Chapter 9
Augmented Lagrangians
(Chapter Fragment)

We give fundamental properties for the augmented Lagrangian:

$$L(x,u,\alpha) := f(x) + \frac{1}{2\alpha}\left[\|(\alpha g(x)+u)_+\|^2 - \|u\|^2\right], \alpha > 0,$$

associated with the nonlinear program:

$$\min\ f(x),\ \text{s.t. } g(x) \leq 0,$$

where $f: R^n \to R$ and $g: R^n \to R^m$, are twice continuously differentiable functions on R^n. Let

$$\nabla L(x,u,\alpha) = \begin{bmatrix} \nabla f(x) + (\alpha g(x)+u)_+ \nabla g(x) \\ \frac{1}{\alpha}\Big(\big(\alpha g(x)+u\big)_+ - u\Big) \end{bmatrix},$$

let $\nabla L(\bar{x},\bar{u},\alpha) = 0$, $\Big|g(\bar{x})\Big| + \bar{u} > 0$, $I = \{i \Big| g_i(\bar{x}) = 0\}$, $J = \{i \Big| g_i(\bar{x}) < 0\}$, $\bar{m} = \text{card}(I)$, $\tilde{m} = \text{card}(J)$, $\bar{m} + \tilde{m} = m$ and:

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$$\nabla^2 L(\bar{x},\bar{u},\alpha) = \begin{bmatrix} \nabla^2 f(\bar{x}) + \bar{u}_I \nabla^2 g(\bar{x}) + \alpha \nabla g_I(\bar{x})^T \nabla g_I(\bar{x}) & \nabla g_I(\bar{x})^T & 0 \\ \nabla g_I(\bar{x}) & 0 & 0 \\ 0 & 0 & -\frac{I}{\alpha} \end{bmatrix}$$

9A Lemma

$$\left.\begin{array}{l} \text{a. Rows of D lin. indep.} \\ \text{b. } Dx = 0,\ x \neq 0 \Longrightarrow xBx > 0 \\ \text{c. } \mu \geq 0 \end{array}\right\rangle \Longrightarrow \begin{bmatrix} B + \mu A^T A & D^T \\ D & 0 \end{bmatrix}^{-1} \text{ exists}$$

Proof.

$$\begin{bmatrix} B + \mu A^T A & D^T \\ D & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0 \Longrightarrow \begin{array}{ll} (B + \mu A^T A)x + D^T y & = 0 \\ Dx & = 0 \end{array}$$

$$\begin{array}{cclcccc} \text{Since} & (B+\mu A^T A)x + D^T y & = 0 & \text{By } b,c & x = 0 & \text{By a} & x = 0 \\ \Longrightarrow & Dx & = 0 & \Longrightarrow & D^T y = 0 & \Longrightarrow & y = 0 \\ yDx = 0 & x(B+\mu A^T A)x & = 0 & & & & \end{array}$$

□

9B Debreu's Lemma (Econometrica 20, 1952, 295–300)

$$\Big\langle Ax = 0,\ x \neq 0 \Longrightarrow xBx > 0 \Big\rangle \Longleftrightarrow \left\langle \begin{array}{l} B + \mu A^T A \text{ pos. def.} \\ \text{for } \mu \geq \bar{\mu} \text{ for some } \bar{\mu} \end{array} \right\rangle$$

Proof. ($\Longleftarrow$) If not $\exists x \neq 0,\ Ax = 0,\ xBx \leq 0$. Then for $\mu \geq \bar{\mu}$:

$$0 < x(B + \mu A^T A)x = xBx \leq 0. \quad \text{Contradiction.}$$

($\Longrightarrow$) If not, $\exists \{x^i\},\ \|x^i\| = 1$ s.t.:

$$x^i(\frac{1}{i}B + A^T A)x^i \leq 0, i = 1,2,3,\ldots\ .$$

Hence $x^i B x^i \leq 0,\ i = 1,2,3,\ldots$.
Letting $\{x^{i_j}\} \to \bar{x} \neq 0$ gives $\bar{x}^T A^T A \bar{x} \leq 0$.
Hence $A\bar{x} = 0,\ \bar{x} \neq 0,\ \bar{x}B\bar{x} \leq 0$. Contradiction.

□

10 Theorem (Positive definiteness of $\nabla_{11}L(\bar{x},\bar{u},\alpha)$ and nonsingularity of $\nabla^2 L(\bar{x},\bar{u},\alpha)$). Let $\nabla L(\bar{x},\bar{u},\alpha)=0,\ \alpha>0$. Then:

(i)

$$\left.\begin{array}{l} \text{(a) } \bar{u}_I > 0 \text{ for } g_I(\bar{x})=0 \\ \quad\text{(Nondegeneracy)} \\ \text{(b) } f,g \in C^2(\bar{x}) \\ \quad \nabla g_I(\bar{x})x=0,\ x\neq 0 \Longrightarrow x\nabla_{11}L_0(\bar{x},\bar{u})x>0 \\ \quad\text{(2nd order sufficiency)} \end{array}\right\rangle \Longrightarrow \left\langle \begin{array}{l} \nabla_{11}L(\bar{x},\bar{u},\alpha) \text{ pos. def.} \\ \forall\alpha\geq\bar{\alpha} \text{ for some } \bar{\alpha} \end{array}\right.$$

(ii) **JNC: Jacobian Nonsingularity Condition**

$$\left.\begin{array}{l} \text{(a)As in (a) above.} \\ \text{(b)As in (b) above.} \\ \text{(c) } \nabla g_{j\in I}(\bar{x}) \text{ are lin. indep.} \end{array}\right\rangle \Longrightarrow \left\langle \begin{array}{l} \nabla^2 L(\bar{x},\bar{u},\alpha)^{-1} \text{ exists} \\ \forall\alpha\geq\bar{\alpha} \end{array}\right.$$

Proof.

(i) Set in Debreu's Theorem: $B=\nabla_{11}L_0(\bar{x},\bar{u}),\ A=\nabla g_I(\bar{x}),\ \alpha=\mu$.

(ii) Set in Lemma 9A: $B=\nabla_{11}L_0(\bar{x},\bar{u}),\ D=A=\nabla g_I(\bar{x}),\ \alpha=\mu$.

□

12 Corollary Under assumptions (a) & (b) of Theorem 10:

$$\begin{array}{rcl} L(\bar{x},u,\alpha) & \leq & L(\bar{x},\bar{u},\alpha) < L(x,\bar{u},\alpha) \\ & \uparrow & \qquad\qquad\quad \uparrow \\ \forall u & \in & R^m \qquad\quad \forall x\in N(\bar{x}),\ x\neq\bar{x} \end{array}$$

where $N(\bar{x})$ is some neighborhood of $\bar{x}$. If f,g are convex, then $N(\bar{x})=R^n$.

Sketch of Proof First inequality flows from concavity in u of $L(\bar{x},u,\alpha)$. Second inequality from pos. def. of $\nabla_{11}L(\bar{x},\bar{u},\alpha)$. The final statement from convexity of $L(x,\bar{u},\alpha)$ in x.