernel.

The RBF kernel as a projection into infinite dimensions

Recall a kernel is any function of the form:

$$K(\mathbf{x}, \mathbf{x}') = \langle \psi(\mathbf{x}), \psi(\mathbf{x}') \rangle$$

where ψ is a function that projections vectors $\mathbf{x}$ into a new vector space. The kernel function computes the inner-product between two projected vectors.

As we prove below, the ψ function for an RBF kernel projects vectors into an infinite dimensional space. For Euclidean vectors, this space is an infinite dimensional Euclidean space.

That is, we prove that

$$\psi_{\text{RBF}} : \mathbb{R}^n \to \mathbb{R}^\infty$$

Proof:

Without loss of generality, let $\gamma = \frac{1}{2}$.

$$K_{\text{RBF}}(\mathbf{x}, \mathbf{x}') = \exp\left[-\frac{1}{2}\left\|\mathbf{x} - \mathbf{x}'\right\|^2\right]$$

$$= \exp\left[-\frac{1}{2}\langle\mathbf{x} - \mathbf{x}', \mathbf{x} - \mathbf{x}'\rangle\right]$$

$$= \exp\left[-\frac{1}{2}(\langle\mathbf{x}, \mathbf{x} - \mathbf{x}'\rangle - \langle\mathbf{x}', \mathbf{x} - \mathbf{x}'\rangle)\right]$$

$$= \exp\left[-\frac{1}{2}(\langle\mathbf{x}, \mathbf{x} - \mathbf{x}'\rangle - \langle\mathbf{x}', \mathbf{x} - \mathbf{x}'\rangle)\right]$$

$$= \exp\left[-\frac{1}{2}(\langle\mathbf{x}, \mathbf{x}\rangle - \langle\mathbf{x}, \mathbf{x}'\rangle - \langle\mathbf{x}', \mathbf{x}\rangle + \langle\mathbf{x}', \mathbf{x}'\rangle)\right]$$

$$= \exp\left[-\frac{1}{2}(\|\mathbf{x}\|^2 + \|\mathbf{x}'\|^2 - 2\langle\mathbf{x}, \mathbf{x}'\rangle)\right]$$

$$= \exp\left[-\frac{1}{2}\|\mathbf{x}\|^2 - \frac{1}{2}\|\mathbf{x}'\|^2\right]\exp\left[-\frac{1}{2} - 2\langle\mathbf{x}, \mathbf{x}'\rangle\right]$$

$$= Ce^{\langle\mathbf{x},\mathbf{x}'\rangle} \qquad C := \exp\left[-\frac{1}{2}\|\mathbf{x}\|^2 - \frac{1}{2}\|\mathbf{x}'\|^2\right] \text{ is a constant}$$

$$= C\sum_{n=0}^{\infty}\frac{\langle\mathbf{x}, \mathbf{x}'\rangle^n}{n!} \qquad \text{Taylor expansion of } e^x$$

$$= C\sum_{n=0}^{\infty}\frac{K_{\text{poly(n)}}(\mathbf{x}, \mathbf{x}')}{n!}$$

We see that the RBF kernel is formed by taking an infinite sum over polynomial kernels.

As proven previously, recall that the sum of two kernels

$$K_c(\mathbf{x}, \mathbf{x}') := K_a(\mathbf{x}, \mathbf{x}') + K_b(\mathbf{x}, \mathbf{x}')$$

implies that the ψ_c function is defined so that it forms vectors of the form

$$\psi_c(\mathbf{x}) := (\psi_a(\mathbf{x}), \psi_b(\mathbf{x}))$$

That is, the vector $\psi_c(\mathbf{x})$ is a tuple where the first element of the tuple is the vector $\psi_a(\mathbf{x})$ and the second element is $\psi_b(\mathbf{x})$. The inner-product on the vector space of ψ_c is defined as

$$\langle \psi_c(\mathbf{x}), \psi_c(\mathbf{x}') \rangle := \langle \psi_a(\mathbf{x}), \psi_a(\mathbf{x}') \rangle + \langle \psi_b(\mathbf{x}), \psi_b(\mathbf{x}') \rangle$$

For Euclidean vector spaces, this means that $\psi_c(\mathbf{x})$ is the vector formed by appending the elements of $\psi_b(\mathbf{x})$ onto the $\psi_a(\mathbf{x})$ and that

$$\begin{aligned}
\langle \psi_c(\mathbf{x}), \psi_c(\mathbf{x}') \rangle &:= \sum_i^{\text{dimension}(a)} \psi_{a,i}(\mathbf{x})\psi_{a,i}(\mathbf{x}') + \sum_j^{\text{dimension}(b)} \psi_{b,j}(\mathbf{x})\psi_{b,j}(\mathbf{x}') \\
&= \sum_i^{\text{dimension}(a)+\text{dimension}(b)} \psi_{c,i}(\mathbf{x})\psi_{c,i}(\mathbf{x}')
\end{aligned}$$

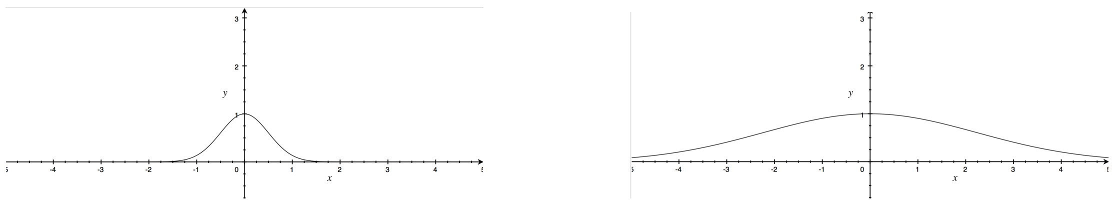
Since the RBF is an infinite sum over such appendages of vectors, we see that the projections is into a vector space with infinite dimension.

□

The γ parameter

Recall a kernel expresses a measure of similarity between vectors. The RBF kernel represents this similarity as a decaying function of the distance between the vectors (i.e. the squared-norm of their distance). That is, if the two vectors are close together then, $||\mathbf{x} - \mathbf{x}'||$ will be small. Then, so long as $\gamma > 0$, it follows that $-\gamma\, ||\mathbf{x} - \mathbf{x}'||^2$ will be larger. Thus, closer vectors have a larger RBF kernel value than farther vectors. This function is of the form of a bell-shaped curve.

The γ parameter sets the width of the bell-shaped curve. The larger the value of γ the narrower will be the bell. Small values of γ yield wide bells. This is illustrated in Figure 1.



(a) **(b)**

Figure 1: (a) Large γ. (b) Small γ.