CS 730 May 10, 2001 Final Exam Solutions

1. $(x^{i+1},\ u^{i+1})$ satisfy the KKT conditions for the IQP:

$$\nabla f(x^i) + G^i(x^{i+1} - x^i) + u^{i+1}\nabla g(x^i) = 0, \qquad u^{i+1} \geq 0$$

$$g(x^i) + \nabla g(x^i)(x^{i+1} - x^i) \leq 0,\ \ u^{i+1}(g(x^i) + \nabla g(x^i)(x^{i+1} - x^i)) = 0.$$

Since $(x^{i+1}-x^i)G^i(x^{i+1}-x^i) = 0$ and G^i is positive definite, it follows that $(x^{i+1}-x^i) = 0$. Hence the KKT conditions of IQP become the KKT condditions of (*) and x^{i+1} solves (*).

2.

$$\nabla f(x) = \begin{bmatrix} -2x_1 + x_2 - 1 \\ x_1 - x_2 \end{bmatrix},\ \ \nabla f(\begin{bmatrix} 1 \\ 1 \end{bmatrix}) = \begin{bmatrix} -2 \\ 0 \end{bmatrix}.$$

Hence:

$$x^1 = P(\begin{bmatrix} 1 \\ 1 \end{bmatrix} - \lambda \begin{bmatrix} -2 \\ 0 \end{bmatrix}) = P(\begin{bmatrix} 1 + 2\lambda \\ 1 \end{bmatrix}) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

The point $\left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]$ thus satifies the gradient projection **necessary** optimality condition. **However**, it is not a global solution because $f(x)$ is not convex since its Hessian $\nabla^2 f(x) = \begin{bmatrix} -2 & 1 \\ 1 & -1 \end{bmatrix}$ is not positive semidefinite, for $e\nabla^2 f(x)e = -1$. In fact $\nabla^2 f(x)$ is negative definite. The global solution is at another vertex:$\left[\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}\right]$ with $f(\left[\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}\right]) = -\frac{7}{2}$.

3. By the Quadratic Bound Lemma

$$\text{(QBL)} \quad f(x) \leq f(x^i) + \nabla f(x^i)(x - x^i) + \frac{K}{2}\|x - x^i\|^2 \qquad \forall x \in R^n$$

The right hand side is a strongly convex quadratic function in x which bounds $f(x)$ from above and attains its global minimum when its gradient vanishes, that is when:

$$\nabla f(x^i) + K(x - x^i) = 0 \ \text{ or } \ x - x^i = -K^{-1}\nabla f(x^i).$$

This gives a minimum value for the quadratic function of the right hand side of QBL of

$$f(x^i) - \frac{1}{2K}\, \|\nabla f(x^i)\|^2,$$

which is the desired upper bound to $\min\limits_{x} f(x)$.