Observations regarding matchings of snake graphs and related phenomena

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1 A combinatorial interpretation of A and B

Let

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

and

$$B = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}.$$

We see that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} A = \begin{pmatrix} a+b & a \\ c+d & c \end{pmatrix}$$

and

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} B = \begin{pmatrix} b & a+b \\ d & c+d \end{pmatrix}.$$

In other words, A puts into the left column a sum of the left and right columns, and puts into the right column a copy of the old left column for future summation. We can call the left column the "accumulation column" and the right column the "summand column"; for B the opposite is true. On the other hand, the two rows never intermix. We can make this even more obvious by beginning every chain of multiplications with the base matrix

$$C = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}.$$

Then the bottom row is always empty, so we don't even need to worry about it. The sum still works out to what it was before. Thus, the bottom row is merely an old copy of the top row, so in some sense, since the individual entries are unnecessary, they are not individually meaningful.

What does switching matrices do exactly? We switch the interpretation of our columns; now what once was our accumulation column becomes our addition column, and vice versa. But with the switching of this interpretations comes a price. If we repeatedly multiply the same matrix, the summand column always holds the previous value of the accumulation matrix, but if we multiply a different matrix, the summand column's value remains the same, although the accumulation column dutifully accumulates the sum of both columns.

2 Cube snake recurrence

Paul and Abby have independently verified that the generating function for the straight cube snake is the following:

$$C(x) = \frac{2+3x-x^2}{1-3x-3x^2+x^3}.$$

This is represented in Sloane's Encyclopedia as A003697. The entry contains a note that the values alternate between squares and twice squares. When we de-multiplex the two sequences, we find:

$$c_n = \begin{cases} 2a^2_{n/2+1} & n \text{ is even} \\ b^2_{(n-1)/2+1} & n \text{ is odd} \end{cases}$$

where a_n is the number of spanning trees of a 2 by n grid, Sloane's A001353, and b_n is the number of perfect matchings of a 3 by $2n$ grid, Sloane's A001835. We call this the straight cube matching theorem.

2.1 Proof of the Straight Cube Matching Theorem

First, we note that both $A = (a_0, a_1, a_2, \ldots)$ and B have the same recurrence: $a_n = 4a_{n-1} - a_{n-2}$ and $b_n = 4b_{n-1} - b_{n-2}$. The only difference is that $a_0 = 0$ and $a_1 = 1$ versus $b_0 = b_1 = 1$. (Intriguingly, even though the b_n sequence begins "bigger," it is actually dominated by the a_n sequence for all $n > 0$.) Let's use this to find a homogeneous recurrence for a_n^2 (and equivalently for b_n^2):

$$\begin{aligned}
a_n &= 4a_{n-1} - a_{n-2} \\
a_n^2 &= (4a_{n-1} - a_{n-2})^2 \\
&= 16a_{n-1}^2 - 8a_{n-1}a_{n-2} + a_{n-2}^2 \\
&= 16a_{n-1}^2 - 8(4a_{n-2} - a_{n-3})a_{n-2} + a_{n-2}^2 \\
a_{n-1}^2 &= 16a_{n-2}^2 - 8a_{n-2}a_{n-3} + a_{n-3}^2 \\
a_n^2 + a_{n-1}^2 &= a_{n-3}^2 + a_{n-2}^2 + 16a_{n-1}^2 - 16a_{n-2}^2 \\
a_n^2 &= 15a_{n-1}^2 - 15a_{n-2}^2 + a_{n-3}^2
\end{aligned}$$

But actually, since we want to interleave these sequences, we really want the following recurrence: $a_n^2 = 15a_{n-2}^2 - 15a_{n-4}^2 + a_{n-6}^2$ This relation will hold true for both interleaved sequences and thus the sum of them, since doubling the entire sequence does not affect the recurrence. We now use Wilf's method to find the generating function for this combined interleaved sequence:

$$\begin{aligned}
-\frac{F(x) - (2 + 9x + 32x^2 + 121x^3 + 450x^4 + 1681x^5)}{x^6} &+ \\
15\frac{F(x) - (2 + 9x + 32x^2 + 121x^3)}{x^4} &+ \\
-15\frac{F(x) - (2 + 9x)}{x^2} &+ \\
F(x) &= 0
\end{aligned}$$

Solving, we find

$$F(x) = -\frac{x^2 - 3x - 2}{x^3 - 3x^2 - 3x + 1} = C(x),$$

as desired.

2.2 A-B Bijection

We would like to find a bijection between members of the interleaved sequence and members of C. To begin with, it may be helpful to unify A and B. They are related in the following way: $a_n = \sum_{i=1}^{n} b_n$ if $n \geq 1$. To prove this, first we show

$$a_n = 3a_{n-1} + 2a_{n-2} + 2a_{n-3} + \cdots + 2a_1 + 2a_0 + 1.$$

Examine a spanning tree T of a 2 by n grid graph, which we call G_n. We consider the subgraph of T induced by columns of G_n, starting from the right and working left, i.e., the rightmost k columns at a time, referring to the graph induced on k columns as T_k ($T_n = T$). If no T_k for $k < n$ is a spanning tree, we note that the graph must look like a giant letter C. If there were any vertical edges earlier than the far left edge, then if we consider the smallest T_k that contains a vertical edge, it must be a C-shaped spanning tree. Otherwise, T_k would not be connected, but this would imply that T is not connected, violating the assumption that it is a spanning tree. If there is such a vertical edge, examine the largest T_k (where $k < n$) is a spanning tree. If $k = n-1$, then we have three possibilities for T_n: either we add a $\mathsf{\Gamma}$ (the "Gamma" case), an L ("Ell"), or a $=$ ("Equals") to connect the last two unconnected vertices to the tree; these are the only three ways to connect them which will not add a loop. Otherwise, $k < n-1$. In that case, if we examine T_{k+1}, there will be one disconnected vertex, either the upper ("Upper") or lower ("Lower"). If both are disconnected, T cannot be connected. In either of these cases, the remainder of T must look like a giant C, but since there are two possible extensions to T_k to make it a spanning tree, for each spanning tree on T_k where $k < n-1$ there are two possible spanning trees on G_n.

Since we have just identified a way of breaking down any spanning tree on G_n into a smaller case, we have covered all cases. This means that if G_n has a_n spanning trees,

$$a_n = 3a_{n-1} + 2a_{n-2} + 2a_{n-3} + \cdots + 2a_1 + 2a_0 + 1.$$

From this, we can easily derive that $a_{n+1} = a_n + 3a_n - a_{n-1} = 4a_n - a_{n-1}$.

Now we go on to show the bijection. We prove this inductively. We assume we have a bijection for all cases through $n-1$. For the base case, since the starting values are both 1, this is trivial: the single spanning tree of the 2 by 1 grid corresponds to the single empty matching of the 3 by 0 grid.

We now show the bijection for the n case. We recall

$$a_n = 3a_{n-1} + 2a_{n-2} + 2a_{n-3} + \cdots + 2a_1 + 1.$$

The first term is different. This corresponds to taking all of the tilings we have which correspond to a_{n-1} and tacking onto the left two extra columns worth of irreducible tiling, of which there are three ways: two vertical dominos atop a horizontal domino, a horizontal domino atop two vertical dominos, or three horizontal dominos. We can see by exhausting all possibilities that these are the only tilings of the 2 by 3 rectangle, and that they are all irreducible. We will correspond these to the Gamma, Ell, and Equals cases, respectively, as the seams between the tiles resemble these shapes to some degree. The remaining a_k terms correspond to taking all of the tilings we have which correspond to a_k and tacking onto the left $2(n-k)$ irreducible tilings. These tilings have the following form: either the top two rows begin and end with a vertical domino and the rest is tiled with horizontal dominos, or the bottom two rows begin and end with a vertical domino and the rest is tiled with horizontal dominos. We will correspond these to the Upper and Lower cases, respectively, to correspond to the positions we placed the vertical dominos. To prove that these are the only two irreducible tilings we could use an argument as we did in problem 1(b) above, but instead we give a direct argument. Starting from the leftmost column, we note that the tiling cannot begin with three horizontal dominos, so it must either begin with a vertical domino stacked atop a horizontal domino, or vice versa. Examining the two rows taken up by the vertical domino, we can either put a vertical domino or two horizontal dominos. However, if we put a vertical domino, the tiling is not irreducible because there is now a break after the first two columns. Furthermore, if we put two horizontal dominos, we must now put a horizontal domino in the remaining row. The edge of the tiling is now the same as before, so we may repeat the argument all the way to the edge of the rectangle, where in order to finish off the tiling we must place a vertical domino in the final column, in the same two rows as the original vertical domino.

We still have one tree remaining. Since all of our constructions above involve adding dominos, this corresponds to the single construction which consists of no dominos: the empty tiling of the empty rectangle.

Since we always tack the irreducible tiling of the $2k$ rectangle onto the leftmost $2k$ columns, we can tell by looking at a tiling which case it corresponds to, by selecting the largest irreducible tiling which includes the leftmost column and then recursing on the remaining part of the tiling. (This could also be the empty tiling, which we have also covered.) Since every tiling fits this pattern, we have covered all possible tilings, and based on the recurrence we are following, we have also covered all possible spanning trees.

Since we have corresponded all of the first $n-1$ cases, and from this built a construction which corresponds the n case, we have completed the proof by induction.

2.3 C Bijection?