CS 730 May 11, 1995 Solutions

1. No. **Any** of the following are valid reasons:

   (i) The necessary optimality condition $0 \leq x \perp \nabla f(x) \geq 0$ is not satisfied at $x = 0$, because:

$$\nabla f(0) = \begin{bmatrix} 0 - 1 + 0 \\ 0 \end{bmatrix} \not\geq 0$$

   (ii) The gradient projection necessary optimality $x = \big(x - \nabla f(x)\big)_+$ is not satisfied at $x = 0$, because

$$0 - \big(0 - \nabla f(0)\big)_+ = \begin{bmatrix} 1 \\ 0 \end{bmatrix}_+ \neq 0$$

   (iii) $f(0) = 0 > -\dfrac{1}{2} = f(1,0)$

   Remark Not Required in Answer: It is nontrivial to give a sufficient optimality condition for this **nonconvex** problem, but it is not difficult either. For each fixed x_2, the objective function is strongly convex in x_1 and hence the global minimum solution with respect to x_1 for each fixed x_2 denoted by $x_1(x_2)$, can be found as follows:

   For $\quad x_2 > 1 : x_1(x_2) = 0,\ f\big(x_1(x_2), x_2\big) = 0.$

   For $1 \geq x_2 \geq 0 : x_1(x_2) = 1 - x_2,\ f\big(x_1(x_2), x_2\big) = -\dfrac{1}{2} - \dfrac{1}{2}x_2^2 + x_2.$

   The minimum of this concave function $f\big(x_1(x_2), x_2\big)$ in x_2 on $\{x_2 | x_2 \geq 0\}$ is at $x_2 = 0$ and so $x_1(0) = 1$ and the global minimum is at $x = (1,0)$ with $f(1,0) = -\dfrac{1}{2}$.

2.

   (a) $P(x,\alpha) = x_1 + x_2 + \dfrac{\alpha}{2}(-x_1 + x_2 + 1)_+^2 + \dfrac{\alpha}{2}(-x_2)_+^2$

$$\nabla P(x,\alpha) = \begin{bmatrix} 1 - \alpha(-x_1 + x_2 + 1)_+ \\ 1 + \alpha(-x_1 + x_2 + 1)_+ - \alpha(-x_2)_+ \end{bmatrix} = 0$$

   Substitution from the first equality in the second gives $2 - \alpha(-x_2)_+ = 0$ and hence $x_2 = -\dfrac{2}{\alpha}$. Substitution for x_2 in the first equation gives:

$$1 - \alpha(-x_1 - \frac{2}{\alpha} + 1)_+ = 0$$

Hence

$$x_1 = -\frac{3}{\alpha} + 1$$

(b) Letting $\alpha \to \infty$ gives a solution of the LP:

$$\bar{x} = lim_{\alpha \to \infty}\Big(-\frac{3}{\alpha} + 1, -\frac{2}{\alpha}\Big) = (1, 0).$$

3. x^i is optimal, because:

$$f(x^i) \geq \inf\{f(x)|g(x) \leq 0\} \geq f(x^i) + \big(\alpha g(x^i) + u^i\big)_+ g(x^i) \geq f(x^i)$$

The first inequality is a consequence of $g(x^i) \leq 0$, the second inequality follows from the Weak Duality Theorem and $\nabla f(x^i) + \big(\alpha g(x^i) + u^i\big)_+ \nabla g(x^i) = 0$, and the third inequality follows from $\big(\alpha g(x^i) + u^i\big)_+ g(x^i) \geq 0$.