1 Review of the recent lectures

• A and B are independent if
  \[ P(AB) = P(A)P(B) \]  

• A and B are mutually exclusive if
  \[ P(AB) = 0 \]  

In other words
\[ P(A \cup B) = P(A) + P(B) \]

(Originally, \( P(A \cup B) = P(A) + P(B) - P(AB) \))

eg1) Coin toss twice.
A = 1st fair coin toss gets Head.
B = 2nd fair coin toss gets Tail.
C = 1st toss is Head
D = 1st toss is Tail
E = 2nd toss is Head
F = 2nd toss is Tail

We know
outcome 1: HH (This is in event A)
outcome 2: HT (This is in event A) (This is in event B)
outcome 3: TH
outcome 4: TT (This is in event B)

By looking at this, we see
A= outcome 1 or outcome 2.
B= outcome 2 or outcome 4.
A and B = outcome 2.

A and B are not mutually exclusive. Proof:
A and B can happen at the same time! (outcome 2 is in common).

A and B are independent. Proof:
After some calculation (see the last lecture)
\[ P(A)P(B)=1/2 \times 1/2=1/4=P(AB) \]

C and D are mutually exclusive. Proof:
P(CD)=P(1st toss= Head and 1st toss= Tail)=0.

C and D are NOT independent. Proof: \( P(CD)=0, P(C)=1/2, P(D)=1/2 \), Hence \( P(C)P(D)=1/4 \).
\[ P(CD) \neq P(C)P(D) \].

Caution: Sometimes, we say Independent (irrelevant) in different meaning in daily life, but this is different from our definition of Event A and event B are independent, i.e., \( P(AB) = P(A)P(B) \).

eg) 7 balls are in the box. 1 ball= Winning ball (W), 6 balls = Looser balls (L). 2 people draw a ball without replacement.
Q1) Show \( P(X_1 = W) = P(X_2 = W) \). That means it is irrelevant to draw a ball 1st or 2nd.

Hint1: Use LEC 8, p5, line 1.
Trick: \( P(A) = P(A \cap B) + P(A \cap \bar{B}) \)

ANS 1:
Q2) Show $X_1 = W$ and $X_2 = W$ are dependent.

ANS 2:


2 [4.6] The Rule of Combinations

- $k! = k(k - 1)(k - 2) \cdots (2)(1)$. (Not in the textbook?)
  - $1! = 1$.
  - $0! \equiv 1$.

eg) You have 5 cards A, B, C, D, and E. You pick up one and make sequence from left to right in front of you. For example, the outcome can be [ABCED] or [EDCAB], etc. You will never observe [AAABC] because there is only one A. [ABCDE] $\neq$ [EDCBA] because LHS (left hand side of the equation), A is the leftest (1st draw =A), while RHS, E is the leftest (1st draw =E).

Now question is : how many different outcomes are there?

**Answer:**

- **Combination (p184):** N choose r ($r \leq N$):
  \[
  \binom{N}{r} = \frac{N(N - 1)(N - 2) \cdots (N - r + 1)}{r(r - 1) \cdots (2)(1)} \quad \text{(Formula 1)}
  \]
  \[
  = \frac{N!}{r!(N - r)!} \quad \text{(Formula 2)}.
  \]

**Formula 2** can be used to prove (p185)

\[
\binom{N}{r} = \binom{N}{N-r}
\]

Sometimes, this modified expression is easier to calculate.

eg1) \[
\binom{150}{146} =
\]

eg2) \[
\binom{23}{11} =
\]

eg3) \[
\binom{100}{98} =
\]

eg4) 5 cats (A,B, C, D, E) and 2 dogs (G,H) in the room. Line them up from left to right. For example, case1):[ABCDEGH],
case2):[BACDEHG],
case3):[HABDECG], etc. If you just focus on position of dogs and cats, how many distinct sequences (outcomes) are there? **Answer:** Using T=cat, O=dog, we can rewrite above 3 cases as: case1):[TTTTTOO]
case2):[TTTTTOO]
case3):[OTTTTO]

Hence case1 = case2. Hence 3 cases are actually 2 distinct outcomes. We want to get number of distinct outcomes. Use "Rule of Combinations".

(Use Formula 1). Or equivalently, using Formula 2,
eg5) Same problem, but now distinct each individual. Hence case1 ≠ case2. How many distinct outcomes?

Answer:

eg6) (Modification of p186 Ex 27) Toss fair coin 7 times. P(3 are H out of 7 tosses)=?

Answer: "3 Hs out of 7 tosses" event includes
[HHTTTTT],
[HTHTTTT],
[TTHTHTT],

etc. (Each of them are called elementary outcomes. Remember?) Hence the number of distinct outcome which satisfies "3 Hs out of 7 tosses" is

(= number of elementary outcomes in "3 Hs out of 7 tosses", if I use a notation in p142).

The total number of elementary outcomes (= number of elementary outcomes in S where S= Sample space in notation p142 ) ,which includes

[TTTTTTT],
[HHHHHHH],
[THHHHHH],

etc is $2^7$. Hence using uniform probability model (p142),

\[ P(3 \text{ Hs out of 7 tosses}) = \]

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eg4) Fair 3-face dice 1 throw (A,B,C):
1:A
2:B
3:C
Hence, $3^1 = 3$ outcomes.

eg5) Fair 3-face dice 2 throws (A,B,C):
1:AA
2:AB
3:AC
4:BA
5:BB
6:BC
7:CA
8:CB
9:CC
Hence, $3^2 = 3$ outcomes.

eg6) Fair 3-face dice 5 throws (A,B,C):
1:AAAAA
2:AAAAAB
...
...
?:CCCCC
I don't want to draw it. Use the formula, $3^5 = 243$ outcomes.
eg7) (Lecture 7 p14) Fair coin 7 throws:
1:HHHHHHH
2:HHHHHHT
...
8:TTTTTTTT

I don’t want to draw it. Use the formula, $2^7 = 128$ outcomes.

eg8) Fair coin toss, (2-face) 3 throws. $A=2$ are head
using 2-face fair dice throw 3 times. $P(A)=?:
1:HHH
2:HHT*
3:HTH*
4:HTT
5:THH*
6:THT
7:TTH
8:TTT

Hence (by just looking at the figure) $m=3$. Or you can use the formula

$$m = \binom{3}{2} = \frac{3!}{2!(3-2)!} = \frac{3 \cdot 2 \cdot 1}{2 \cdot 1 \cdot 1} = 3.$$  

Remember, eg3 result shows $k = 2^3 = 8$. Hence $P(A) = \frac{m}{k} = \frac{3}{8}$.

eg9) Fair 3-face dice 5 throw (A,B,C).(Same as eg6)
M= B appears 4 times. $P(M)=?:
1:AAAAA
2:AAAAAB
...
9:BBABB*
...
15:BCBBB*
...
21:CCCCC

Some events in $M$ are marked by “∗”. But it is tedious to mark all outcomes in $M$. So, just let’s use the formula

$$m = \binom{5}{4} (3-1)^{(5-4)} = \frac{5!}{4!(5-4)!} = \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{4 \cdot 3 \cdot 2 \cdot 1} \cdot 1 = 10.$$  

Hence $P(M) = \frac{m}{k} = \frac{10}{243} = 0.041$. (Remember, we got $k=243$ in eg6).
eg10) Fair 2-face dice 5 throws (A,B). (similar to eg9) M = B appears 4 times. P(M) = ?:
1:AAAAA
2:AAAAB
...
?:BBABB*
...
?:BABBB*
...
?:BBBBB
Some events in M are marked by “*”.

\[
m = \binom{5}{4} = \frac{5!}{4!(5-4)!} = \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{4 \cdot 3 \cdot 2 \cdot 1} = 5.
\]  

\[
k = 2^5 = 32 \text{ Hence } P(M) = \frac{m}{k} = \frac{5}{32}.
\]

3 Ch5 Random Variables and Probability Distribution

- Random Variable: A random variable, usually denoted by X or Y, associates a numerical value with each outcome of an experiment.

- Probability Distributions for Discrete Random Variables:
  The probability distribution of a discrete random variable X is described as the function,
  \[ f(x_i) = P(X = x_i) \]
  which gives the probability for each value and satisfies:
  1) \( f(x_i) \geq 0 \), for each value \( x_i \) of \( X \).
  2) \( \sum_{i} f(x_i) = 1 \).

eg) Cast a fair 5-face dice value \( \in \{1, 2, 3, 4, 5\} \). Define
\[ X = 20 \text{ if ith observation is even number.} \]
\[ X = 30 \text{ if ith observation is odd number.} \]
Then,
\[ P(X = 20) = \frac{2}{5} \text{ (dice = 2 or 4).} \]
\[ P(X = 30) = \frac{3}{5} \text{ (dice = 1 or 3 or 5).} \]

In this case, \( x_1 = 20 \), \( x_2 = 30 \). (Or you can define \( x_2 = 20 \), \( x_1 = 30 \). It does not matter.)

- population mean or the Expected Value of X: \( \mu = E(X) = \sum (\text{value} \times \text{probability}) = \sum x_i f(x_i) \)
The mean of a random variable X is also called its expected value, denoted by E(X).
Here the sum extends over all the distinct values \( x_i \) of X.
Remember, sample mean is random variable, while population mean is fixed value.

- (Population) Variance and (Population) Standard Deviation of X:
  Population variance Formula 1:
  \[ \sigma^2 = \text{Var}(X) = \sum (\text{value} - \text{mean})^2 \times \text{probability} = \sum [(x_i - \mu)^2 f(x_i)] \]
  Population variance Formula 2
  \[ \sigma^2 = \left( \sum x_i^2 \cdot f(x_i) \right) - \mu^2 = E(X^2) - [E(X)]^2 \]

Population standard Deviation (SD):
\[ \sigma = sd(X) = \sqrt{\text{Var}(X)} \]
Note: SD is nothing to do with Nintendo DS.
Remember:
Sample mean = \( \frac{\sum x_i}{n} = \bar{X} \)

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4) Sample Variance
\[ = \frac{\sum (X_i - \bar{X})^2}{n-1} \]
Sample SD
\[ = \sqrt{\frac{\sum (X_i - \bar{X})^2}{n-1}} \]

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Then,
\[ P(X = 20) = \]
\[ P(X = 30) = \]
What is population mean?
Answer:
\[ \mu = \bar{X} = \]
What is population variance?
Answer 1: Use Formula 1,
Answer 2: User Formula 2.

What is population SD?
Answer: $\sigma = \sqrt{24}$.

Now let $i$th observation $= X_i$. This means $X_i = x_1$ or $X_i = x_2$, which means $X_i = 20$ or $X_i = 30$.

I throw it 3 times and faces are 3, 5, 2. Hence $X_1 = 30$, $X_2 = 30$, $X_3 = 20$. Then,
Sample mean =
Sample Variance =

Mike throws it 2 times and gets face 1, 4. Hence $X_1 = 30$, $X_2 = 20$. Then
Sample mean $= (30 + 20)/2 = 25$
Sample Variance =

Remember, sample mean, sample variance are random (changes all the time). Population mean and Population variance are fixed value.

- Linear Transformation of $E(X)$ and $Var(X)$ (Useful rules)
Suppose that $X$, $Y$ are random variables and $a$, $b$ are constants. Then,
$E(aX + b) = aE(X) + b$
$Var(aX + b) = a^2Var(X)$,
$E(aX + bY) = aE(X) + bE(Y)$
$Var(aX + bY) = a^2Var(X) + b^2Var(Y)$ when $X$ and $Y$ are independent.