1 Reminder

Mid1 (Oct 7(Thur) in class 75 min) = Ch1, 2, 4, 5, 6 (up to Oct 5 lecture).

- Seating is alphabetical order. I will show it on Tuesday.
- Try be in the class 10 minutes earlier.
- You will assigned to solve a specific exam. We write sequential numbers on the top of the exam. If you solve another person’s exam, you will get 0 point.
- 1 letter size cheating paper, both side, typed one is fine.
- Bring a hand calculator (no cell phone allowed)
- Bring your student ID and pencils.
- All multiple choice questions. About 60% from HWs and suggested problems. About 40% from lecture notes and R. Mainly numerical value will be modified.
- Suggestion: Try to solve all R related problems. If you did HWs, they should be easy. (I mean not time consuming).
- See the last year mid 1 posted on the course HP.
• Remember, the final grade is relative scale. If you see difficult problems in the exam, skip it! No one will solve it. (Don’t skip all problems.)

• All solutions of end of chapter are posted on learnuw. (I got the request...) If you want to solve beyond HWs, suggested problems. try them. They will be pretty similar.

• **Write your name, ID first.**

• When the time is up (12:15pm), I will ask you to stop. Please stop. Otherwise, you will get **-4 points as penalty.** NO EXCEPTION. Everyone knows who is which seat... It is just for the matter of fairness.
2 Tips

Q) What is common in Avatar, music, sports, and Statistics?
3 Review of the last lecture: Binomial Distribution (p227)

- The following $X$ is called a **Binomial Random Variable** and its distribution is called **Binomial distribution**.
  
  $n =$ # of Bernoulli trials
  
  $p =$ probability of Success
  
  $X =$ # of Success

- Binomial Distribution function with $n$ trials and success probability $p$ is

  $$f(x) = P(X = x) = \binom{n}{x} p^x (1 - p)^{n-x}, \quad x = 0, 1, 2, 3, \ldots, n \quad (1)$$

- $\mu$ (population mean) and $\sigma^2$ (population variance) of Binomial Distribution:

  $$\mu = E(X) = np$$
  $$\sigma^2 = Var(X) = np(1 - p)$$
  $$\sigma = SD(X) = \sqrt{np(1 - p)}$$
Q) For the binomial distribution with \( n=4 \) and \( p=.45 \) find the probability of

(a) Three or more successes
(b) At most three successes
(c) Two or more failures

ANS)
eg1) Fair coin toss, (2-face) 3 throw.
Let \( A=2 \) are heads out of 2-face fair dice (coint) throw 3 times. In other words, \( X=2 \) in Binomial distribution. \( P(A)=P(X=2)=? \):
outcomes which are in A is marked by " * ":

outcome 1: HHH
outcome 2: HHT*
outcome 3: HTH*
outcome 3: HTT
outcome 5: THH*
outcome 6: THT
outcome 7: TTH
outcome 8: TTT
Hence (by just looking at the figure) \( m=3 \). Or you can use the formula

\[
m = \binom{3}{2} = \frac{3!}{2!(3-2)!} = \frac{3 \cdot 2 \cdot 1}{2 \cdot 1 \cdot 1} = 3.
\]

(2)
We got \( P(A) \) using uniform discrete model \((m/k=3/8)\). But now, let’s calculate \( P(A) \) using Binomial formula.
Let $Y_i = i$-th result. In other words, 

$$X = Y_1 + Y_2 + Y_3.$$  

(3) 

$X = \{0, 1, 2, 3\}$. $Y_i = \{0, 1\}$. Observe, 

$$P(outcome 2) = P([Y_1 = H] \cap [Y_2 = H] \cap [Y_3 = T])$$  

(rule : independent) 

$$= P(Y_1 = H)P(Y_2 = H)P(Y_3 = T)$$  

(4) 

$$= (1/2)(1/2)(1/2) = 1/8.$$ 

$$P(outcome 3) = P([Y_1 = H] \cap [Y_2 = T] \cap [Y_3 = H])$$  

(rule : independent) 

$$= P(Y_1 = H)P(Y_2 = T)P(Y_3 = H)$$  

(5) 

$$= (1/2)(1/2)(1/2) = 1/8.$$ 

$$P(outcome 5) = P([Y_1 = T] \cap [Y_2 = H] \cap [Y_3 = H])$$  

(rule : independent) 

$$= P(Y_1 = T)P(Y_2 = H)P(Y_3 = H)$$  

(6) 

$$= (1/2)(1/2)(1/2) = 1/8.$$ 

So they have same probability. 

$$P(A) = P(outcome 2 \cup outcome 3 \cup outcome 5)$$  

$$= P(outcome 2) + P(outcome 3) + P(outcome 5) \text{ (rule: mutually exclusive)}$$
=3*P(outcome 2) (rule: we just showed they have same probability) =3*(1/8)=3/8.

Note ”3” (the number of outcome of our interest) comes from $\binom{3}{2} = 3$.

\[
P(A) = \binom{3}{2} \cdot \left(\frac{1}{2}\right)^2 \left(1 - \frac{1}{2}\right)^{(3-2)} = 3 \cdot \frac{1}{8}
\]

(7)

This is Binomial formula. $\left(\frac{1}{2}\right)^2$ is for Head (you observe 2 Heads). $\left(1 - \frac{1}{2}\right)^{(3-2)}$ is for Tail (you observe 3-2=1 Tail).

Confusing? Let’s look at unfair coin toss.
eg2) **Unfair** coin toss, (2-face) 3 throws.

P(Head) = P(Success) = 0.3

Let **A = 2 are head out of 2-face unfair dice throw 3 times**. P(A) = ?:

outcomes which are in A is marked by “∗”:

- outcome 1: HHH
- outcome 2: HHT∗
- outcome 3: HTH∗
- outcome 3: HTT
- outcome 5: THH∗
- outcome 6: THT
- outcome 7: TTH
- outcome 8: TTT

Hence (by just looking at the figure) m=3. Or you can use the formula

\[
m = \binom{3}{2} = \frac{3!}{2!(3 - 2)!} = \frac{3 \cdot 2 \cdot 1}{2 \cdot 1 \cdot 1} = 3. \tag{8}
\]

Let \(Y_i = i\)-th result. We already know

\[
A = [\text{outcome2} \cup \text{outcome3} \cup \text{outcome5}]
\]

Observe,
\[
P(\text{outcome}2) = P([Y_1 = H] \cap [Y_2 = H] \cap [Y_3 = T])
\]
\[
\text{(rule : independent)}
\]
\[
= P(Y_1 = H)P(Y_2 = H)P(Y_3 = T)
\]
\[
= (0.3)(0.3)(0.7) = 0.63.
\]  

\[
P(\text{outcome}3) = P([Y_1 = H] \cap [Y_2 = T] \cap [Y_3 = H])
\]
\[
\text{(rule : independent)}
\]
\[
= P(Y_1 = H)P(Y_2 = T)P(Y_3 = H)
\]
\[
= (0.3)(0.7)(0.3) = 0.63.
\]  

\[
P(\text{outcome}5) = P([Y_1 = T] \cap [Y_2 = H] \cap [Y_3 = H])
\]
\[
\text{(rule : independent)}
\]
\[
= P(Y_1 = T)P(Y_2 = H)P(Y_3 = H)
\]
\[
= (0.7)(0.3)(0.3) = 0.63.
\]  

Now you see, even if we use unfair coin, each outcome of our interest has same probability. So,

\[
P(A) = \text{(number of outcome in A) \times (outcome 2’s probability)}
\]

will give us an answer in this case. Or you could use

\[
P(A) = \text{(number of elementary outcome in A) \times (outcome 3’s probability)}
\]
You don’t have to calculate \( P(\text{outcome } 2) \) AND \( P(\text{outcome } 3) \) AND \( P(\text{outcome } 5) \). Because they are same.

Note: \( P(\text{outcome } 1) \neq P(\text{outcome } 2) \).
\[ P(\text{outcome } 1) = 0.3^3 = 0.027. \]

Summary:

we know : the number of elementary outcome in \( A = m = \binom{3}{2} = 3 \).

we know:
\[ P(\text{outcome } 2) = P([Y_1 = H] \cap [Y_2 = H] \cap [Y_3 = T]) \]
\[ = P(Y_1 = H)P(Y_2 = H)P(Y_3 = T) \]
\[ = (0.3)(0.3)(0.7) = 0.63 \]
\[ = 0.3^2(1 - 0.3)^{3-2}. \]

Hence,
\[ P(A) = (\text{number of elementary outcome in } A) \times (\text{outcome } 2\text{'s probability}) \]
\[ = \binom{3}{2} \times (0.3^2(1 - 0.3)^{3-2}) \]
This is the Binomial formula.
eg4) **Unfair** 3-face dice 5 throw (A,B,C). Let $Y_i = i$-th throw.

$P(Y_i=A) = 0.2,$ 
$P(Y_i=B) = 0.3,$ 
$P(Y_i=C) = 0.5.$

Event $M =$”B appears 4 times.” Question: $P(M) =$?

Can we use Binomial formula? Yes, we can.

Hint: What is $P(Y_i = \overline{B}) =$?
4 Ch6 Normal Distributions

- Continuous Random Variable (r.v.)
  Probability density function for continuous r.v.
  Total area under the curve should be 1.
  \( P(a \leq X \leq b) = \text{area under the curve between } a \text{ and } b. \)
  \( f(x) \geq 0 \text{ for all } x. \)

- If \( X \) is continuous r.v. then \( P(X = x) = 0. \) This fact is much more complicated issue than it looks. So just memorize it for this course. (you need high level math, Measure Theory.)

  Hence
  \( P(a \leq X \leq b) = P(a < X \leq b) = P(a \leq X < b) = P(a < X < b) \)

- Normal Distribution (one kind of continuous distribution)

  If random variable \( X \) has (is) normal distribution, we usually write \( X \sim N(\mu, \sigma^2). \) The following expressions are same:

  "\( X \) is Normal distribution."
  "\( X \) is Normal distribution with mean \( \mu \) and variance \( \sigma^2. \)."
  "\( X \sim N(\mu, \sigma^2). \)."
"X \sim N(\mu, \sigma)."

It is
1) **roughly speaking** bell-shaped.
2) **perfectly** symmetric density.
3) (population) mean \( \mu \) and (population) standard deviation \( \sigma \) (or you can say population variance \( \sigma^2 \)).

- **Standard Normal Distribution**: It is usually denoted by \( N(\mu = 0, \sigma^2 = 1) \). Hence following expressions are same:

"X is standard Normal distribution."
"X is Normal distribution with (population) mean \( \mu = 0 \) and (population) variance \( \sigma^2 = 1 \)."
"X \sim N(\mu = 0, \sigma^2 = 1)."
"X \sim N(\mu = 0, \sigma^2 = 1^2)."
"X \sim N(\mu = 0, \sigma = 1)." (because \( \sigma^2 = 1 \) means \( \sigma = 1 \))
"X \sim N(0, 1)."

"Z" (In many situation, r.v. \( Z \) is \( N(0,1) \). But usually mentioned this at the beginning of the problem.)

Why do we need this distribution? Because it is in the exam? Of course. Moreover because it is used in our life pretty often.
eg1) IQ-distribution (See Wikipedia), \( N(\mu = 100, sd = 15) \).

eg2) Buy same type of cookies. Measure 100 cookies’ weight. The weight distribution will be normal distribution.

counter eg) You run 100 meter 100 times. Measure the speed. (Maybe around 13 sec for the 1st run?) But they are not normal distribution. Because you are getting tired. 100th run will be like 20 sec.

- Let’s look at typical figures of normal distribution in the textbook. Typical questions in the exam will be

Suppose \( Z \) is standard normal.

1) Given location, find area.

eg1) (small) \( z=3 \) is given, \( P(Z \leq z) = P(Z \leq 3) =? \)

eg2) Suppose I got IQ test =100. How many are below me?

Answer: 50% (We will do it next time, for non-standard normal distribution case).

2) Given area, find location.

eg) 0.95 (95%) is given, also define \( P(Z \leq z) = 0.95. \) what is (small) \( z \)?
5 Software R for Ch4, Ch5, Ch6

- $X \sim \text{Binomial}(n=10, p=0.2)$.
  
  $P(X = 3) : \text{dbinom}(3, \text{size}=10, \text{prob}=0.2)$
  
  $P(X \leq 3) : \text{pbinom}(3, \text{size}=10, \text{prob}=0.2)$

- $X \sim \text{Normal}(\mu = 5, \sigma^2 = 16)$.
  
  $P(X \leq 3) : \text{pnorm}(3, \text{mean}=5, \text{sd}=4)$

  Remember for continuous r.v.,
  
  $P(X = 3) = 0$
  
  But $f(3) \neq 0$.

  Here $f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{(x-\mu)^2}{\sigma^2}\right)}$ (see p261) You don’t need to memorize this. Not in the exam in this course. $f(x)$ is called density function.
**What are parameters?**
Parameters determine its distribution function uniquely in the model.

eg1) Suppose your model is Binomial distribution. If you specify $n=1$, $p=0.5$ (actually this is Bernoulli distribution), then the distribution is like:

eg2) Suppose your model is Binomial distribution. If you specify $n=1$, $p=0.1$ (actually this is Bernoulli distribution), then the distribution is like:
eg3) Suppose your model is Binomial distribution. If you specify \( n=2, \ p=0.3 \), then the distribution is like:

eg4) Suppose your model is Binomial distribution. If you specify \( n=1 \), then the distribution is like: like what?

eg5) Suppose your model is \( N(\text{mean}=0, \ \text{sd}=1) \). (standard normal) Then, its distribution is like:

eg6) \( W_1 \sim N(\text{mean} = -2, \ \text{sd} = 1) \), \( W_2 \sim N(\text{mean} = -2, \ \text{sd} = 2) \), \( W_3 \sim N(\text{mean} = -2, \ \text{sd} = 0.5) \), \( W_4 \sim N(\text{mean} = \)
3, $sd = 1$). Then these distributions are:

\begin{align*}
\text{eg7)} \quad & \text{Suppose } Y \text{ is from normal distribution with } EY = 0, Var(y) = 1. \text{ Then the distribution is like:} \\
\text{eg8)} \quad & \text{Suppose } Y \text{ is from normal distribution with } EY = 1, Var(y) = 0.5. \text{ Then the distribution is like:} \\
\text{eg9)} \quad & \text{Suppose } T \text{ is from Binomial distribution with } ET = 1, Var(T) = 0.5. \text{ Then the distribution is like:}
\end{align*}
6 Ch6 More on Normal distribution

- (general result) Difference of median and mean (p258):
  (A1) Symmetric density, (A2) left skewed (A3) right skewed. Note: Normal distribution is always (A1) type.

- Relation between location and area (p257):
  Assume general continuous distribution. It can be Normal distribution, Uniform distribution etc. But the concept of area, location is same. Let’s take a look.

  eg) (A1) $P[a < X < b] = ?$ (A2) $P[a < X] = ?$
  (A3) $P[X < a] = ?$ (a, b are given value) Show me those area.
• Relation between location and area for $N(\mu, \sigma)$ (p261):

Typically 2 kind of questions in this section.
Type 1) Given location, find area:
eg1) (small) $z=3$ is given, $P(Z \leq z) = P(Z \leq 3) =$?
eg2) Suppose I got IQ test =100. How many are below me?

Answer: Let $W \sim N(mean = 100, sd = 15)$ (Wikipedia said this is the IQ distribution), and solve,
$P(W \leq 100) =$?

50% because normal distribution is symmetric. Below mean (in this case, $\mu = 100$) is always 50% This means 50% of population will be below me.

Type 2) Given area, find location:
eg1) 0.95 (95%) is given, also define $P(Z \leq z) = 0.95$. what is (small) $z$?

eg2) To be (95%) percentile, what IQ score you should take? Let $W \sim N(100, sd = 15)$, and solve small $w$; $P(W \leq w) = 0.95$?

Answer: Using R program:

```r
> qnorm(0.95,100,15)
124.6728
```

• Find (A1) $P[Z \leq 1.37] = \ ?$ (A2) $P[Z \geq 1.37] = \ ?$ using Appendix B talbe 3. (p264, example 1)

  step1) $z =$

  step2) Look at column and raw.

  step3) Find correspondent entry.

• Find (A3) $P[Z \leq -1.37] = \ ?$ (A4) $P[-1.37 \leq Z \leq 1.37] = \ ?$
• \( P[-0.15 \leq Z \leq 1.60] =? \) (p265, Eg2)

• \( P(Z > b) = 0.25 \). Find b. (p266 Eg4)

• \( P[-b \leq Z \leq b] = 0.90 \) Find b where \( b > 0 \). (p267 eg5)
  Hint: Remember, Normal distribution is symmetric around mean. In this case, mean=0.
7 Ch6.5 Transform non-standard normal dist. to $N(0, 1)$

- If $X \sim N(\mu, \sigma)$ then $Z$ becomes standard normal where,

$$Z = \frac{X - \mu}{\sigma}.$$  

(12)

eg1) Suppose $X \sim N(\mu = 3, \sigma = 1)$, find $P[2.85 \leq X \leq 4.60] =$?

eg2) Suppose $X \sim N(\mu = 3, \sigma = 2)$, find $P[2.7 \leq X \leq 6.2] =$?