On Symmetric Circuits and FPC

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Context / Motivation

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Characterises first-order logic FO by uniform constant-depth poly-size symmetric Boolean circuits.

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Theorem

P-*uniform poly-size symmetric threshold circuits* $=$ FPC.

Structures / Logic

Vocabulary τ

Finite τ-structures $\mathrm{fin}[\tau]$

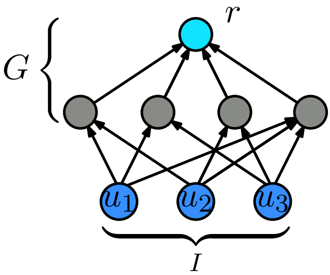
FPC Inflationary fixed-point logic extended with the ability to express the size of definable sets.

- Assume standard syntax and semantics.
- Expresses properties invariant to isomorphisms of structures.

Colored DAGs

A $\mathbb{C}$-Colored Directed Acyclic Graph (CDAG) over a set U:

- Gates G
- Inputs I
- Directed edges E – form acyclic graph on $G \uplus I$ with leaves I with a single root gate r.
- Coloring $\xi : G \uplus I \to \mathbb{C}$
- Input Tuples $\lambda : I \to U^k$



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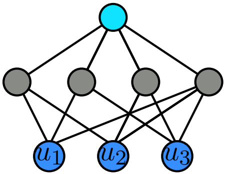

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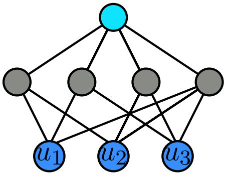

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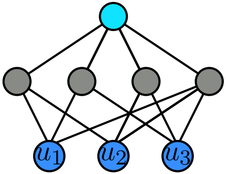

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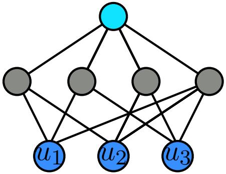

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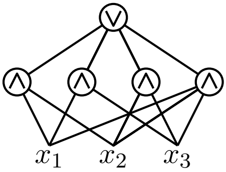

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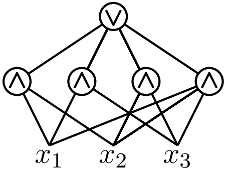

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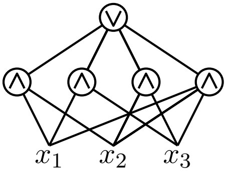

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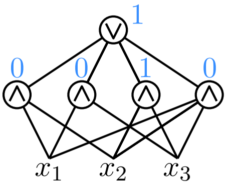

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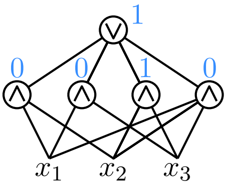

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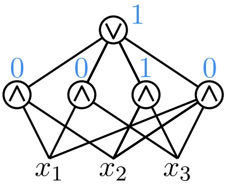

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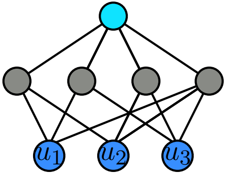
- A circuit is invariant if the value computed at the root is independent of isomorphisms of the structure.
- A family of invariant Boolean circuits on τ-structures for all sizes of U defines a function $\text{fin}[\tau] \to \{0, 1\}$.

Symmetric CDAGs

Let $C = (G, I, E, \xi, \lambda)$ be a CDAG over U.



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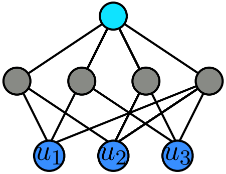

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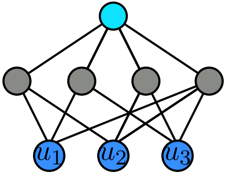

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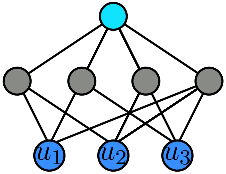

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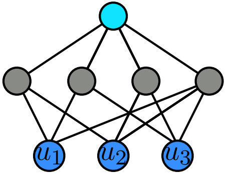

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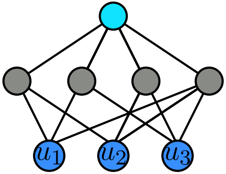

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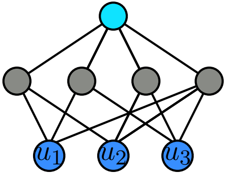

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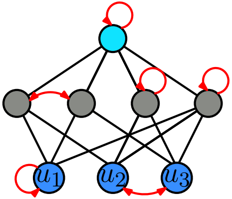

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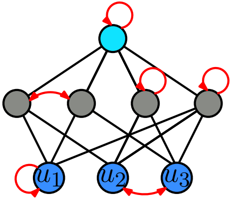

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Call C symmetric if $\forall\sigma \in \mathrm{Sym}_U$, σ induces an automorphism of C.

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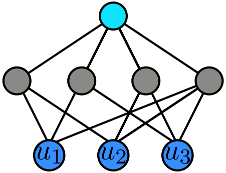

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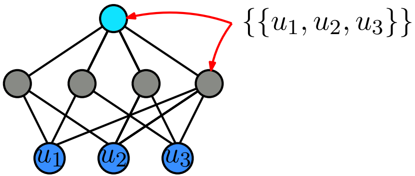

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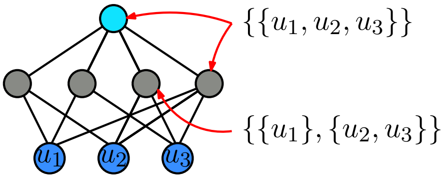

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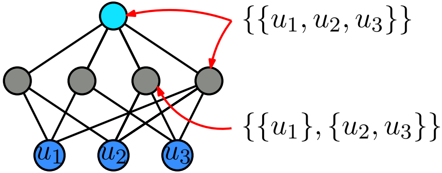

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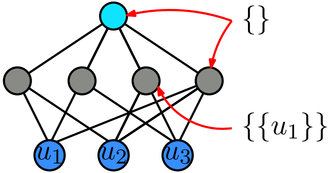

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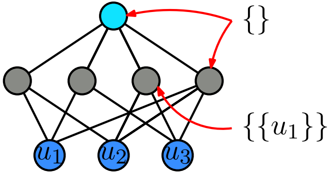

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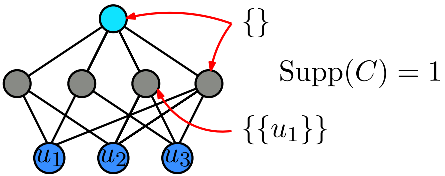

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For any $1 > \epsilon \geq \frac{2}{3}$, let C be a symmetric s-node CDAG over U with $\log |U| \geq \frac{56}{\epsilon^2}$, and $s \leq 2^{|U|^{1-\epsilon}}$. Then

$$\mathrm{Supp}(C) \leq \frac{33}{\epsilon} \frac{\log s}{\log |U|}.$$

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Corollary

Poly-size symmetric CDAGs have constant support.

Application: symmetric threshold circuits = FPC

The corollary leads to a characterisation of FPC.

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Proof Idea

1. Generate the P-uniform circuit over the number sort, using the Immerman-Vardi theorem.
2. Label gates with their support partition.
3. Transform labels into tuples by duplicating gates.
4. Determine equality test indicating edges.
5. Evaluate circuit w.r.t. unordered universe using equality test.

Bonus Application: Circuit Lower Bounds

Consider arithmetic circuits whose inputs are matrices $X \in \mathbb{F}^{U \times U}$:

- Constants 0, and 1.
- Basis $+, -,$ and $\times$.
- Variables $X = \{x_{u,v}\}_{u,v \in U}$.

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One of the most efficient, i.e., size $2^{O(|U|)}$, ways of computing $\mathrm{Per}(X)$ known is as a symmetric multilinear formula [Ryser '57].

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The permanent $\mathrm{Per}(X)$ is the invariant polynomial:

$$\mathrm{Per}(X) := \sum_{\sigma \in \mathrm{Sym}_U} \prod_{u \in U} x_{u,\sigma(u)} = \sum_{S \subseteq U} (-1)^{|U \setminus S|} \prod_{u \in U} \sum_{v \in S} x_{u,v}.$$

One of the most efficient, i.e., size $2^{O(|U|)}$, ways of computing $\mathrm{Per}(X)$ known is as a symmetric multilinear formula [Ryser '57].

Bonus Application: Circuit Lower Bounds

Consider arithmetic circuits whose inputs are matrices $X \in \mathbb{F}^{U \times U}$:

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Context: $2^{\Omega(\log^2 |U|)}$ size for multilinear formulas [Raz '08].

Summary

Support Theorem

For any $1 > \epsilon \geq \frac{2}{3}$, let C be a symmetric s-node CDAG over U with $\log|U| \geq \frac{48}{\epsilon}$, and $s \leq 2^{|U|^{1-\epsilon}}$. Then

$$\mathrm{Supp}(C) \leq \frac{24}{\epsilon} \frac{\log s}{\log|U|}.$$

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Applications:

Theorem

P-*uniform poly-size symmetric threshold circuits* $=$ FPC.

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Open Questions

- Can the notion of support be generalised:
  - to multi-sorted domains,
  - to subgroups of Sym_U, or
  - to larger ranges of ϵ?
- Are there other applications in logic or circuit complexity?
- Is there a similar circuit characterisation of $\tilde{\mathrm{C}}\mathrm{PT(Card)}$?

Thanks!